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ON THE SALAS THEOREM AND HYPERCYCLICITY OF $f(T)$

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Abstract. We study hypercyclicity properties of functions of Banach space operators. Generalizations of the results of Herzog-Schmoeger and Bermudez-Miller are obtained. As a corollary we also show that each non-trivial operator commuting with a generalized backward shift is supercyclic. This gives a positive answer to a conjecture of Godefroy and Shapiro.

Furthermore, we show that the norm-closures of the set of all hypercyclic (mixing, chaotic, frequently hypercyclic, respectively) operators on a Hilbert space coincide. This implies that the set of all hypercyclic operators that do not satisfy the hypercyclicity criterion is rather small - of first category (in the norm-closure of hypercyclic operators).

1. Introduction

In [S], H. Salas proved that for any backward weighted shift T on $\ell_p \quad (1 \le p < \infty)$ or c_0, the operator $I+T$ is hypercyclic. Surprisingly, the proof was direct and did not use the hypercyclicity criterion (or any of its variants), which is the usual tool for showing the hypercyclicity.

However, in [LMo] it was shown that $I+T$ does satisfy the hypercyclicity criterion. In [G2] it was shown that $I+T$ is even mixing for all backward weighted shifts.

Moreover, the Salas theorem is true not only for weighted backward shifts but for a much larger class of operators, see e.g. [A], [B], [BM2].

In Section 2 we study the hypercyclicity properties of functions $f(T)$ of an operator T with large generalized kernel. We generalize the results of [HS] and [BMi]. As a corollary we also show that each non-trivial operator commuting with a generalized backward shift is supercyclic. This gives a positive answer to a conjecture of Godefroy and Shapiro [GS], Remark 4.10 (c).

The results become stronger if we consider operators with closed ranges. In Section 3 we show that if T is Kato with dense generalized kernel and $|f(0)| = 1$ then $f(T)$ is not only mixing, but also chaotic and frequently hypercyclic.

In Section 4 we consider approximation in the algebra of all operators on a separable Hilbert space. Recall that the norm-closure of hypercyclic operators was characterized by D. Herrero [H2]. Analyzing the proof of [H2] we show that norm-closures of hypercyclic, mixing, chaotic and frequently hypercyclic operators coincide.

It was shown recently by Bayart and Matheron [BM1] (modifying an earlier result of Read and De La Rosa [DR]) that there are hypercyclic Hilbert space operators which do not satisfy the hypercyclicity criterion. We show that the set of such operators is relatively small, of first category in the norm-closure of hypercyclic operators.

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2. Hypercyclicity of $f(T)$

Let X be a complex Banach space. Denote by $B(X)$ the algebra of all bounded linear operators acting on X. For $T \in B(X)$ denote by $N(T)$ and $R(T)$ the kernel and range of T, respectively. Write $N^\infty(T) = \bigcup_{k=1}^\infty N(T^k)$ and $R^\infty(T) = \bigcap_{k=1}^\infty R(T^k)$. We denote by $\bigvee$ the closed linear span.

An operator $T \in B(X)$ is called hypercyclic if there exists a vector $x \in X$ which is hypercyclic for T, i.e., such that the set $\{T^n x : n = 0, 1, \dots\}$ is dense in X. It is well known [GS] that T is hypercyclic if and only if for all non-empty open sets $U, V \subset X$ there exists $n \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$.

An operator $T \in B(X)$ is called supercyclic if there exists a vector $x \in X$ supercyclic for T, i.e., such that the set $\{\lambda T^n x : \lambda \in \mathbb{C}, n = 0, 1, \dots\}$ is dense in X.

A usual tool for showing hypercyclicity of an operator is the hypercyclicity criterion. There are many equivalent formulations of this condition; the simplest one is that $T \in B(X)$ satisfies the hypercyclicity criterion if and only if $T \oplus T$ is hypercyclic. These operators are sometimes called weakly mixing.

Furthermore, $T \in B(X)$ is called mixing if for all non-empty open subsets $U, V \subset X$ we have $T^n U \cap V \neq \emptyset$ for all n sufficiently large.

The following implications are true:

$$T \text{ mixing} \Longrightarrow T \oplus T \text{ hypercyclic} \Longrightarrow T \text{ hypercyclic} \Longrightarrow T \text{ supercyclic};$$

the reverse implications are not true.

The following generalization of the Salas theorem follows from [BM2], Theorem 2.2 (the result is attributed there to S. Grivaux and S. Shkarin).

Theorem 1. Let X be a separable Banach space, $T \in B(X)$. Let

$$\bigvee_{z \in \mathbb{C}, |z|=1} \big(N^\infty(T - z) \cap R^\infty(T - z)\big) = X.$$

Then T is mixing.

The previous theorem implies the following generalization of [BMi].

Theorem 2. Let X be a separable Banach space, $T \in B(X)$, let f be a function analytic on a neighbourhood of $\sigma(T)$ which is non-constant on each component of its domain of definition. Let

$$\bigvee \{N^\infty(T - w) \cap R^\infty(T - w) : w \in \sigma(T), |f(w)| = 1\} = X.$$

Then $f(T)$ is mixing.

Proof. Let $w \in \sigma(T)$, $|f(w)| = 1$. We show first that

$$N^\infty(T - w) \cap R^\infty(T - w) \subset N^\infty\big(f(T) - f(w)\big) \cap R^\infty\big(f(T) - f(w)\big).$$

Let $f(z) = \sum_{i=0}^\infty \alpha_i (z - w)^i$ be the Taylor expansion of f in a neighbourhood of w. Then $|\alpha_0| = 1$. Since f is non-constant in a neighbourhood of w, there exists $n \geq 1$ such that $\alpha_n \neq 0$ and $\alpha_i = 0 \quad (1 \leq i \leq n - 1)$. Write $f(z) = \alpha_0 + \alpha_n (z - w)^n + (z - w)^{n+1} g(z)$ where g is a function analytic on a neighbourhood of $\sigma(T)$.

Let $M=N^\infty(T-w)\cap R^\infty(T-w)$ and let

$$S=f(T)-f(w)=\alpha_n(T-w)^n+(T-w)^{n+1}g(T).$$

Clearly $SN\bigl((T-w)^k\bigr)\subset N\bigl((T-w)^{k-1}\bigr)$ for all $k\in\mathbb{N}$. So $S^kN\bigl((T-w)^k\bigr)=\{0\}$ and $M\subset N^\infty(S)$.

We show that $M\subset R^\infty(S)$. To this end we must show that $M\subset R(S^m)$ for each m. Fix $m\in\mathbb{N}$. By induction on k we show that $N\bigl((T-w)^k\bigr)\cap M\subset R(S^m)$. This is clear for $k=0$. Let $k\ge 0$ and suppose that $N\bigl((T-w)^k\bigr)\cap M\subset R(S^m)$. Let $x\in N\bigl((T-w)^{k+1}\bigr)\cap M$. Then there exists $u\in X$ such that $x=\alpha_n^m(T-w)^{mn}u$. Clearly $u\in N\bigl((T-w)^{mn+k+1}\bigr)$. We have $S^m=\alpha_n^m(T-w)^{mn}+h(T)(T-w)^{mn+1}$ for some function h analytic on a neighbourhood of $\sigma(T)$. So

$$S^mu=x+h(T)(T-w)^{mn+1}u=x+\alpha_n^{-m}h(T)(T-w)x\in x+\bigl(N((T-w)^k)\cap M\bigr).$$

By the induction assumption, $x\in R(S^m)$ and $N((T-w)^{k+1})\cap M\subset R(S^m)$. Hence $M\subset R(S^m)$. Since m was arbitrary, we have $M\subset R^\infty(S)$.

Hence $N^\infty(S)\cap R^\infty(S)\supset M$.

So

$$\begin{aligned}&\bigvee\bigl\{N^\infty(f(T)-z)\cap R^\infty(f(T)-z):z\in\mathbb{C},|z|=1\bigr\}\\ \supset&\bigvee\bigl\{R^\infty(f(T)-f(w))\cap N^\infty\bigl(f(T)-f(w)\bigr):w\in\sigma(T),|f(w)|=1\bigr\}\\ \supset&\bigvee\bigl\{R^\infty(T-w)\cap N^\infty(T-w):w\in\sigma(T),|f(w)|=1\bigr\}=X.\end{aligned}$$

Hence $f(T)$ is mixing by Theorem 1. □

Corollary 3. (cf. [HS]) Let X be a separable Banach space, $T\in B(X)$, let $N^\infty(T)\cap R^\infty(T)$ be dense in X. Let f be a function analytic on a neighbourhood of $\sigma(T)$ which is non-constant on the component of its domain of definition containing $\sigma(T)$. Suppose that $|f(0)|=1$. Then $f(T)$ is mixing.

Note in the previous corollary that $0\in\sigma(T)$, and so $f(0)$ is defined. Note also that $\sigma(T)$ is connected. Indeed, suppose that $\sigma(T)=K_1\cup K_2$ where K_1 and K_2 are disjoint compact sets. We may assume that $0\in K_1$. Let X_1,X_2 be the spectral subspaces corresponding to K_1,K_2. Then $T|X_2$ is invertible, and so $N^\infty(T)\subset X_1$. Hence $X_2=\{0\}$ and $K_2=\emptyset$.

If T satisfies a slightly stronger assumption, namely that $N^\infty(T)$ is dense and contained in $R(T)$, then we can formulate an additional result.

Theorem 4. Let X be a separable Banach space, $T\in B(X)$, let $N^\infty(T)$ be dense and contained in $R(T)$. Let f be a non-constant function analytic on a neighbourhood of $\sigma(T)$. Then:

(i) $f(T)$ is supercyclic;

(ii) if $|f(0)|=1$ then $f(T)$ is mixing.

Proof. It is easy to show (see e.g. [M], p. 117) that the inclusion $N^\infty(T)\subset R(T)$ implies that $N^\infty(T)\subset R^\infty(T)$. So T satisfies conditions of Corollary 3. This proves (ii).

If $f(0)\ne 0$ then $f(0)^{-1}f(T)$ is mixing by (ii), and so $f(T)=f(0)\cdot\bigl(f(0)^{-1}f(T)\bigr)$ is supercyclic. So it is sufficient to consider the case $f(0)=0$.

Since f is non-constant, there exist $n \geq 1$ and a function g analytic on a neighbourhood of $\sigma(T)$ such that $\alpha_n \neq 0$ and $f(z) = \alpha_n z^n + z^{n+1} g(z)$. Hence $f(T) = \alpha_n T^n + T^{n+1} g(T)$.

For each $k \geq 0$ we have $f(T)N(T^{k+n}) \subset N(T^k)$. We show by induction on k that $f(T)N(T^{k+n}) = N(T^k)$. This is clear for $k = 0$. Let $k \geq 0$ and $f(T)N(T^{k+n}) = N(T^k)$. Let $x \in N(T^{k+1})$. Since $N(T^{k+1}) \subset N^\infty(T) \subset R(T^n)$, there exists $u \in X$ such that $x = T^n u$. Clearly $u \in N(T^{k+n+1})$ and $f(T)(\alpha_n^{-1} u) = x + \alpha_n^{-1} T^{n+1} g(T) u \in x + N(T^k)$. By the induction assumption, $x \in f(T)N(T^{k+n+1})$. So for each k, $f(T)$ maps $N(T^{k+n})$ onto $N(T^k)$. In particular, $f(T)N(T^{(s+1)n}) = N(T^{sn}) \quad (s \in \mathbb{N})$.

By the open mapping theorem, there exist constants $c_s > 0$ such that

$$f(T)\bigl(N(T^{(s+1)n}) \cap B_X\bigr) \supset c_s\bigl(N(T^{sn}) \cap B_X\bigr),$$

where B_X denotes the closed unit ball in X. Without loss of generality we may assume that $1 \geq c_1 \geq c_2 \geq \cdots$.

Consider the operators $A_s = 2^s c_{2s}^{-s} \bigl(f(T)\bigr)^s$. Clearly $A_s x \to 0$ for each $x \in N^\infty(T)$.

Furthermore, for $s \geq 1$ we have

$$N(T^{sn}) \cap B_X \subset c_s^{-1} f(T)\bigl(N(T^{(s+1)n}) \cap B_X)\bigr) \subset$$

$$(c_s c_{s+1})^{-1} (f(T))^2 \bigl(N(T^{(s+2)n}) \cap B_X\bigr) \subset \cdots$$

$$\cdots \subset (c_s \cdots c_{2s-1})^{-1} (f(T))^s \bigl(N(T^{(2s-1)n}) \cap B_X\bigr) \subset \frac{c_{2s}^s A_s B_X}{2^s c_s \ldots c_{2s-1}} \subset 2^{-s} A_s B_X.$$

So $\bigcup_{s=1}^\infty A_s B_X \supset \bigcup_{s=1}^\infty 2^s \bigl(N(T^{sn}) \cap B_X\bigr) = N^\infty(T)$ and $\bigcup_{s=1}^\infty A_s B_X$ is dense in X. By the hypercyclicity criterion (see e.g. [LM2], Theorem 4), the sequence (A_s) is hypercyclic. Consequently, the operator $f(T)$ is supercyclic. □

Conditions of Corollary 3 and Theorem 4 are clearly satisfied by any backward weighted shift on $\ell_p \quad (1 \leq p < \infty)$ or c_0 (we always assume that the shift is non-degenerated, with all weights non-zero).

Another example of an operator satisfying the conditions of Theorem 4 is any Hilbert space operator which has in some orthonormal basis the form $T = (a_{ij})$ with $a_{ij} = 0 \quad (i \geq j)$ and $a_{i,i+1} \neq 0$ (the entries a_{ij} with $j \geq i+2$ are arbitrary).

Recall that an operator T acting on an infinite-dimensional Banach space X is called a generalized backward shift if $N^\infty(T)$ is dense and $\dim N(T) = 1$. By [SG], Proposition 3.3 there exists a sequence $(x_k)_{k=0}^\infty \subset X$ such that $Tx_0 = 0$, $Tx_k = x_{k-1} \quad (k \geq 1)$, $\bigvee_{k=0}^\infty x_k = X$ and $N(T^k) = \bigvee_{i=0}^{k-1} x_i$ for all $k \geq 1$. Clearly the conditions of Theorem 4 are satisfied by any generalized backward shift or, more generally, by any finite direct sum of generalized backward shifts.

Moreover, for generalized backward shifts it is possible to formulate Theorem 4 not only for analytic functions $f(T)$ but also for operators in the commutant of T. The next result gives a positive answer to the conjecture [SG], Remark 4.10 (c).

Theorem 5. Let $T \in B(X)$ be a generalized backward shift and let $A \in B(X)$ be an operator commuting with T which is not a scalar multiple of the identity. Then A is supercyclic.

Proof. Let $x_0, x_1, \cdots \in X$ be the vectors satisfying $Tx_0 = 0$, $Tx_k = x_{k-1} \quad (k \geq 1)$ and $\bigvee_{k=0}^\infty x_k = X$.

Let $A\in B(X)$, $AT=TA$. Since $AN(T^k)\subset N(T^k)$ and $N(T^k)=\bigvee_{i=0}^{k-1}x_i$ for all k, we can express $Ax_k=\sum_{i=0}^k\beta_{k,i}x_i$ for some complex coefficients $\beta_{k,i}$ $(0\le i\le k)$. We have $\sum_{i=0}^k\beta_{k,i}x_i=Ax_k=ATx_{k+1}=TAx_{k+1}=\sum_{i=1}^{k+1}\beta_{k+1,i}x_{i-1}$. Since the vectors x_i are linearly independent, we have $\beta_{k,i}=\beta_{k+1,i+1}$ for all $0\le i\le k$. Hence there exist complex numbers $\alpha_0,\alpha_1,\dots$ such that

$$Ax_k=\sum_{i=0}^k\alpha_ix_{k-i}\qquad(k=0,1,\dots).$$

Since A is not a scalar multiple of the identity, there exist $n\ge 1$ such that $\alpha_n\ne 0$ and $\alpha_i=0\quad(1\le i\le n-1)$.

Suppose first that $\alpha_0=1$. We show that A is even mixing. Set $S=A-I$, so $Sx_k=\sum_{i=1}^k\alpha_ix_{k-i}\in\bigvee_{i=0}^{k-1}x_i$ for all k. Thus $S^{k+1}x_k=0$.

Let M be the set of all finite linear combinations of the vectors $x_0,x_1,\dots$. Then $M\subset N^\infty(S)$.

Clearly $SM\subset M$. We show by induction on k that $x_k\in SM$ for all k. We have $S(\alpha_n^{-1})x_n=x_0$, so $x_0\in SM$.

Let $k\ge 1$ and suppose that $x_0,\dots,x_{k-1}\in SM$. We have $S(\alpha_n^{-1}x_{n+k})\in x_k+\bigvee_{i=0}^{k-1}x_i$, so by the induction assumption $x_k\in SM$.

Hence $SM=M$ and $M\subset N^\infty(S)\cap R^\infty(S)$. So $A=S+I$ is mixing by Theorem 1.

If $\alpha_0\ne 0$ then the operator $\alpha_0^{-1}A$ is mixing and so A is supercyclic.

If $\alpha_0=0$ then $Ax_0=0$ and $N(A)\supset N(T)$. By [GS], Proposition 3.6, A is supercyclic. □

We finish this section by the following observation which is an analogy to the result of [LM1]. Denote by $\mathbb{T}$ the unit circle in the complex plane, $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$.

Proposition 6. Let $T\in B(X)$ be mixing, $\lambda\in\mathbb{C}$, $|\lambda|=1$. Then λT is mixing.

Proof. Let $U,V\subset X$ be non-empty open subsets of X. Choose $y\in V, y\ne 0$. Let $\varepsilon>0$ satisfy $B(y,\varepsilon)\subset V$, where $B(y,\varepsilon)$ denotes the open ball centered at y with radius ε.

Let $\mu_1,\dots,\mu_k\in\mathbb{T}$ be a finite $\frac{\varepsilon}{2\|y\|}$ - net in $\mathbb{T}$. Since T is mixing, for each $j=1,\dots,k$ there exists $n_j\in\mathbb{N}$ such that $T^nU\cap B(\mu_jy,\varepsilon/2)\ne\emptyset$ for all $n\ge n_j$.

Let $n\ge\max\{n_j:j=1,\dots,k\}$. Find j such that $|\lambda^{-n}-\mu_j|\le\frac{\varepsilon}{2\|y\|}$. Find $u\in U$ such that $\|T^nu-\mu_jy\|<\varepsilon/2$. Then

$$\|(\lambda T)^nu-y\|=\|T^nu-\lambda^{-n}y\|\le\|T^nu-\mu_jy\|+\|\mu_jy-\lambda^{-n}y\|<\varepsilon/2+\|y\|\cdot|\mu_j-\lambda^{-n}|\le\varepsilon.$$

Hence $(\lambda T)^nU\cap V\ne\emptyset$ and λT is mixing. □

3. Operators with closed range

The results from the previous section can be improved if we consider operators with closed ranges.

Operators $T\in B(X)$ with closed range and satisfying $N^\infty(T)\subset R(T)$ are called Kato (sometimes also semiregular) and have been studied intensely by a number of authors. For $T\in B(X)$ denote by $\sigma_K(T)=\{z\in\mathbb{C}:T-z \text{ is not Kato}\}$ the Kato spectrum of T.

Recall that $\sigma_K(T)$ is always a non-empty compact subset of $\mathbb{C}$, $\partial\sigma(T)\subset\sigma_K(T)\subset\sigma(T)$. The functions $z\mapsto\overline{N^\infty(T-z)}$ and $z\mapsto R^\infty(T-z)$ are constant on each connected subset of $\mathbb{C} \setminus \sigma_K(T)$. For a survey of results concerning Kato operators and the corresponding Kato spectrum see e.g. [M], pp. 117–130.

We need the following lemma, cf. [H1],7, Chap. 3.

Lemma 7. Let X be a Banach space, let $T \in B(X)$ be a Kato operator, let G be a connected open set such that $0 \in G$ and $T - z$ is Kato for each $z \in G$. Let $\alpha_k \in G \setminus \{0\}$, $\alpha_k \to 0$. Then $\overline{N^\infty(T)} = \bigvee_{k=1}^\infty N(T - \alpha_k)$.

Proof. By [M], p.124, $\overline{N^\infty(T)} = \overline{N^\infty(T - \alpha_k)}$ for all k, so $\overline{N^\infty(T)} \supset \bigvee_{k=1}^\infty N(T - \alpha_k)$.

Conversely, let $n \in \mathbb{N}$ and $x \in N(T^n)$. Set $x_1 = T^{n-1}x$, $x_2 = T^{n-2}x, \dots, x_{n-1} = Tx, x_n = x$. Let $\gamma(T) = \inf\{\|Tx\| : \operatorname{dist}\{x, N(T)\} = 1\}$ denote the reduced minimum modulus. Since $R(T)$ is closed, we have $\gamma(T) > 0$.

Since $N^\infty(T) \subset R(T)$, we can find inductively vectors $x_{n+1}, x_{n+2}, \dots \in X$ such that $Tx_j = x_{j-1}$ and $\|x_j\| \le 2\gamma(T)^{-1}\|x_{j-1}\| \quad (j \ge n+1)$. So $\|x_j\| \le (2\gamma(T)^{-1})^{j-n}\|x\|$ for all $j \ge n$, and so the series $f(z) := \sum_{j=1}^\infty x_j z^{j-1}$ is convergent in a certain neighbourhood U of 0. It is easy to check that $(T - z)f(z) = 0$ for all $z \in U$. Thus $f(\alpha_k) \in N(T - \alpha_k)$ for all k large enough.

We prove by induction on j that $x_j \in \bigvee_k N(T - \alpha_k)$. We have $x_1 = f(0) = \lim_{k\to 0} f(\alpha_k) \in \bigvee_k N(T - \alpha_k)$.

Suppose that $j \ge 1$ and $x_i \in \bigvee_k N(T - \alpha_k)$ for $i = 1, \dots, j$. Then

$$x_{j+1} = \lim_{k\to\infty} \frac{f(\alpha_k) - (x_1 + x_2\alpha_k + \dots + x_j\alpha_k^{j-1})}{\alpha_k^j} \in \bigvee_k N(T - \alpha_k)$$

by the induction assumption. In particular, $x = x_n \in \bigvee_k N(T-\alpha_k)$. Hence $\overline{N^\infty(T)} = \bigvee_k N(T-\alpha_k)$. □

If we replace the condition of Theorem 2 by the condition that $\bigcup\{N^\infty(T - w) : w \notin \sigma_K(T), |f(w)| = 1\}$ is dense, then the operator $f(T)$ is not only mixing but also chaotic and frequently hypercyclic.

Recall that an operator $T \in B(X)$ is called chaotic if it is hypercyclic and the set of its periodic vectors (i.e., the vectors $x \in X$ satisfying $T^n x = x$ for some $n \ge 1$) is dense.

$T \in B(X)$ is called frequently hypercyclic if there exists $x \in X$ such that for each non-empty open subset $U \subset X$ the set $\{n \in \mathbb{N} : T^n x \in U\}$ has a positive lower density.

Theorem 8. Let X be a separable Banach space, $T \in B(X)$, let f be a function analytic on a neighbourhood of $\sigma(T)$ which is non-constant on each component of the domain of its definition. Suppose that

$$\bigvee\{N^\infty(T - w) : w \in \sigma(T) \setminus \sigma_K(T), |f(w)| = 1\} = X.$$

Then $f(T)$ is mixing and chaotic. If X is a Hilbert space then $f(T)$ is also frequently hypercyclic.

Proof. $f(T)$ is mixing by Theorem 2.

Let $w \in \sigma(T) \setminus \sigma_K(T)$ and $|f(w)| = 1$. Since f is non-constant in a neighbourhood of w, there exists $n \ge 1$ and a function g analytic on a neighbourhood of $\sigma(T)$ such that $f(z) = f(w) + (z - w)^n g(z)$ and $g(w) \ne 0$.

Let U be an open disc centered at w such that $U \cap \sigma_K(T) = \emptyset$ and $|g|$ is bounded below on U. Then $V := f(U)$ is an open neighbourhood of $f(w) \in \mathbb{T}$. Find a sequence $(\alpha_k) \subset (V \setminus \{f(w)\}) \cap \mathbb{T}$

such that $\alpha_k \to f(w)$ and each $\alpha_k = e^{2\pi i t_k}$ for some rational t_k. Choose $w_k \in U$ such that $f(w_k) = \alpha_k$. This implies that $w_k \to w$.

By Lemma 7, we have $\bigvee_k N(T - w_k) = \overline{N^\infty(T - w)}$. Moreover, if $x \in N(T - w_k)$ then $x \in N\big(f(T) - f(w_k)\big) = N\big(f(T) - \alpha_k\big)$, and so x is periodic for $f(T)$.

Clearly the set of all periodic vectors for $f(T)$ is a linear manifold. Its closure contains $N(T - w_k)$ for each k, and so also $\overline{N^\infty(T - w)}$ whenever $w \notin \sigma_K(T)$, $|f(w)| = 1$. Hence the set of all vectors periodic for $f(T)$ is dense and $f(T)$ is chaotic.

Let X be a Hilbert space. Since $\partial\sigma(T) \subset \sigma_K(T) \subset \sigma(T)$, the set $\sigma(T) \setminus \sigma_K(T)$ is open. So $A := f\big(\sigma(T) \setminus \sigma_K(T)\big) \cap \mathbb{T}$ is open in $\mathbb{T}$. Let μ be the normalized Lebesgue measure on A, i.e., $\mu(A) = 1$.

Let $B \subset A$ be a measurable subset with $\mu(B) = 1$. We show that $\bigvee_{z\in B} N(f(T) - z) = X$.

Let $w \in \sigma(T) \setminus \sigma_K(T)$, $|f(w)| = 1$. Let U be an open disc centered at w chosen as above. Let $A_0 = f(U) \cap \mathbb{T}$ and $B_0 = B \cap A_0$. Then $\mu(B_0) = \mu(A_0) > 0$, and so there exists a sequence $(z_k) \subset B$ converging to a point $z' \in A_0$ such that $z_k \neq z'$ $(k \in \mathbb{N})$. Let $w', w_k \in U$, $f(w') = z'$, $f(w_k) = z_k$ for all $k \in \mathbb{N}$. As above, $w_k \to w'$, and so $\bigvee_k N(T - w_k) = \overline{N^\infty(T - w')} = \overline{N^\infty(T - w)}$. Moreover, for each k we have $N(T - w_k) \subset N\big(f(T) - z_k\big) \subset \bigvee_{z\in B} N\big(f(T) - z\big)$. So $\bigvee_{z\in B} N\big(f(T) - z\big) \supset \overline{N^\infty(T - w)}$. Hence

$$\bigvee_{z\in B} N\big(f(T) - z\big) \supset \bigvee\{N^\infty(T - w) : w \in \sigma(T) \setminus \sigma_K(T), |f(w)| = 1\} = X,$$

and so $f(T)$ is frequently hypercyclic by [BM2], Corollary 6.24. □

Corollary 9. Let $T \in B(X)$ be a Kato operator such that $N^\infty(T)$ is dense, let f be a function analytic on a neighbourhood of $\sigma(T)$ which is non-constant on each component of its domain of definition. Suppose that $|f(0)| = 1$. Then $f(T)$ is mixing and chaotic. If X is a Hilbert space then $f(T)$ is also frequently hypercyclic.

Remark 10. By a recent result of S. Grivaux (personal communication), Corollary 6.24 of [BM2] remains true also for Banach space operators. So the operator $f(T)$ in Theorem 8 and Corollary 9 is frequently hypercyclic even in the Banach space case.

4. Norm-closures of Hilbert space operators

Let H be an infinite-dimensional separable complex Hilbert space. Denote by $hc(H)$ the set of all hypercyclic operators on H. The norm-closure $\overline{hc(H)}$ was characterized in [H2] as the set of all operators $T \in B(H)$ satisfying:

(a) $\sigma_0(T) = \emptyset$,

where $\sigma_0(T)$ is the set of all $\lambda \in \mathbb{C}$ such that λ is an isolated point of the spectrum $\sigma(T)$ and the corresponding Riesz spectral space is finite-dimensional;

(b) $\operatorname{ind}(T - \lambda) \geq 0$ for all complex numbers λ such that $T - \lambda$ is semi-Fredholm;

(c) $\mathbb{T} \cup \sigma_W(T)$ is connected,

where $\sigma_W(T) = \bigcap\{\sigma(T + K) : K \in B(H) \text{ is compact}\}$ is the Weyl spectrum.

The proof that conditions (a)–(c) characterize the norm-closure $\overline{hc(H)}$ consists of two parts:

A. the set of all operators satisfying (a)–(c) is norm-closed and every hypercyclic operator satisfies these conditions. This part is simple and is true for any Banach space.

B. The second inclusion that every operator satisfying (a)–(c) can be approximated by hypercyclic operators is much more difficult and uses deep approximating techniques that are available only for Hilbert space operators. Using several approximations it is shown that for each $T \in B(H)$ satisfying (a)–(c) and every $\varepsilon > 0$ there exists a hypercyclic operator S such that $\|S - T\| < \varepsilon$.

In fact, it is proved more in [H2]. It is proved that for each operator $T \in B(H)$ satisfying (a)–(c) and every $\varepsilon > 0$ there exists $S \in B(H)$ satisfying $\|S - T\| < \varepsilon$ and S is a finite direct sum $S = \bigoplus_{j=1}^n S_j$, where for each $S_j \in B(H_j)$ there exists $\alpha_j \in \mathbb{T}$ such that $S_j - \alpha_j$ is a semi-Fredholm operator, and $S_j - \alpha_j$ is upper-triangular (in a certain orthonormal basis) with zero main diagonal.

Clearly by an arbitrarily small perturbation we can achieve that all the entries in the first diagonal over the main diagonal are nonzero. Thus we may assume that $N^\infty(S_j - \alpha_j)$ is dense in H_j and $S_j - \alpha_j$ is surjective.

By Corollary 9, S_j is mixing, chaotic and frequently hypercyclic. Moreover, S is mixing by Theorem 1.

Furthermore, S has a dense set of periodic vectors, so S is chaotic.

For $j = 1, \dots, n$ choose a non-empty open connected subset $A_j \subset \mathbb{T} \setminus \sigma_K(S_j)$. Without loss of generality we may assume that the sets A_j are mutually disjoint. Let μ be the normalized Lebesgue measure on the set $A : \bigcup_{j=1}^n A_j$. Let $B \subset A$ be a measurable subset satisfying $\mu(B) = \mu(A) = 1$. For each j we have $\mu(B \cap A_j) = \mu(A_j) > 0$. As above, $\bigvee_{z \in B \cap A_j} N(S_j - z) = H_j$, and so $\bigvee_{z \in B} N(S - z) = H$. Hence S is frequently hypercyclic by [BM2], Corollary 6.24.

Hence for each $T \in B(H)$ satisfying (a)–(c) and each $\varepsilon > 0$ there exists a mixing, chaotic and frequently hypercyclic operator $S \in B(H)$ with $\|S - T\| < \varepsilon$. So we have proved the following theorem.

Theorem 11. Let H be a separable infinite-dimensional Hilbert space. Then the set of all operators satisfying (a)–(c) is equal to the norm-closure of any of the following sets:

(i) the set of all hypercyclic operators;

(ii) the set of all operators satisfying the hypercyclicity criterion;

(iii) the set of all mixing operators;

(iv) the set of all chaotic operators;

(v) the set of all frequently hypercyclic operators;

(vi) the set of all mixing, chaotic and frequently hypercyclic operators.

Corollary 12. The set of all operators satisfying the hypercyclicity criterion is residual in $\overline{hc(H)}$.

Proof. Denote by $\mathcal{L}$ the set of all operators on H satisfying the hypercyclicity criterion, i.e., operators S such that $S \oplus S$ is hypercyclic. By Theorem 11, $\mathcal{L}$ is dense in $\overline{hc(H)}$.

Moreover, $\mathcal{L}$ is a G_δ set. Indeed, let (U_j) be a countable base of open subsets of H. It is easy to see that

$$\mathcal{L} = \bigcap_{j_1, j_2, j_3, j_4 \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} \{T \in B(H) : T^n U_1 \cap U_3 \neq \emptyset, T^n U_2 \cap U_4 \neq \emptyset\},$$

which is obviously a G_δ set. Hence $\mathcal{L}$ is residual in $\overline{hc(H)}$. □

Remark 13. By [BMP] and [Sk], there are Banach spaces without chaotic operators and frequently hypercyclic operators. On the other hand, each Banach space admits hypercyclic operators, [A], [B]. So Theorem 11 can not be true for general Banach spaces.

Remark 14. By [P], the set of all operators in $B(H)$ satisfying (a)–(c) is also equal to the norm-closure of the set of all weakly hypercyclic operators.

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