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Sigma-finite dual dentability indices

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Dedicated to Professor Isaac Namioka on the occasion of his 80th birthday

Abstract

A characterization of Banach spaces admitting uniformly Gâteaux smooth norms in terms of σ-finite dual dentability indices is given. Some applications in the area of weak compactness are discussed. We also study σ-locally uniformly rotund dual renormings in connection with σ-countable dual dentability indices.

1 Introduction

Banach spaces that can be renormed by uniformly Fréchet smooth norms were characterized by Enflo, James, and Pisier in terms of Walsh-Paley martingales (see, e.g., [3, Chapter IV]). For a more elementary approach see, e.g., [7, Chapter 9]. This result was extended to spaces admitting uniformly Gâteaux smooth norms by Troyanski in [25] (see, e.g., [3, Theorem IV.6.8]). A characterization of spaces that admit uniformly Fréchet smooth norms in terms of dual dentability indices was given by Lancien in [19] (see, e.g., [16], [20]).

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In this note we extend Lancien's result to spaces that admit uniformly Gâteaux smooth norms. As a byproduct, we will encounter a notion, strictly stronger than that of weak compactness, which we will briefly discuss. We will show that this approach leads to a characterization of uniform Eberlein compacts in terms of dual dentability indices in the space of Borel measures on them. We also study σ-countable dual dentability indices with respect to renorming by σ-locally uniformly rotund norms and the weak compact generating.

An asset of our approach is its transparent elementary character. We believe that this may help solving some problems in this area, for example Question 5.11a in [2] on the so called three space problem for weakly uniformly rotund renormings.

Our notation is standard. Let $(X, \|\cdot\|)$ be a Banach space (we write just X if mentioning of the norm is not necessary). The dual norm on the dual space X^* will be denoted again $\|\cdot\|$ if there is no possibility of misunderstanding. B_X (or, more precisely, $B_{(X,\|\cdot\|)}$) is the closed unit ball of X, and S_X (or $S_{(X,\|\cdot\|)}$) its unit sphere. Unexplained concepts can be found, for example, in [7].

Let M be a bounded subset of X. Given $f \in X^*$, we denote $|f|_M := \sup_{x\in M} |f(x)|$ and, for a bounded set $S \subset X^*$, we let $\operatorname{diam}_M(S) := \sup\{|f - g|_M; f, g \in S\}$, the M-diameter of S.

Let M be a bounded set in a Banach space $(X, \|\cdot\|)$ and let $\varepsilon > 0$ be given. We say that the dual norm $\|\cdot\|$ on X^* is (M, ε)-*LUR* if $\limsup_n |f_n - f|_M \le \varepsilon$ whenever $f, f_n \in S_{X^*}$ are such that $\lim_n \|f_n + f\| = 2$. The dual norm $\|\cdot\|$ on X^* is called σ-*LUR* if for every $\varepsilon > 0$, there is a decomposition $B_X = \bigcup_{k=1}^\infty M_k^\varepsilon$ such that $\|\cdot\|$ is $(M_k^\varepsilon, \varepsilon)$-LUR for every $k \in \mathbb{N}$. We say that the dual norm $\|\cdot\|$ on X^* is M-*LUR* if it is (M, ε)-LUR for every $\varepsilon > 0$. The dual norm $\|\cdot\|$ on X^* is called *weak**-*LUR* if it is M-LUR for every finite subset M of X. We say that the norm $\|\cdot\|$ on X is M-*uniformly Gâteaux smooth* if $\lim_n |f_n - g_n|_M = 0$ whenever $f_n, g_n \in S_{X^*}$ are such that $\lim_n \|f_n + g_n\| = 2$. We say that the norm $\|\cdot\|$ on X is *strongly uniformly Gâteaux smooth* if it is M-uniformly Gâteaux smooth for some bounded linearly dense set M in X. Using the Šmulyan duality (see, e.g., [3, Section I.1]), we can also define that $\|\cdot\|$ on X is *uniformly Gâteaux smooth* [3, Definition II.6.5] if it is M-uniformly Gâteaux smooth for every finite subset M of X [3, Lemma II.6.6].

The notion of dual σ-LUR norms represents a sort of a common roof over uniformly Gâteaux smooth and Fréchet smooth norms (see Theorem 4 and Theorem 7 below). It is closely related to weak compactness (see [9] and [13]). In particular, the existence of such a norm in a weakly Lindelöf determined space implies that this space is necessarily a subspace of a weakly compactly generated space [9]. We recall that a Banach space X is *weakly Lindelöf determined* if (B_{X^*}, w^*) is a Corson compact space (for definitions see, e.g., [3, Chapter VI], [4], and [7, Chapter 12]). By a *weak***-slice* of a set $D \subset X^*$ we understand the intersection of D with a weak*-open halfspace in X^*. Given a bounded set $M \subset X$, $\varepsilon > 0$, and $D \subset B_{X^*}$, we introduce the (M, ε)*-dentability derivative of* D by

$$D'_{(M,\varepsilon)} := \{f \in D;\ \mathrm{diam}_M(S) \geq \varepsilon \text{ for each weak}^*\text{-slice } S \text{ of } D \text{ containing } f\}$$

Let $\alpha > 1$ be an ordinal number and assume that we already defined a dentability derivatve $D^{(\beta)}_{(M,\varepsilon)}$ for every ordinal $\beta < \alpha$. If $\alpha - 1$ exists, we define the α-th (M, ε)*-dentability derivative of* D as $D^{(\alpha)}_{(M,\varepsilon)} = (D^{(\alpha-1)}_{(M,\varepsilon)})'_{(M,\varepsilon)}$. Otherwise, we put $D^{(\alpha)}_{(M,\varepsilon)} = \bigcap_{\beta<\alpha} D^{(\beta)}_{(M,\varepsilon)}$. We observe a simple fact that, if D is convex and weak*-closed, then so is $D'_{(M,\varepsilon)}$.

Definition 1 *Let $(X, \|\cdot\|)$ be a Banach space. Let a bounded set $M \subset X$ and $\varepsilon > 0$ be given. We say that M* has finite *(resp.* countable) ε-dual index *if $(B_{X^*})^{(\alpha)}_{(M,\varepsilon)} = \emptyset$ for some finite (resp. countable) ordinal number. The first ordinal with this property, if it exists, is called* the ε-dual index of M.

Definition 2 *We say that a Banach space $(X, \|\cdot\|)$* has σ-finite *(resp.* σ-countable) dual index *if, for every $\varepsilon > 0$, there is a decomposition $B_X = \bigcup_{k=1}^{\infty} M_k^{\varepsilon}$ such that each set M_k^{ε} has finite (resp. countable) ε-dual index.*

Remark 3

1. The property of a bounded set in a Banach space X to have finite (resp. countable) ε-dual index is invariant under equivalent renormings of the space X. Therefore, the concept of a Banach space having a σ-finite (resp. σ-countable) index is also invariant under equivalent renormings.

2. It follows from the statement (and the proof) of Theorem 4 that the set B_X in the definition of a Banach space having σ-finite (resp. σ-countable) dual index can be substituted in the very definition by any

bounded and linearly dense set $\Gamma \subset X$. Now Γ can be written, for every $\varepsilon > 0$, as $\bigcup_{k=1}^{\infty} \Gamma_k^{\varepsilon}$, where Γ_k^{ε} has finite (resp. countable) ε-dual index.

2 The results

Theorem 4 *Let $(X, \|\cdot\|)$ be a Banach space. Then the following assertions are equivalent:*
(i) *X admits an equivalent uniformly Gâteaux smooth norm.*
(ii) *X has σ-finite dual index.*

Theorem 5 *Let $(X, \|\cdot\|)$ be a Banach space. Let M be a bounded subset of X. Then the following assertions are equivalent:*
(i) *X admits an equivalent M-uniformly Gâteaux smooth norm.*
(ii) *M has finite ε-dual index for every $\varepsilon > 0$.*
Thus, X admits an equivalent strongly uniformly Gâteaux smooth norm if and only if there exists a bounded linearly dense set $M \subset X$ that has finite ε-dual index for every $\varepsilon > 0$.

Remark 6

1. In view of Remark 11 below, every Banach space with a strongly uniformly Gâteaux smooth norm is weakly compactly generated [8].

2. Note that any norm compact subset K of an arbitrary Banach space $(X, \|\cdot\|)$ has finite ε-dual index for every $\varepsilon > 0$. Indeed, let $\{x_i;\ i \in \mathbb{N}\}$ be a dense subset of K, and consider the dual norm in X^* given by

$$|||f|||^2 = \|f\|^2 + \sum_{i=1}^{\infty} \frac{1}{2^i} f^2(x_i), \quad f \in X^*.$$

   Then it is standard to check (see, e.g., [3, Chapter II]) that the norm $|||\cdot|||$ is K-uniformly Gâteaux smooth and thus, by Theorem 5, K has the proclaimed property.

3. By using Enflo's renorming result (see, e.g., [7, Theorem 9.18]) and Theorem 5, the unit ball in any superreflexive space has finite ε-dual index for every $\varepsilon > 0$.

Theorem 7 *Assume that X has σ-countable dual index. Then X^* admits an equivalent dual σ-LUR, and hence weak*-LUR norm.*

Theorem 8 *Assume that a bounded set M in a Banach space X has countable ε-dual index for every $\varepsilon > 0$. Then X^* admits an equivalent dual M-LUR norm.*

Examples

1. A Banach space X is said to be *strongly generated by a Banach space Z* if there exists a bounded linear operator $T : Z \to X$ such that, for every weakly compact subset M of X and for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $M \subset nT(B_Z) + \varepsilon B_X$ (see [24]). Every Banach space strongly generated by a superreflexive Banach space admits an equivalent norm that is M-uniformly Gâteaux smooth for every weakly compact set $M \subset X$ (see, e.g., [12]); thus such a norm is then uniformly Gâteaux smooth. For a finite measure μ, the space $L_1(\mu)$ is strongly generated by the Hilbert space $L_2(\mu)$. Let X_0 be the Rosenthal subspace of $L_1(\mu)$, for a certain finite measure μ, that is not weakly compactly generated ([23]). By Theorem 4, X_0 has σ-finite dual index. The space X_0 is weakly Lindelöf determined as it is a subspace of the weakly compactly generated space $L_1(\mu)$ (see, e.g., [7, Chapters 11 and 12]). Assume that X_0 contained a bounded linearly dense set M that had countable ε-dual index for every $\varepsilon > 0$. By Theorem 8, ${X_0}^*$ would then admit an equivalent dual M-locally uniformly rotund norm. Thus X_0 would be weakly compactly generated ([9, Theorem 1]). Therefore, X_0 *is a space that has σ-finite dual index but for no $\varepsilon > 0$, X_0 contains a bounded linearly dense set having countable ε-dual index.*

2. Let X be the Ciesielski-Pol space $C(K)$, where K is a scattered compact of finite height (see e.g., [3, Chapter VI]). Thus B_X has countable ε-dual index for every $\varepsilon > 0$ ([20]). However, X does not admit any equivalent uniformly Gâteaux smooth norm. Indeed, otherwise X would be a subspace of a weakly compactly generated Banach space ([6], see, e.g., [7, Theorem 12.18]). However, this is not the case as there is no bounded linear injection of X into any $c_0(\Gamma)$ ([3, Chapter VI]). Thus *the Ciesielski-Pol space does not have σ-finite dual index* by Theorem 4.

3. The space X in [1, page 421] admits a dual weak*-LUR norm ([22]) but does not have σ-countable dual index. Indeed, otherwise, it would admit an equivalent dual σ-LUR norm by Theorem 7. Thus X would be a subspace of a weakly compactly generated space as X is weakly Lindelöf determined ([9]). However, as it is proved in [1], X is not a subspace of a weakly compactly generated space.

4. If M is the unit ball of the space $C[0,\omega_1]$, then for every $\varepsilon > 0$ there is an ordinal α such that $(B_{X^*})^{(\alpha)}_{(M,\varepsilon)} = \emptyset$. This is so as $C[0,\omega_1]$ is an Asplund space (see, e.g., [3, Theorem 12.29]), and hence its dual is weak* dentable. However, $C[0,\omega_1]$ does not have σ-countable dual index as otherwise $C[0,\omega_1]$ would admit an equivalent dual strictly convex norm by Theorem 5, which is not the case by a classical Talagrand's result (see, e.g., [3, page 313]).

3 Proofs

A main tool is the following lemma, which is an adjustment of results in [18] and [19].

Lemma 9 *Let $(X, \|\cdot\|)$ be a Banach space. Let $M \subset X$ be a bounded set, and $\varepsilon > 0$, $\Delta > 0$ be given.*
(i) *Assume that M has finite ε-dual index. Then X^* admits a dual norm $|||\cdot|||$ such that $\|\cdot\| \le |||\cdot||| \le (1+\Delta)\|\cdot\|$, and $\limsup_n |f_n - g_n|_M \le 2\varepsilon$ whenever $f_n, g_n \in B_{(X^*,|||\cdot|||)}$, $n \in \mathbb{N}$, satisfy that $\lim_n |||f_n + g_n||| = 2$.*
(ii) *Assume that M has countable ε-dual index. Then X^* admits a dual norm $|||\cdot|||$ such that $\|\cdot\| \le |||\cdot||| \le (1+\Delta)\|\cdot\|$, and $\limsup_n |f_n - f|_M \le 2\varepsilon$ whenever $f, f_n \in B_{(X^*,|||\cdot|||)}$, $n \in \mathbb{N}$, satisfy that $\lim_n |||f + f_n||| = 2$.*
(iii) *Assume that the dual norm $\|\cdot\|$ on X^* satisfies $\limsup_n |f_n - g_n|_M < \varepsilon$ whenever $f_n, g_n \in B_{(X^*,\|\cdot\|)}$, $n \in \mathbb{N}$, are such that $\lim_n \|f_n + g_n\| = 2$. Then M has finite ε-dual index.*

Proof.
(i) Put $D_0 := B_{(X^*,\|\cdot\|)}$. Let r denote the ε-dual index of M. For $j \in \{1, 2, \dots, r\}$ put $D_j = (D_0)^{(j)}_{(M,\varepsilon)}$. Define $F : X^* \to [0, +\infty)$ by

$$F(f) := \|f\| + \Delta \sum_{j=0}^{r-1} \frac{1}{2^{j+1}} \operatorname{dist}(f, D_j), \quad f \in X^*, \tag{1}$$

where the distance function is considered in the original dual norm $\|\cdot\|$ on X^*. Clearly, the function F is symmetric. It is also weak*-lower semicontinuous and convex since each D_j is a weak*-closed and convex set. We shall need the following

`Claim.` *Let* (f_n) *and* (g_n) *be sequences in* $B_{(X^*,\|\cdot\|)}$ *such that*

$$\tfrac{1}{2}F(f_n) + \tfrac{1}{2}F(g_n) - F\left(\tfrac{1}{2}(f_n + g_n)\right) \to 0 \quad \textit{as} \quad n \to \infty. \tag{2}$$

Then $\limsup_n |f_n - g_n|_M \leq 2\varepsilon$.

Proof of the Claim. Assume, by contradiction, that this is not so. Then $\limsup_n |f_n - g_n|_M > 2\varepsilon + \delta$ for a suitable $\delta > 0$. Hence $|f_n - g_n|_M > 2\varepsilon + \delta$ for infinitely many $n \in \mathbb{N}$. Assume, for simplicity, that this inequality holds for all $n \in \mathbb{N}$. We shall prove the following

`Subclaim.` *For every* $j \in \{0, 1, \ldots, r-1\}$ *we have*

$$\operatorname{dist}(f_n, D_j) \to 0 \quad \textit{and} \quad \operatorname{dist}(g_n, D_j) \to 0 \quad \textit{as} \quad n \to \infty. \tag{3}$$

Proof of the Subclaim. For $j = 0$ the statement (3) is trivial. So further asssume that $r > 1$. Fix $k \in \{0, 1, \ldots, r-2\}$ and assume that (3) was already proved for $j = k$. Fix for a while any $n \in \mathbb{N}$. Find $f_n',\ g_n' \in D_k$ so that

$$\|f_n' - f_n\| \leq 2\operatorname{dist}(f_n, D_k) \quad \text{and} \quad \|g_n' - g_n\| \leq 2\operatorname{dist}(g_n, D_k).$$

Then

$$\begin{aligned} |f_n' - g_n'|_M &\geq |f_n - g_n|_M - |f_n' - f_n|_M - |g_n' - g_n|_M \\ &> 2\varepsilon + \delta - (2\operatorname{dist}(f_n, D_k) + 2\operatorname{dist}(g_n, D_k)) \sup\{\|m\|;\ m \in M\}. \end{aligned}$$

Hence $|f_n' - g_n'|_M > 2\varepsilon$ for all large $n \in \mathbb{N}$; assume for simplicity that this inequality holds for all $n \in \mathbb{N}$. Now, since any weak*-slice S of D_k, containing $\frac{1}{2}(f_n' + g_n')$, contains either f_n' or g_n', we have from the above estimate that

$$\operatorname{diam}_M(S) \geq \left| f_n' - \frac{f_n' + g_n'}{2} \right|_M = \left| g_n' - \frac{f_n' + g_n'}{2} \right|_M = \frac{1}{2} |f_n' - g_n'| > \varepsilon.$$

Therefore, $\frac{1}{2}(f_n' + g_n') \in D_{k+1}$. This holds for every $n \in \mathbb{N}$. From (2), using convexity, we get that

$$\tfrac{1}{2}\operatorname{dist}(f_n, D_{k+1}) + \tfrac{1}{2}\operatorname{dist}(g_n, D_{k+1}) - \operatorname{dist}\left(\tfrac{1}{2}(f_n + g_n), D_{k+1}\right) \to 0$$

for $n \to \infty$. And, as $\|f_n' - f_n\| \to 0$ and $\|g_n' - g_n\| \to 0$, we can conclude that (3) holds for $j = k+1$. This proves the Subclaim.

Now, (3) for $j = r-1$ means that

$$\operatorname{dist}(f_n, D_{r-1}) \to 0 \quad \text{and} \quad \operatorname{dist}(g_n, D_{r-1}) \to 0 \quad \text{as} \quad n \to \infty.$$

For $n \in \mathbb{N}$ find $f_n', g_n' \in D_{r-1}$ so that

$$\|f_n' - f_n\| \leq 2\operatorname{dist}(f_n, D_{r-1}) \quad \text{and} \quad \|g_n' - g_n\| \leq 2\operatorname{dist}(g_n, D_{r-1}).$$

Fix any $n \in \mathbb{N}$. Since $D_r = \emptyset$, there must exist a weak*-slice S of D_{r-1}, containing $\frac{1}{2}(f_n' + g_n')$, so that $\operatorname{diam}_M(S) < \varepsilon$. Hence, as $\{f_n', g_n'\} \cap S \neq \emptyset$, we have

$$\left|f_n' - \tfrac{1}{2}(f_n' + g_n')\right|_M = \left|g_n' - \tfrac{1}{2}(f_n' + g_n')\right|_M < \varepsilon,$$

and so $|f_n' - g_n'|_M < 2\varepsilon$. Thus

$$\limsup_n |f_n - g_n|_M = \limsup_n |f_n' - g_n'|_M \leq 2\varepsilon,$$

a contradiction. The Claim is proved.
Let $|||\cdot|||$ be the Minkowski functional of the set $\{f \in X^*;\ F(f) \leq 1\}$. From the properties of F it easily follows that $|||\cdot|||$ is a dual norm on X^* and that $\|f\| \leq |||f||| \leq (1+\Delta)\|f\|$ for every $f \in X^*$. Let $f_n,\ g_n,\ n \in \mathbb{N}$, be as in (i). Then $F(f_n) \leq 1,\ F(g_n) \leq 1,\ n \in \mathbb{N}$. Further, the uniform continuity of the function F on bounded sets yields that

$$F\left(\frac{f_n + g_n}{2}\right) - F\left(\frac{f_n + g_n}{|||f_n + g_n|||}\right) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus (2) is satisfied, and the Claim guarantees that $\limsup_n |f_n - g_n|_M \leq 2\varepsilon$.

(ii) Denote by β the ε-dual index of M; we know that it is a countable ordinal. Choose an indexed family $\{a_\alpha;\ 0 \leq \alpha < \beta\}$ of positive numbers such that $\sum_{0\leq\alpha<\beta} a_\alpha < 1$. Put $D_0 = B_{(X^*,\|\cdot\|)}$ and $D_\alpha = (D_0)^{(\alpha)}_{(M,\varepsilon)}$ for $0 < \alpha \leq \beta$; thus $D_\beta = \emptyset$. Define $G : X^* \to [0, +\infty)$ by

$$G(f) = \|f\|^2 + \Delta \sum_{0\leq\alpha<\beta} a_\alpha \operatorname{dist}^2(f, D_\alpha), \quad f \in X^*$$

where the distance functions are considered in the original dual norm on X^*. Clearly, G is symmetric, weak*-lower semicontinuous, and $\|f\|^2 \le G(f) \le (1+\Delta)\|f\|^2$ for all $f \in X^*$. It is also convex, since the square of a convex non-negative function is convex.

`Claim.` *Let $f, f_n \in B_{(X^*,\|\cdot\|)}$, $n \in \mathbb{N}$, be such that*

$$\tfrac{1}{2}G(f) + \tfrac{1}{2}G(f_n) - G\left(\tfrac{1}{2}(f+f_n)\right) \to 0 \quad as \quad n \to \infty. \tag{4}$$

Then $\limsup_n |f - f_n|_M \le 2\varepsilon$.

Proof of the Claim. Let α ($\le \beta$) be the first ordinal such that $f \notin D_\alpha$. A simple weak*-compactness argument reveals that α has a predecessor, $\alpha - 1$. Convexity and (4) yield that

$$\tfrac{1}{2}\mathrm{dist}^2(f, D_\alpha) + \tfrac{1}{2}\mathrm{dist}^2(f_n, D_\alpha) - \mathrm{dist}^2\left(\tfrac{1}{2}\left(f+f_n\right), D_\alpha\right) \to 0$$

and

$$\tfrac{1}{2}\mathrm{dist}^2(f, D_{\alpha-1}) + \tfrac{1}{2}\mathrm{dist}^2(f_n, D_{\alpha-1}) - \mathrm{dist}^2\left(\tfrac{1}{2}\left(f+f_n\right), D_{\alpha-1}\right) \to 0$$

as $n \to \infty$. Hence, the convexity of the functions $\mathrm{dist}(\cdot, D_\alpha)$ and $\mathrm{dist}(\cdot, D_{\alpha-1})$ yields

$$\mathrm{dist}(f_n, D_\alpha) \to \mathrm{dist}(f, D_\alpha), \quad \mathrm{dist}(\tfrac{1}{2}(f+f_n), D_\alpha) \to \mathrm{dist}(f, D_\alpha),$$

$$\mathrm{dist}(f_n, D_{\alpha-1}) \to \mathrm{dist}(f, D_{\alpha-1}), \quad \text{and} \quad \mathrm{dist}(\tfrac{1}{2}(f+f_n), D_{\alpha-1}) \to \mathrm{dist}(f, D_{\alpha-1})$$

as $n \to \infty$. Observe that $\mathrm{dist}(f, D_\alpha) > 0$ and $\mathrm{dist}(f, D_{\alpha-1}) = 0$ as $f \in D_{\alpha-1}$. Put $\delta = \frac{1}{2}\mathrm{dist}(f, D_\alpha)$. Then, for all large $n \in \mathbb{N}$ we have

$$\mathrm{dist}(f_n, D_\alpha) > \delta, \quad \mathrm{dist}\left(\tfrac{1}{2}(f+f_n), D_\alpha\right) > \delta,$$

$$\mathrm{dist}(f_n, D_{\alpha-1}) < \delta, \quad \text{and} \quad \mathrm{dist}\left(\tfrac{1}{2}(f+f_n), D_{\alpha-1}\right) < \delta.$$

For $n \in \mathbb{N}$ find $f_n' \in D_{\alpha-1}$ so that $\|f_n' - f_n\| \le \min\{\delta, 2\mathrm{dist}(f_n, D_{\alpha-1})\}$. Then, for all $n \in \mathbb{N}$ sufficiently large we have

$$f, \ f_n', \ \tfrac{1}{2}(f+f_n') \in D_{\alpha-1} \backslash D_\alpha;$$

the latter inclusion holds because of the following estimate:

$$\mathrm{dist}(\tfrac{1}{2}(f+f_n'), D_\alpha) \ge \mathrm{dist}(\tfrac{1}{2}(f+f_n), D_\alpha) - \tfrac{1}{2}\|f_n' - f_n\| > \delta - \tfrac{1}{2}\delta > 0.$$

Hence, for all $n \in \mathbb{N}$ large enough there exists a weak*-slice S of $D_{\alpha-1}$ such that $S \ni \frac{1}{2}(f+f'_n)$ and $\operatorname{diam}_M(S) < \varepsilon$. Therefore,

$$|f - f'_n|_M = 2\left|f - \tfrac{1}{2}(f+f'_n)\right|_M = 2\left|f'_n - \tfrac{1}{2}(f+f'_n)\right|_M < 2\varepsilon,$$

for all $n \in \mathbb{N}$ large enough. Then, finally,

$$\limsup_n |f - f_n|_M \le \limsup_n |f - f'_n|_M + \lim_n |f_n - f'_n|_M \le 2\varepsilon.$$

The Claim is thus proved.

Let $|||\cdot|||$ be the Minkowski functional of the set $\{f \in X^*;\ G(f) \le 1\}$. From the properties of G it easily follows that $|||\cdot|||$ is a dual norm on X^* and that $\|f\|^2 \le |||f|||^2 \le (1+\Delta)\|f\|^2$ for every $f \in X^*$. Let $f,\ f_n,\ n \in \mathbb{N}$, be as in (ii). Then, as in the proof of (i), we can verify the validity of (4). Now, by the Claim, we conclude that $\limsup_n |f - f_n|_M \le 2\varepsilon$.

(iii) From the premise here, find $\delta > 0$ so small that $|f-g|_M < \varepsilon-\delta$ whenever $f, g \in B_{(X^*,\|\cdot\|)}$ and $\|f+g\| > 2-2\delta$. Then

$$(B_{X^*})'_{(M,\varepsilon)} \subset (1-\delta)B_{X^*}. \tag{5}$$

Indeed, assume, there is $f_0 \in B_{X^*} \setminus (1-\delta)B_{X^*}$. Find $x_0 \in S_X$ so that $f_0(x_0) > 1-\delta$. Put $S = \{f \in B_{X^*};\ f(x_0) > 1-\delta\}$. This is a weak*-slice of B_{X^*} and $S \ni f_0$. On the other hand, if $f, g \in S$, then $\|f+g\| \ge f(x_0)+g(x_0) > 2-2\delta$, and hence, $|f-g|_M < \varepsilon - \delta$. Therefore $f_0 \notin (B_{X^*})'_{(M,\varepsilon)}$. This proves (5). Now, from (5), a homogeneity argument yields that

$$(B_{X^*})^{(2)}_{(M,\varepsilon)} \subset (1-\delta)^2 B_{X^*}, \quad (B_{X^*})^{(3)}_{(M,\varepsilon)} \subset (1-\delta)^3 B_{X^*}, \ldots .$$

However, once $k \in \mathbb{N}$ is big enough, then $\operatorname{diam}_M((1-\delta)^k B_{X^*}) < \varepsilon$ and so the ε-dual index of M must be equal to k at most. ∎

Proof of Theorem 4. (i)$\Longrightarrow$(ii). Because of Remark 3, we may and do assume that the original norm $\|\cdot\|$ on X is already uniformly Gâteaux smooth. Fix any $\varepsilon > 0$. Put

$$M_k^\varepsilon := \big\{x \in B_X;\ |f(x) - g(x)| < \tfrac{\varepsilon}{2} \\ \text{whenever } f, g \in B_{X^*} \text{ and } \|f+g\| > 2 - \tfrac{1}{k}\big\}, \quad k \in \mathbb{N}.$$

From the Šmulyan duality [3, Theorem II.6.7] and the uniform Gâteaux smoothness, it follows that $\bigcup_{k=1}^{\infty} M_k^\varepsilon = B_X$. Moreover, for every $\varepsilon > 0$ and every $k \in \mathbb{N}$, we can immediately see that the premise of (iii) in Lemma 9 is satisfied with $M := M_k^\varepsilon$. Therefore, each set M_k^ε has finite ε-dual index. We thus proved (ii).

(ii)$\Longrightarrow$(i). For every $\varepsilon > 0$ we have the decomposition $B_X = \bigcup_{k=1}^{\infty} M_k^\varepsilon$ where each M_k^ε has finite dual index. For $m \in \mathbb{N}$ and $k \in \mathbb{N}$ let $|||\cdot|||_{m,k}$ be the dual norm on X^* found in Lemma 9 (i) for the set $M := M_k^{1/m}$ and for $\Delta := 1$; thus $\|f\| \le |||f|||_{m,k} \le 2\|f\|$ for all $f \in X^*$. Define

$$|||f|||^2 := \|f\|^2 + \sum_{m,k=1}^{\infty} 2^{-m-k} |||f|||_{m,k}^2, \quad f \in X^*; \tag{6}$$

this is a dual norm on X^* and $\|f\| \le |||f||| \le 3\|f\|$ for all $f \in X^*$. Let $|||\cdot|||$ be the corresponding predual norm on X. We shall show that this norm on X is uniformly Gâteaux smooth. So, consider sequences (f_n), (g_n) in $B_{(X^*,|||\cdot|||)}$ such that $|||f_n + g_n||| \to 2$ as $n \to \infty$. According to Šmulyan duality, [3, Theorem II.6.7], we have to show that $f_n - g_n \to 0$ in the weak* topology of X^*. Assume that this is not the case. Find then $\varepsilon > 0$, $x \in B_X$, and an increasing sequence(n_i) in $\mathbb{N}$ so that $|f_{n_i}(x) - g_{n_i}(x)| > \varepsilon$ for every $i \in \mathbb{N}$. Take $m \in \mathbb{N}$ such that $m > \frac{\varepsilon}{4}$. Finally, find $k \in \mathbb{N}$ such that $M_k^{1/m} \ni x$. Equation (6) and convexity yield that

$$2|||f_{n_i}|||_{m,k}^2 + 2|||g_{n_i}|||_{m,k}^2 - 2|||f_{n_i} + g_{n_i}|||_{m,k}^2 \to 0 \qquad \text{as} \qquad i \to \infty,$$

and hence

$$|||f_{n_i}|||_{m,k} - |||g_{n_i}|||_{m,k} \to 0 \qquad \text{and} \qquad |||f_{n_i} + g_{n_i}|||_{m,k} - 2|||f_{n_i}|||_{m,k} \to 0$$

as $i \to \infty$. Note that $|||f_{n_i}||| \le 3\|f_{n_i}\| \le 3|||f_{n_i}|||_{m,k}$ for every $i \in \mathbb{N}$, and that $|||f_{n_i}||| \to 1$ as $i \to \infty$. Put $f_i' := f_{n_i}/|||f_{n_i}|||_{m,k}$ and $g_i' := g_{n_i}/|||g_{n_i}|||_{m,k}$, $i \in \mathbb{N}$. The sequences (f_i'), (g_i') lie in $B_{(X^*,|||\cdot|||_{m,k})}$ and $|||f_i' + g_i'|||_{m,k} \to 2$ as $i \to \infty$. Therefore, by Lemma 9 (i),

$$\limsup_{i\to\infty} |f_i'(x) - g_i'(x)| \le \limsup_{i\to\infty} |f_i' - g_i'|_{M_k^{1/m}} \le 2 \cdot \tfrac{1}{m}.$$

However,

$$|||f_{n_i}|||_{m,k} \le 2\|f_{n_i}\| \le 2|||f_{n_i}||| \le 2,$$

and so $\limsup_{i\to\infty} |f_{n_i}(x) - g_{n_i}(x)| \leq 2 \cdot \frac{2}{m} < \varepsilon$, a contradiction. ∎

Proof of Theorem 5

(i)$\Longrightarrow$(ii). Fix any $\varepsilon > 0$. Let $|\cdot|$ be an equivalent M-uniformly Gâteaux smooth norm on X. By Lemma 9 (iii), we have that $\big(B_{(X^*,|\cdot|)}\big)^{(k)}_{(M,\varepsilon)} = \emptyset$ for some $k \in \mathbb{N}$. Then, by Remark 3, we have also $\big(B_{(X^*,\|\cdot\|)}\big)^{(k')}_{(M,\varepsilon)} = \emptyset$ for a suitable $k' \in \mathbb{N}$.

(ii)$\Longrightarrow$(i). For $m \in \mathbb{N}$, let $\|\cdot\|_m$ be a dual norm on X^* such that $\|f\| \leq |||f|||_m \leq 2\|f\|$ for every $f \in X^*$ and with the property that $\lim_n |f_n - g_n|_M \leq \frac{2}{m}$ whenever $f_n, g_n \in B_{(X^*,|||\cdot|||_m)}$, $n \in \mathbb{N}$, satisfy that $\lim_n |||f_n + g_n|||_m = 2$. The existence of such a norm is guaranteed by Lemma 9 (i). Define

$$|||f|||^2 := \sum_{m=1}^{\infty} 2^{-m} |||f|||_m^2, \quad f \in X^*.$$

Then $|||\cdot|||$ is a dual norm on X^* and $(1/2)\|f\|^2 \leq |||f|||^2 \leq 4\|f\|^2$ for every $f \in X^*$. It remains to prove that the corresponding predual norm on X^* is M-uniformly Gâteaux smooth. So let $f_n, g_n \in B_{(X^*,|||\cdot|||)}$, $n \in \mathbb{N}$, satisfy that $\lim_n |||f_n + g_n||| = 2$. An argument very similar to the proof of the implication (ii)$\Longrightarrow$(i) in Theorem 4 yields that $\limsup_n |f_n - g_n|_M \leq \frac{4}{m}$ for every $m \in \mathbb{N}$. And this is what we wanted to prove. ∎

Proof of Theorem 7 follows the same lines as those in the proof of (ii)$\Longrightarrow$(i) in Theorem 4 (this time part (ii) of Lemma 9 is used, instead) and hence is omitted. ∎

Proof of Theorem 8 follows the same lines as those in the proof of (ii)$\Longrightarrow$(i) in Theorem 5 and hence is omitted. ∎

4 Applications

In [10], Banach spaces that are subspaces of WCG Banach space were characterized in terms of *ε-weakly relatively compact sets*, i.e., subsets M of a Banach space X that satisfy $\overline{M}^{w^*} \subset X + \varepsilon B_{X^{**}}$ (see also [9] and [11]). Here we prove that sets with finite ε-dual index have a more precise property than being ε-weakly relatively compact.

Theorem 10 *Let M be a bounded closed convex subset of a Banach space $(X, \|\cdot\|)$, and $\varepsilon > 0$ be given.*
If M has finite ε-dual index, then for every $\varepsilon' > \varepsilon$ we have $\overline{M}^{w^*} \subset M + 2\varepsilon' B_{X^{**}}$, where $\overline{M}^{w^*}$ denotes the closure of M in (X^{**}, w^*).
In particular, if M has finite ε-dual index for every $\varepsilon > 0$, then M is relatively weakly compact.

Proof Fix any $\Delta > 0$. By Lemma 9 (i), there exists a dual norm $|||\cdot|||$ on X^* such that $\|\cdot\| \le |||\cdot||| \le (1+\Delta)\|\cdot\|$ and $\limsup_n |f_n - g_n|_M \le 2\varepsilon$, whenever $f_n, g_n \in B_{(X^*,|||\cdot|||)}$, are such that $\lim_n |||f_n + g_n||| = 2$. Its predual norm $|||\cdot|||$ on X then satisfies $|||x||| \le \|x\| \le (1+\Delta)|||x|||$ for all $x \in X$. Using an elementary argument based on Goldstine's theorem, we get that whenever $F_n, G_n \in B_{(X^{***},|||\cdot|||)}$, $n \in \mathbb{N}$, are such that $\lim_n |||F_n + G_n||| = 2$, then $\limsup_n |F_n - G_n|_M \le 2\varepsilon$.
Fix any $z_0^{**} \in \overline{M}^{w^*}$. Assume that $M \cap (z_0^{**} + rB_{(X^{**},|||\cdot|||)}) = \emptyset$ for some $r > 2\varepsilon$. Separate M and $z_0^{**} + rB_{(X^{**},|||\cdot|||)}$ by some $F \in S_{(X^{***},|||\cdot|||)}$. This means that for some γ we have $F(x) \le \gamma$ for all $x \in M$ and $F(z^{**}) \ge \gamma$ for all $z^{**} \in (z_0^{**} + rB_{(X^{**},|||\cdot|||)})$. Note that then $F(z_0^{**}) - r \ge \gamma$. Find $z_1^{**}, z_2^{**}, \ldots \in B_{(X^{**},|||\cdot|||)}$ such that $\lim_k F(z_k^{**}) = 1$. By Goldstine's theorem, there is a sequence (f_n) in $B_{(X^*,|||\cdot|||)}$ such that $F(z_k^{**}) = \lim_n f_n(z_k^{**})$ for every $k = 0, 1, 2, \ldots$ Then, clearly, $\lim_n |||f_n + F||| = 2$ and thus $\limsup_n |f_n - F|_M \le 2\varepsilon$. Then

$$\begin{aligned} F(z_0^{**}) = \lim_{n\to\infty} f_n(z_0^{**}) &\le \limsup_{n\to\infty} \sup_M f_n \\ &\le \sup_M F + 2\varepsilon \le \gamma + 2\varepsilon \le F(z_0^{**}) - r + 2\varepsilon < F(z_0^{**}), \end{aligned}$$

a contradiction. Thus, for every $r > 2\varepsilon$, there exists $m_r \in M$ such that $|||z_0^{**} - m_r||| \le r$, and hence $\|z_0^{**} - m_r\| \le r(1+\Delta)$. This proves $\overline{M}^{w^*} \subset M + r(1+\Delta)B_{(X^{**},\|\cdot\|)}$. Here $r > 2\varepsilon$ and $\Delta > 0$ were arbitrary. Hence the proclaimed inclusion follows.
The proof of the second part is immediate. ∎

Remark 11 In a Banach space, every bounded subset M with finite ε-dual index for every $\varepsilon > 0$ is weakly relatively compact. Indeed, it is clear that the closed convex hull of M has also ε-dual index finite for every $\varepsilon > 0$, hence the conclusion follows from Theorem 10. It is worth noticing that there are weakly compact sets that do not have finite ε-dual index for some $\varepsilon > 0$; see the next theorem.

Theorem 12 *If K is an infinite compact set, then $C(K)$ contains a weakly compact set W that does not have finite ε-dual index for some $\varepsilon > 0$.*

Proof (Sketch) We follow the notation in e.g., [7, Chapter 9]. Let $(X, \|\cdot\|) = (c_0, \|\cdot\|_\infty)$, and put, for $n = 1, 2 \ldots$,

$$W_n := \left\{ \sum_{i=0}^{2^{n-1}-1} \varepsilon_i e_{2^n+i} \, ; \, \varepsilon_i = \pm 1 \right\}$$

where e_i, $i \in \mathbb{N}$, are the canonical unit vectors in X. Then let

$$W := \left(\bigcup_{n=1}^{\infty} W_n \right) \cup \{0\}.$$

It is standard to check that W is a weakly compact set in c_0. In order to see that W does not have finite ε-dual index for some $\varepsilon > 0$, consider first the element $y_0 := (\frac{1}{2}, \frac{1}{2}, 0, 0, \ldots)$ in ℓ_1. Then $y_0 = \frac{1}{2}(y_1 + y_2)$, where $y_1 := (1, 0, 0, \ldots,)$ and $y_2 := (0, 1, 0, 0, \ldots)$. Every w^*-slice of the unit ball of ℓ_1 that contains y_0, contains either y_1 or y_2, thus it has W-diameter ≥ 1. It follows that $y_0 \in (B_{\ell_1})'_{(W,1)}$. Now, let $v_0 := (0, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, \ldots)$. Then we have $v_0 = \frac{1}{2}(v_1 + v_2)$, where $v_1 := (0, 0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, \ldots)$ and $v_2 := (0, 0, 0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, \ldots)$. Proceed similarly to prove that $v_0 \in (B_{\ell_1})^{(2)}_{(W,1)}$.
Using this observation, one can construct, for each $n \in \mathbb{N}$, an n-tree T_n in the unit ball of ℓ_1 so that the supremum over W of the difference of two next elements stays bounded below uniformly by 1. It follows that the root of T_n lies in the n-th $(W, 1)$-derivative of B_{ℓ_1}. Thus W does not have finite 1-dual index.
As it is well known (see, e.g., [7, Theorem 12.30]), the space c_0 is isomorphic to a subspace of $C(K)$. From the previous paragraph and Remark 3, a weakly compact subset having the sought property can be found in $C(K)$. ∎

Theorem 13 *Every weakly compact subset of a Banach space X that is strongly generated by a superreflexive space* (for definition, see Example 1) *has finite ε-dual index for every $\varepsilon > 0$. In particular, if μ is a finite measure, this happens for the space $L_1(\mu)$.*

Proof It is shown in [15] (see also [12] and [17, Chapter 6]) that X admits an equivalent norm that is M-uniformly Gâteaux smooth for every weakly compact set $M \subset X$. Now, it is enough to apply Theorem 5. ∎

A compact space K is called a *uniform Eberlein compact* if K is homeomorphic to a weakly compact set in a Hilbert space endowed with its weak topology.

Theorem 14 *Let K be a compact space. Then the following are equivalent.*
(i) *K is a uniform Eberlein compact.*
(ii) *There is a bounded linearly dense set in $C(K)$ that has finite ε-dual index for every $\varepsilon > 0$.*
(iii) *$C(K)$ has σ-finite dual index.*

Proof. If K is a compact set, then K is a uniform Eberlein compact if and only if $C(K)$ admits a uniformly Gâteaux smooth norm if and only if $C(K)$ admits a strongly uniformly Gâteaux smooth norm (see, e.g., [7, Theorem 12.18]). Now, apply Theorem 4 and Theorem 5. ∎

Remark 15 Theorem 14 should be compared with the Amir-Lindenstrauss result that *K is an Eberlein compact if and only if $C(K)$ contains a weakly compact linearly dense set* (see, e.g., [7, Theorem 12.12]), and with a result that *K is a Radon-Nikodým compact if and only if $C(K)$ contains a bounded linearly dense Asplund set M* (see, e.g., [5]), i.e., a set such that each continuous convex function on $C(K)$ is differentiable at points of a dense set in $C(K)$ uniformly in the directions from M ([21], see, e.g., [4, Theorem 1.5.4]). A compact space is called *Radon-Nikodým compact* if it is homeomorphic to a weak* compact set in the dual of some Asplund space. ([21]).

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