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# INTERPOLATION CHARACTERIZATION OF THE REARRANGEMENT-INVARIANT HULL OF A GENERALIZED BESOV SPACE

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Abstract. Let $X$ be a rearrangement-invariant Banach function space. We calculate the $K$-functionals for the pairs $(X, V^1X)$ and $(X, S_X(t^{-\frac{1}{n}}))$, where $V^1X$ is the reduced Sobolev space built upon $X$ and $S_X(t^{-\frac{1}{n}})$ is a particular instance of the space $S_X(w)$, determined, for a measurable nonnegative function (weight) $w$ by the norm

$$\|f\|_{S_X(w)} = \|(f^{**} - f^*)w\|_{\overline{X}},$$

where $\overline{X}$ is the representation space of $X$. Using this result, we characterize the rearrangement-invariant hull of a generalized Besov space built upon a pair of r.i. spaces.

## 1. Introduction and main results

In this paper we establish sharp estimates of rearrangements of functions in terms of moduli of continuity. Such estimates have been studied by many authors including classics. The modern approach to such estimates comes from the effort to solve a problem posed by Ul'yanov [23]. Various estimates, mostly in the setting of Lebesgue spaces, were obtained for example by Ul'yanov [23], Kolyada [11], Kolyada and Lerner [14], Storozhenko [22] and others. The subject was studied by classics such as Hardy and Littlewood and plenty of results found its way to modern monographs such as [3]. An excellent survey can be found in [13]. Some results in this direction can be also found in [15].

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Our approach is different. First, we study the problem in the far more general setting of rearrangement-invariant spaces, and second, we apply the methods of interpolation. To this end we introduce new function spaces and characterize their $K$-functionals. We thereby obtain new results which yield sharp embeddings of spaces of Besov type into spaces of Lorentz type. We also recover and improve some known results by completely different methods.

Let $\Omega$ be a domain (open and connected set) in $\mathbb{R}^n$ and let $\mathcal{M}(\Omega)$ be the set of all extended complex–valued measurable functions on $\Omega$. By $\mathcal{M}^+(\Omega)$ we denote the set of all non-negative functions from $\mathcal{M}(\Omega)$. For $f \in \mathcal{M}(\Omega)$ and $t \in (0,\infty)$, we define the *distribution function* of $f$ by $f_*(t) = |\{x \in \Omega; |f(x)| > t\}|$, where, as usual, $|\cdot|$ denotes the $n$-dimensional Lebesgue measure. The *non-increasing rearrangement* of $f$ is defined by

$$f^*(t) = \inf\{s > 0; f_*(s) \le t\}, \quad t \in [0,\infty).$$

Also, we define the *maximal non-increasing rearrangement* of $f$ by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds, \qquad t \in (0,\infty).$$

Recently, a lot of attention has been paid to the study of function spaces whose norms are defined in terms of the functional $f^{**} - f^*$. It has been shown to be useful in various parts of analysis including the interpolation theory (see [3] for classical results and, for instance, [6] for some history and references). Needless to say that it vanishes on constant functions and the operation $f \to f^{**} - f^*$ is not subadditive. Therefore, quantities involving $f^{**} - f^*$ do not necessarily have norm properties, which makes the study of the corresponding function spaces difficult. On the other hand, structures involving $f^{**} - f^*$ appear quite regularly as natural function classes in various situations. To name just few, let us mention the problem of characterization of the optimal partner norms in Sobolev-type embeddings ([10], [17], [1]), the duality problem for classical Lorentz spaces of type $\Gamma$ ([21]) or boundedness of maximal Calderón–Zygmund singular integral operators on classical Lorentz spaces ([4]).

In [6], the following function spaces were introduced (particular cases had been treated in earlier works, cf. e.g. [17] or [1]). Let $0 < p < \infty$ and let $w$ be a weight on $(0,\infty)$, that is, a measurable non-negative function. Then, the space $S^p(w)$ is the collection of all measurable functions on $(0,\infty)$ such that $\|f\|_{S^p(w)} < \infty$, where

$$\|f\|_{S^p(w)} := \left(\int_0^\infty (f^{**}(t) - f^*(t))^p\, w(t)\,dt\right)^{\frac{1}{p}}.$$

Basic properties of the spaces $S^p(w)$ including linearity were investigated in [6]. Their relations to other function spaces were further studied in [7].

Our aim is to introduce a more general version of these function spaces, based on a concept of a rearrangement-invariant space.

A space $X$ of functions in $\mathcal{M}(\Omega)$, equipped with the norm $\|\cdot\|_X$, is said to be a *rearrangement-invariant Banach function space* (shortly *r.i. space*) if the following five axioms hold:

(P1) $0 \le g \le f$ a.e. implies $\|g\|_X \le \|f\|_X$;
(P2) $0 \le f_n \nearrow f$ a.e. implies $\|f_n\|_X \nearrow \|f\|_X$;
(P3) $\|\chi_E\|_X < \infty$ for every $E \subset \Omega$ of finite measure;
(P4) a constant $C$ exists such that $\int_\Omega |f(x)|\,d\mu(x) \le C\|f\|_X$ for every $f \in X$;
(P5) $\|f\|_X = \|g\|_X$ whenever $f^* = g^*$.

Given an r.i. space $X$ on $\Omega$, the set

$$X' = \left\{ f \in \mathcal{M}(\Omega) : \int_\Omega |f(x)g(x)|\,dx < \infty \text{ for every } g \in X \right\},$$

equipped with the norm

$$\|f\|_{X'} = \sup_{\|g\|_X \le 1} \int_\Omega |f(x)g(x)|\,dx,$$

is called the *associate space* of $X$. It turns out that $X'$ is again an r.i. space and that $X'' = X$. Furthermore, the *Hölder inequality*

$$\int_\Omega |f(x)g(x)|\,dx \le \|f\|_X \|g\|_{X'}$$

holds for every $f$ and $g$ in $\mathcal{M}(\Omega)$. It will be useful to note that

$$\|f\|_X = \sup_{\|g\|_{X'} \le 1} \int_\Omega |f(x)g(x)|\,dx. \tag{1.1}$$

For every r.i. space $X$ on $\Omega$, there exists a unique r.i. space $\overline{X}$ over $(0,\infty)$ with respect to the one-dimensional Lebesgue measure, satisfying

$$\|f\|_X = \|f^*\|_{\overline{X}}$$

for every $f \in X$ (cf. [3, Chapter 2, Theorem 4.10]). This space, equipped with the norm

$$\|f\|_{\overline{X}} = \sup_{\|g\|_{X'} \le 1} \int_0^\infty f^*(t)g^*(t)\,dt,$$

is called the *representation space* of $X$.

Given a function $f \in \mathcal{M}(\Omega)$, $t \in (0,\infty)$ and an r.i. space $X$, we define the *modulus of continuity* of $f$ at $t$ with respect to $X$ by

$$\omega_X(f,t) := \sup_{|h| \le t} \|\Delta_h f\|_X,$$

where

$$\Delta_h f(x) := f(x+h) - f(x)$$

is the first-order difference. When, in particular, $X = L^p$, $1 \le p \le \infty$, then we write $\omega_p(f,t)$ instead of $\omega_{L^p}(f,t)$. We note that we use a simplified notation compared e.g. to [3], since we use only the first-order differences.

Consider a pair $(X_0, X_1)$ of Banach spaces which are compatible in the sense that they are continuously embedded into a common Hausdorff topological vector space $H$. Their *$K$-functional* is defined for each $f$ in the vector sum $X_0 + X_1$ by

$$K(t, f; X_0, X_1) = \inf_{f=g+h} (\|g\|_{X_0} + t\|h\|_{X_1}) \qquad \text{for } t > 0.$$

Then $K(t, f; X_0, X_1)$ is, as a function of $t$, *quasiconcave* on $(0, \infty)$, that is, it is non-decreasing in $t$, and the function $t^{-1}K(t, f; X_0, X_1)$ is non-increasing in $t$.

Let X be an rearrangement-invariant space on $\Omega$ and let $w$ be a non-negative measurable (*weight*) function on $[0, \infty)$. We define the function space

$$S_X(w) := \{f : \ f^*(\infty) = 0, \ \|f\|_{S_X(w)} := \|(f^{**} - f^*)w\|_{\overline{X}} < \infty\}.$$

Given a locally-integrable function $u : \Omega \to \mathbb{R}$ having all the weak derivatives of the first order, denote by $D_j u = \frac{\partial u}{\partial x_j}$, $j = 1, \dots, n$, and by $\nabla u$ the *gradient* of $u$, that is, the $n$–vector $\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right)$ of all such derivatives of $u$ and by $|\nabla u|$ the Euclidean length of this vector as an element of $\mathbb{R}^n$. We now define the *Sobolev space*

$$W^1X(\Omega) := \{u \in X : \|u\|_{W^1X} := \|u\|_X + \sum_{j=1}^n \|D_j u\|_X < \infty\}.$$

We shall also work with the class

$$V^1X(\Omega) := \{u \in X : \|u\|_{V^1X} := \sum_{j=1}^n \|D_j u\|_X < \infty\}.$$

We note that $\|\cdot\|_{W^1X}$ is a norm while $\|\cdot\|_{V^1X}$ is only a seminorm.

Let the *Hardy averaging operator*, $P$, be defined at a locally integrable function $h$ on $(0, \infty)$ and for $t \in (0, \infty)$ by

$$(Ph)(t) := \frac{1}{t}\int_0^t h(s)ds.$$

Its dual, $Q$, is defined at a locally integrable function $h$ on $(0, \infty)$ and for $t \in (0, \infty)$ by

$$(Qh)(t) := \int_t^\infty \frac{h(s)}{s}\, ds.$$

The *dilation operator* $E_s$, is defined for $s \in [0, \infty)$ and at $g \in \mathcal{M}^+(0, \infty)$, $t \in (0, \infty)$, by

$$(E_s f)(t) := \begin{cases} f\left(\frac{t}{s}\right) & 0 < t < s \\ 0 & s < t < \infty. \end{cases}$$

It is known that, for any $s \in (0, \infty)$, $E_s$ is bounded on any r.i. space ([3, Proposition 5.11, p. 148]. Using the norm of $E_s$ on $X$, denoted by $h_X(s)$, we define the *lower and upper Boyd indices* of $X$ as

$$i_X := \lim_{s\to 0+} \frac{\log\frac{1}{s}}{\log h_X(s)}, \qquad \text{and} \qquad I_X := \lim_{s\to\infty} \frac{\log\frac{1}{s}}{\log h_X(s)},$$

respectively. They satisfy

$$1 \le i(X) \le I(X) \le \infty.$$

Let us recall [3, Chapter 3, Theorem 5.15] that, given an r.i. space $X$, then the operator $P$ is bounded on $\overline{X}$ if and only if $i_X > 1$, while the operator $Q$ is bounded on $\overline{X}$ if and only if $I_X < \infty$.

Our basic idea can be outlined as follows. By the inequality of Kolyada ([12, Lemma 5.1]), see also [13, Lemma 3.1]), we have

$$t^{-\frac{1}{n}}(f^{**}(t) - f^*(t)) \lesssim |\nabla f|^{**}(t) \tag{1.2}$$

for every weakly-differentiable function $f$ on $\Omega$ and every $t \in (0,\infty)$.

Here, and throughout the paper, we write by $A \lesssim B$ when there is a positive constant $C$ independent of all essential quantities taking part in $A$ and $B$ and such that $A \le CB$. We will write $A \approx B$ when both $A \lesssim B$ and $B \lesssim A$.

Now, given an r.i. space $X$ and wrapping the norm of its representation space $\|\cdot\|_{\overline{X}}$ around both sides of (1.2), we get

$$\left\| t^{-\frac{1}{n}}(f^{**}(t) - f^*(t)) \right\|_{\overline{X}} \lesssim \| |\nabla f|^{**}(t) \|_{\overline{X}}. \tag{1.3}$$

First we have to get rid of the double star at the right hand side. It is well-known ([3, Chapter 3, Theorem 5.15] that

$$\|g^{**}\|_{\overline{X}} \lesssim \|g^*\|_{\overline{X}}$$

with the constant independent of $g$ and $t \in (0,\infty)$, if and only if $i_X > 1$. Therefore, assuming $i_X > 1$, we obtain from (1.3)

$$\left\| t^{-\frac{1}{n}}(f^{**}(t) - f^*(t)) \right\|_{\overline{X}} \lesssim \| |\nabla f|^{*}(t) \|_{\overline{X}} \approx \|\nabla f\|_X. \tag{1.4}$$

By the definition of the spaces $S_X(w)$ and $V^1X$, this can be interpreted as the embedding

$$V^1X \hookrightarrow S_X(t^{-\frac{1}{n}}).$$

Complementing this with the trivial inclusion

$$X \hookrightarrow X,$$

and using the definition of the $K$-functional, we obtain

$$K\left(f, t; X, S_X(t^{-\frac{1}{n}})\right) \lesssim K\left(f, t; X, V^1X\right). \tag{1.5}$$

So, in order to carry out our analysis any further we need a reasonable characterization of the $K$-functionals of the pairs $(X, V^1X)$ and $(X, S_X(t^{-\frac{1}{n}}))$. This problem is solved in our first two theorems.

**Theorem 1.1.** *Let $X$ be an r.i.-space. Then*

$$K(f,t,X,W^1X) \approx \min(1,t)\|f\|_X + \omega_X(f,t)$$

*for all $f \in X + W^1X$ and $t > 0$. Similarly,*

$$K(f,t,X,V^1X) \approx \omega_X(f,t)$$

*holds for all $f \in X + V^1X$ and $t > 0$.*

**Theorem 1.2.** *Let $X$ be an r.i. space satisfying $1 < i_X \le I_X < \infty$. Then*

$$\begin{aligned} K(f,t,X,S_X(t^{-\frac{1}{n}})) &\approx \\ &\approx \|(f^*(s)-f^*(t))\chi_{(0,t^n)}(s)\|_{\overline{X}} + t\|s^{-\frac{1}{n}}(f^{**}(s)-f^*(s))\chi_{(t^n,\infty)}(s)\|_{\overline{X}} \\ &\approx \|(f^{**}(s)-f^*(s))\chi_{(0,t^n)}(s)\|_{\overline{X}} + t\|s^{-\frac{1}{n}}(f^{**}(s)-f^*(s))\chi_{(t^n,\infty)}(s)\|_{\overline{X}} \end{aligned} \tag{1.6}$$

*for all $f \in X + S_X(t^{-\frac{1}{n}})$ and $t > 0$.*

From the $K$-functional inequality (1.5) and from Theorems 1.1 and 1.2, we get the following result.

**Theorem 1.3.** *Let $X$ be an r.i. space satisfying $1 < i_X \le I_X < \infty$. Then*

$$\|(f^{**}-f^*)\chi_{(0,t^n)}\|_{\overline{X}} + t\left\|\frac{(f^{**}(s)-f^*(s))\chi_{(t^n,\infty)}(s)}{s^{\frac{1}{n}}}\right\|_{\overline{X}} \lesssim \omega_X(f,t) \tag{1.7}$$

*for all measurable $f$ and $t > 0$.*

In particular, when $X = L^p$, we have

**Corollary 1.4.** *Let $1 < p < \infty$. Then, for $t \in (0,\infty)$,*

$$\int_0^{t^n} (f^{**}(s)-f^*(s))^p \, ds \lesssim \omega_p^p(f,t) \tag{1.8}$$

*and*

$$\int_{t^n}^\infty \left(\frac{f^{**}(s)-f^*(s)}{s^{\frac{1}{n}}}\right)^p ds \lesssim \frac{\omega_p^p(f,t)}{t^p}. \tag{1.9}$$

The inequalities (1.8) and (1.9) are well known (modulo the equivalence mentioned below in (2.8)). They were first proved by Kolyada [11, Theorem 1 and Corollary 6] (see also [14, Theorem 2.7] and [13, Lemma 3.6 and Theorem 3.7]), improving an earlier result of Ul'yanov [23], which states that

$$f^{**}(t) - f^*(t) \lesssim \frac{\omega_p(f,t^{\frac{1}{n}})}{t^{\frac{1}{p}}}, \qquad t \in (0,\infty).$$

Our results recover these inequalities using different methods.

One of our principal goals is to prove the sharpness of (1.7). An analogous assertion for the case when $X = L^p$ was obtained by Caetano, Gogatishvili and Opic ([5, Proposition 3.5 (ii)]).

We denote by $\kappa_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$ the measure of the unit sphere in $\mathbb{R}^n$.

**Theorem 1.5.** *Let $X$ be an r.i. space satisfying $1 < i_X \le I_X < \infty$. Let $f$ be a measurable function and set, for $x \in \Omega$, $F(x) := f^{**}(\kappa_n |x|^n)$. Then*

$$\omega_X(F,t) \lesssim \left\| (f^{**} - f^*)\chi_{(0,t^n)} \right\|_{\overline{X}} + t \left\| \frac{(f^{**}(s) - f^*(s))}{s^{\frac{1}{n}}} \chi_{(t^n,\infty)}(s) \right\|_{\overline{X}}$$

*for all $t > 0$.*

The fact that this theorem implies the sharpness of Theorem 1.3 is clearly visible from the following corollary.

**Corollary 1.6.** *Let $X$ be an r.i. space satisfying $1 < i_X \le I_X < \infty$. Let $f$ be a measurable function and set, for $x \in \Omega$, $F(x) := f^{**}(\kappa_n |x|^n)$. Then*

$$\omega_X(F,t) \lesssim \left\| (F^{**} - F^*)\chi_{(0,t^n)} \right\|_{\overline{X}} + t \left\| \frac{(F^{**}(s) - F^*(s))}{s^{\frac{1}{n}}} \chi_{(t^n,\infty)}(s) \right\|_{\overline{X}} \tag{1.10}$$

*for all $t > 0$.*

One of the most interesting consequences of the sharpness result in Theorem 1.5 is the characterization of the r.i. hull of a generalized Besov space.

**Definition 1.7.** Given two r.i. spaces, $X$ and $Y$, we define the *generalized Besov space*, $B(X,Y)$, by

$$B(X,Y) = \{ f \text{ measurable} : \omega_X(f,t) \in Y \},$$

endowed with the norm

$$\|f\|_{B(X,Y)} := \|f\|_X + \|\omega_X(f,t)\|_{\overline{Y}}.$$

**Theorem 1.8.** *Let $X$ and $Y$ be two r.i. spaces, $X$ satisfying $1 < i_X \le I_X < \infty$. Let us define the class $Z$ of measurable functions defined on $\Omega$ having finite $\|f\|_Z$, where*

$$\begin{aligned} \|f\|_Z : = \|f\|_X + \Big\| & \left\| (f^{**} - f^*)\, \chi_{(0,t^n)} \right\|_{\overline{X}} \\ & + t \left\| s^{-\frac{1}{n}} \left( f^{**}(s) - f^*(s) \right) \chi_{(t^n,\infty)}(s) \right\|_{\overline{X}} \Big\|_{\overline{Y}}. \end{aligned} \tag{1.11}$$

*Then, $Z$ is the r.i. hull of the Besov space $B(X,Y)$ in the following sense: whenever $V$ is an r.i. space on $\Omega$, then*

$$B(X,Y) \hookrightarrow V$$

*if and only if*

$$Z \hookrightarrow V.$$

**Remark 1.9.** The class $Z$ defined in the preceding theorem is not necessarily linear. Conditions of linearity of such classes can be found in [6].

When $X$ is an r.i. space over $\Omega$ we can define the so-called *fundamental function*, $\varphi_X$, of $X$, by

$$\varphi_X(t) := \|\chi_{(0,t)}\|_{\overline{X}}, \qquad t \in (0,\infty).$$

It follows from the properties of an r.i. space that $\varphi_X$ is well defined. The fundamental function satisfies (see [3])

$$\varphi_X(t)\varphi_{X'}(t) = t, \qquad t \in (0,\infty). \tag{1.12}$$

Our next result is a general estimate of a rearrangement in terms of modulus of continuity and the fundamental function.

**Theorem 1.10.** *Let $X$ be an r.i. space satisfying $1 < i_X \le I_X < \infty$. Then*

$$f^{**}(t) - f^*(t) \lesssim \frac{\omega_X(f, t^{\frac{1}{n}})}{\varphi_X(t)} \tag{1.13}$$

*for all measurable $f$ and $t > 0$. Moreover,*

$$f^{**}(t) \lesssim \int_t^\infty \frac{\omega_X(f, s^{\frac{1}{n}})}{\varphi_X(s)} \frac{ds}{s} \tag{1.14}$$

*for all measurable $f$ and $t > 0$.*

Again, for $X = L^p$, this is a classical result; (1.13) is the above-mentioned result of Ul'yanov, and for (1.14) see e.g. [3, Chapter 5, Theorem 4.19]. In general, it was proved by Gol'dman and Kerman [9], by entirely different methods using best approximation and the Jackson inequality.

**Definition 1.11.** Given an r.i. space $X$, we define the *generalized Lorentz Gamma space*, $\Gamma_X$, by

$$\Gamma_X = \left\{ f : \Omega \to \mathbb{R} \text{ measurable} : f^{**} \in \overline{X} \right\},$$

endowed with the norm

$$\|f\|_{\Gamma_X} := \|f^{**}\|_{\overline{X}}.$$

Applying an r.i. norm to the pointwise inequality (1.14), we immediately obtain the following result.

**Corollary 1.12.** *Let $X$ and $Y$ be two r.i. spaces and assume that $1 < i_X \le I_X < \infty$. Then,*

$$\|f^{**}\|_{\overline{Y}} \lesssim \left\| \int_t^\infty \frac{\omega_X(f, s^{\frac{1}{n}})}{\varphi_X(s)} \frac{ds}{s} \right\|_Y \tag{1.15}$$

*for every $f : \Omega \to \mathbb{R}$ measurable.*

We note that (1.15) can be interpreted as an embedding of a Besov space into a generalized Lorentz Gamma space. We can further simplify the resulting expression for example when we know that the operator $Q$ is bounded and thus can be peeled off.

**Example 1.13.** Let $X$ and $Y$ be two r.i. spaces and assume that $1 < i_X \le I_X < \infty$ and $I_Y < \infty$. Let $V$ be another r.i. space such that $Q : \overline{V} \to \overline{Y}$. Then we have

$$\|f^{**}\|_{\overline{Y}} \lesssim \left\| \frac{\omega_X(f, t^{\frac{1}{n}})}{\varphi_X(t)} \right\|_{\overline{V}}. \tag{1.16}$$

Conditions under which the above simplification works for various special cases (for example, weighted Lebesgue spaces), have been extensively studied by many authors, let us name for example Gol'dman [8], Gol'dman–Kerman [9] or Kolyada [11]. It is not our aim to present such results in this paper. We leave this to further research.

Finally, for two Banach spaces $X, Y$ and an r.i. space V, we define the interpolation space

$$(X,Y)_V^K := \left\{ f \text{ measurable}, \ \|f\|_{(X,Y)_V^K} := \|K(t,f;X,Y)\|_{\overline{V}} < \infty \right\}.$$

With this notation, we can formulate the result of Theorem 1.8 in the following way.

**Theorem 1.14.** *Let $X$ and $Y$ be two r.i. spaces and assume that $1 < i_X \le I_X < \infty$. Then, the rearrangement-invariant hull of the Besov space $B(X,Y)$ is the interpolation space $\left(X, S_X(t^{-\frac{1}{n}})\right)_Y^K$.*

## 2. Proofs

We will make use of the identities ([3, Chapter 5, (7.29), page 384])

$$f^{**}(t) = \int_t^\infty \frac{f^{**}(s) - f^*(s)}{s}\, ds \tag{2.1}$$

and ([3, Chapter 5, (7.11), page 379])

$$f^{**}(t) - f^{**}(s) = \int_t^s \frac{f^{**}(s) - f^*(s)}{s}\, ds, \qquad t \in (0,\infty). \tag{2.2}$$

We start with a collection of auxiliary results of independent interest, pointing out the connection between several expressions used in the literature in order to measure mean oscillation of functions in rearrangement-invariant environment. In particular, we shall consider the quantities $f^{**}(t) - f^*(t)$, studied e.g. in [3] or [1], $f^*\left(\frac{t}{2}\right) - f^*(t)$, introduced in [16], or $f^{**}\left(\frac{t}{2}\right) - f^{**}(t)$, see e.g. [18], [19] or [20].

**Proposition 2.1.** (i) *For every $f$ measurable and every $t > 0$,*

$$f^*\left(\frac{t}{2}\right) - f^*(t) \le 2\left(f^{**}(t) - f^*(t)\right), \tag{2.3}$$

$$f^{**}(t) - f^*(t) \le \frac{1}{t}\int_0^t \left(f^*\left(\frac{s}{2}\right) - f^*(s)\right) ds, \tag{2.4}$$

$$f^{**}(t) - f^*(t) \le 2\left(f^{**}(t) - f^{**}(2t)\right), \tag{2.5}$$

$$f^{**}(t) - f^*(t) \le 2\int_t^{2t} \frac{f^{**}(s) - f^*(s)}{s} ds, \tag{2.6}$$

$$f^{**}(t) - f^{**}(2t) = \frac{1}{t}\int_0^t \left(f^*(s) - f^*(2s)\right) ds. \tag{2.7}$$

(ii) *Let further $X$ be an r.i. space whose Boyd indices satisfy $1 < i_X \leq I_X < \infty$. Then, for every $f$ measurable and every $t > 0$,*

$$\|(f^*(s) - f^*(t))\chi_{(0,t)}(s)\|_{\overline{X}} \approx \|(f^{**}(s) - f^*(s))\chi_{(0,t)}(s)\|_{\overline{X}}. \tag{2.8}$$

*Proof.* (i) Both (2.3) and (2.4) are due to Bastero, Milman and Ruiz ([1]). Next,

$$\begin{aligned} f^{**}(t) - f^{**}(2t) &= \frac{1}{t}\int_0^t f^*(s)\,ds - \frac{1}{2t}\int_0^{2t} f^*(s)\,ds \\ &= \frac{1}{2t}\int_0^t f^*(s)\,ds - \frac{1}{2t}\int_t^{2t} f^*(s)\,ds \\ &\geq \frac{1}{2}\left(f^{**}(t) - f^*(t)\right), \qquad t \in (0,\infty), \end{aligned}$$

proving (2.5). To prove (2.6), given $t \in (0,\infty)$, we apply (2.2) to $s = 2t$. This yields

$$f^{**}(t) - f^{**}(2t) = \int_t^{2t} \frac{f^{**}(y) - f^*(y)}{y}\,dy, \qquad t \in (0,\infty),$$

and thus (2.6) now follows from (2.5). Again, (2.7) follows from the simple calculation

$$\begin{aligned} f^{**}(t) - f^{**}(2t) &= \frac{1}{t}\int_0^t f^*(s)\,ds - \frac{1}{2t}\int_0^t f^*(2s)\,ds \\ &= \frac{1}{t}\int_0^t \left(f^*(s) - f^*(2s)\right)\,ds, \qquad t \in (0,\infty). \end{aligned}$$

(ii) First note that, for $t \in (0,\infty)$ and $s \in (0,t)$,

$$f^{**}(s) - f^*(s) \leq f^{**}(s) - f^*(t) = \frac{1}{s}\int_0^s \left(f^*(y) - f^*(t)\right)\,dy,$$

whence, by the boundedness of $P$ on $\overline{X}$,

$$\begin{aligned} \left\|\left(f^{**}(s) - f^*(s)\right)\chi_{(0,t)}(s)\right\|_{\overline{X}} &\leq \left\|P\left(\left(f^*(s) - f^*(t)\right)\chi_{(0,t)}(s)\right)\right\|_{\overline{X}} \\ &\lesssim \left\|\left(f^*(s) - f^*(t)\right)\chi_{(0,t)}(s)\right\|_{\overline{X}}. \end{aligned}$$

Conversely, again with $t \in (0,\infty)$ and $s \in (0,t)$, using (2.2) we have

$$\begin{aligned} f^*(s) - f^*(t) &\leq f^{**}(s) - f^*(t) \\ &= f^{**}(s) - f^{**}(t) + f^{**}(t) - f^*(t) \\ &= \int_s^t \frac{f^{**}(y) - f^*(y)}{y}\,dy + f^{**}(t) - f^*(t) \\ &= Q\left(\left(f^{**}(s) - f^*(s)\right)\chi_{(0,t)}(s)\right) + f^{**}(t) - f^*(t), \end{aligned}$$

whence, by the boundedness of $Q$ on $\overline{X}$, the monotonicity of $s\,(f^{**}(s)-f^*(s))$ and by the boundedness of the dilation operator on $\overline{X}$, we finally get

$$\begin{aligned}
&\left\|(f^*(s)-f^*(t))\,\chi_{(0,t)}(s)\right\|_{\overline{X}}\\
&\quad\le \left\|Q\left((f^{**}(s)-f^*(s))\,\chi_{(0,t)}(s)\right)\right\|_{\overline{X}}+(f^{**}(t)-f^*(t))\,\varphi_X(t)\\
&\quad\lesssim \left\|(f^{**}(s)-f^*(s))\,\chi_{(0,t)}(s)\right\|_{\overline{X}}+(f^{**}(t)-f^*(t))\left\|\chi_{(t,2t)}\right\|_{\overline{X}}\\
&\quad\le \left\|(f^{**}(s)-f^*(s))\,\chi_{(0,t)}(s)\right\|_{\overline{X}}+\frac{1}{t}\left\|s\,(f^{**}(s)-f^*(s))\,\chi_{(t,2t)}(s)\right\|_{\overline{X}}\\
&\quad\lesssim \left\|(f^{**}(s)-f^*(s))\,\chi_{(0,2t)}(s)\right\|_{\overline{X}}\\
&\quad\lesssim \left\|(f^{**}(s)-f^*(s))\,\chi_{(0,t)}(s)\right\|_{\overline{X}},
\end{aligned}$$

finishing the proof. □

We note that for $X=L^p$, an analogous result to (ii) was obtained in [5, Proposition 4.5].

*Proof of* Theorem 1.1. We follow the idea of Bennett and Sharpley, who proved a similar result in the case $X=L_p$, see [3, section 5, Theorem 4.12].

*Step 1* We first prove that there exists a positive constant $c$ such that

$$\min(1,t)\|f\|_X+\omega_X(f,t)\le cK(f,t,X,W^1X)$$

holds. To this end, let $f$ be decomposed as $f=b+g$, with $b\in X$ and $g\in W^1X$. Then

$$\min(1,t)\|f\|_X\le \|b\|_X+t\|g\|_X\le \|b\|_X+t\|g\|_{W^1X}.$$

Since this inequality holds for all such decompositions, it is also true for the infimum, that is the $K$-functional on the right-hand side. Thus, it will be enough to establish

$$\omega_X(f,t)\lesssim K(f,t,X,W^1X).$$

When $f$ is decomposed as above, then for $|h|\le t$

$$\|\Delta_h f\|_X\le \|\Delta_h b\|_X+\|\Delta_h g\|_X\le 2\|b\|_X+\|\Delta_h g\|_X.$$

By [3, Chapter 5, (4.16), p. 336], we get

$$\|\Delta_h g\|_X\le t\|g\|_{V^1X}. \tag{2.9}$$

Combining the last two estimates, we arrive at

$$\|\Delta_h f\|_X\le 2\inf_{f=b+g}(\|b\|_X+t\|g\|_{W^1X})=2K(f,t,X,W^1X),$$

which completes Step 1.

*Step 2* Now we prove the existence of a positive constant $C$ such that

$$K(f,t,X,W^1X)\lesssim(\min(1,t)\|f\|_X+\omega_X(f,t)).$$

Since $K(f,t,X,W^1X)\le\|f\|_X$, we only need to consider the case $0<t<1$. We define functions $b$ and $g$ by

$$b(x)=-\int_U f(x+tu)-f(x)du,$$

where $U = [0,1]^n$ is the $n$-dimensional unit cube and

$$g(x) = f(x) - b(x) = \int_U f(x+tu)du.$$

It is easy to see that

$$\|b\|_X \le \omega_X(f,t). \tag{2.10}$$

On the other hand we have

$$\|g\|_{W^1X} = \|g\|_X + \sum_{j=1}^n \|D_j g\|_X,$$

where $\|g\|_X \le \|f\|_X$ and

$$t\|D_j g\|_X \le c\omega_X(f,t), \tag{2.11}$$

using here an analogue of [3, Chapter 5, (4.40), p. 341] for r.i. spaces, which can be proved almost verbatim. Combining the estimates, we arrive at

$$K(f,t,X,W^1X) \le \|b\|_X + t\|g\|_{W^1X} \le t\|f\|_X + c\omega_X(f,t),$$

which completes the proof of the first part of the assertion.

The second part follows from (2.9), (2.10) and (2.11). $\square$

*Proof of* Theorem 1.2. Fix $f \in X + S_X(t^{-\frac{1}{n}})$ and $t > 0$.

*Step 1* Let $E$ be a set satisfying

$$\{x \in \Omega : |f(x)| > t\} \subset E \subset \{x \in \Omega : |f(x)| \ge t\},$$

then we decompose $f = b + g$, where

$$b(x) = (|f(x)| - f^*(t)) \operatorname{sgn}(f(x))\chi_E(x), \qquad x \in \Omega.$$

This gives

$$K(f, t^{\frac{1}{n}}, X, S_X(t^{-\frac{1}{n}})) \le \|b\|_X + t^{\frac{1}{n}} \|g\|_{S_X(t^{-\frac{1}{n}})}$$

$$= \|(f^*(s) - f^*(t))\chi_{(0,t)}(s)\|_{\overline{X}} + t^{\frac{1}{n}} \|s^{-\frac{1}{n}}(g^{**}(s) - g^*(s))\chi_{(t,\infty)}(s)\|_{\overline{X}}.$$

We calculate the second term of the last line. By construction we have

$$g^*(s) = \min(f^*(t), f^*(s)), \qquad s \in (0,\infty),$$

therefore,

$$g^{**}(s) = f^*(t)\chi_{(0,t)}(s) + \frac{1}{s}\left(tf^*(t) + \int_t^s f^*(y)dy\right)\chi_{(t,\infty)}(s).$$

Hence, we can estimate

$$\begin{aligned} g^{**}(s) - g^*(s) &= \left(\frac{t}{s}f^*(t) + \frac{1}{s}\int_t^s f^*(y)dy - f^*(s)\right)\chi_{(t,\infty)}(s) \\ &\le \left(\frac{1}{s}\int_0^s f^*(y)dy - f^*(s)\right)\chi_{(t,\infty)}(s) \\ &= (f^{**}(s) - f^*(s))\chi_{(t,\infty)}(s), \end{aligned}$$

which proves one inequality in the desired assertion.

*Step 2* Suppose $f = b + g$ be any decomposition, then we want to show that the middle term in (1.6) is smaller than or equal to

$$\|b\|_X + t^{\frac{1}{n}} \|g\|_{S_X(t^{-\frac{1}{n}})}.$$

We start with the term

$$\|(f^*(s) - f^*(t))\chi_{(0,t)}(s)\|_{\overline{X}}.$$

Observe that

$$f^*(t) \le b^*(\frac{t}{2}) + g^*(\frac{t}{2})$$

and

$$f^*(t) \ge g^*(2t) - b^*(t).$$

Therefore, using the boundedness of $P$ on $\overline{X}$, we have

$$\begin{aligned}
(2.12)\quad & \|(f^*(s) - f^*(t))\chi_{(0,t)}(s)\|_{\overline{X}} \\
& \le \|(b^*\left(\frac{s}{2}\right) + g^*\left(\frac{s}{2}\right) - g^*\left(2t\right) + b^*\left(t\right)\chi_{(0,t)}(s)\|_{\overline{X}} \\
& \le \|b^*\left(\frac{s}{2}\right)\chi_{(0,t)}(s)\|_{\overline{X}} + \|b^*(t)\chi_{(0,t)}(s)\|_{\overline{X}} \\
& \quad + \|\left(g^*\left(\frac{s}{2}\right) - g^*(2t)\right)\chi_{(0,t)}(s)\|_{\overline{X}} \\
& \le \|b^{**}\left(\frac{s}{2}\right)\chi_{(0,t)}(s)\|_{\overline{X}} + \|b\|_X + \|\left(g^*\left(\frac{s}{2}\right) - g^*(2t)\right)\chi_{(0,t)}(s)\|_{\overline{X}} \\
& \lesssim \|P(b^*)(s)\chi_{(0,t)}(s)\|_{\overline{X}} + \|b\|_X + \|\left(g^*\left(\frac{s}{2}\right) - g^*(2t)\right)\chi_{(0,t)}(s)\|_{\overline{X}} \\
& \lesssim \|b\|_X + \|\left(g^*\left(\frac{s}{2}\right) - g^*(2t)\right)\chi_{(0,t)}(s)\|_{\overline{X}}.
\end{aligned}$$

We calculate the second term of the right-hand side pointwise. Let $m \in \mathbb{N}$ be the smallest number, such that $2^m s \ge 2t$, then we write

$$g^*\left(\frac{s}{2}\right) - g^*(2t) \le \sum_{k=0}^{m}\left(g^*\left(2^{k-1}s\right) - g^*\left(2^k s\right)\right),$$

then by (2.3) and (2.6) we can estimate further

$$\begin{aligned}
g^*\left(\frac{s}{2}\right) - g^*(2t) &\lesssim \sum_{k=0}^{m} g^{**}\left(2^k s\right) - g^*\left(2^k s\right) \\
&\lesssim \sum_{k=0}^{m} \int_{2^k s}^{2^{k+1}s} \frac{g^{**}(\tau) - g^*(\tau)}{\tau} d\tau \\
&\approx \int_{s}^{2^{m+1}s} \frac{g^{**}(\tau) - g^*(\tau)}{\tau} d\tau.
\end{aligned}$$

Putting this pointwise inequality into the norm, we arrive at

$$\begin{aligned}\|g^*\left(\frac{s}{2}\right) - g^*(2t)\chi_{(0,t)}(s)\|_{\overline{X}} &\lesssim \left\| \int_s^{2^{m+1}s} \frac{g^{**}(\tau) - g^*(\tau)}{\tau} d\tau \chi_{(0,t)}(s) \right\|_{\overline{X}} \\ &\lesssim \left\| \int_s^{8t} \frac{g^{**}(\tau) - g^*(\tau)}{\tau} d\tau \right\|_{\overline{X}} \\ &\lesssim t^{\frac{1}{n}} \left\| \int_s^{\infty} \tau^{-\frac{1}{n}} \frac{g^{**}(\tau) - g^*(\tau)}{\tau} d\tau \right\|_{\overline{X}} \\ &\lesssim t^{\frac{1}{n}} \|\tau^{-\frac{1}{n}}(g^{**}(\tau) - g^*(\tau))\|_{\overline{X}},\end{aligned}$$

where we used the boundedness of the operator $Q$ on the last line. Looking at (2.12) we see that we are done with the first term of the right-hand side in (1.6). It remains to treat the second one, that is

$$t^{\frac{1}{n}} \|s^{-\frac{1}{n}}(f^{**}(s) - f^*(s))\chi_{(t,\infty)}(s)\|_{\overline{X}}.$$

Let $f$ be decomposed as above, then we have

$$f^{**}(s) - f^*(s) \leq b^{**}(s) + g^{**}(s) - g^*(2s) + b^*(s)$$

and get for the norm

$$\begin{aligned}&t^{\frac{1}{n}} \|s^{-\frac{1}{n}}(f^{**}(s) - f^*(s))\chi_{(t,\infty)}(s)\|_{\overline{X}} \\ &\leq \|b^{**}\|_{\overline{X}} + \|b^*\|_{\overline{X}} + t^{\frac{1}{n}} \|s^{-\frac{1}{n}}(g^{**}(s) - g^*(2s))\chi_{(t,\infty)}(s)\|_{\overline{X}} \\ &\leq 2\|b\|_X + t^{\frac{1}{n}} \|s^{-\frac{1}{n}}(g^{**}(s) - g^*(s))\chi_{(t,\infty)}(s)\|_{\overline{X}} \\ &\qquad + t^{\frac{1}{n}} \|s^{-\frac{1}{n}}(g^*(s) - g^*(2s))\chi_{(t,\infty)}(s)\|_{\overline{X}}.\end{aligned}$$

If we use (2.3) for the last term on the right-hand side, we finally arrive at

$$\begin{aligned}&t^{\frac{1}{n}} \|s^{-\frac{1}{n}}(f^{**}(s) - f^*(s))\chi_{(t,\infty)}(s)\|_{\overline{X}} \leq \\ &\leq c\left(\|b\|_X + ct^{\frac{1}{n}} \|\tau^{-\frac{1}{n}}(g^{**}(\tau) - g^*(\tau))\|_{\overline{X}}\right),\end{aligned}$$

which completes the proof of the first equivalence.

The second one follows from Proposition 2.1. □

*Proof of* Theorem 1.5. Let $h \in \mathbb{R}^n$. Then,

$$\begin{aligned}\|\Delta_h F\|_X &= \|F(x+h) - F(x)\|_X \\ &\leq \|\left(F(x+h) - F(x)\right)\chi_{\{0<|x|\leq 2h\}}(x)\|_X \\ &\qquad + \|\left(F(x+h) - F(x)\right)\chi_{\{|x|>2h\}}(x)\|_X \\ &=: I + J,\end{aligned}$$

say. Next, we have

$$
\begin{aligned}
I &\le \| \left(F(x+h) - F(3h)\right) \chi_{\{0<|x|\le 2h\}}(x)\|_X \\
&\quad + \| \left(F(3h) - F(x)\right) \chi_{\{0<|x|\le 2h\}}(x)\|_X \\
&\le 2\| \left(F(y) - F(3h)\right) \chi_{\{0<|x|\le 3h\}}(x)\|_X \\
&= 2\| \left(f^{**}(s) - f^{**}(3^n\kappa_n|h|^n)\right) \chi_{(0,3^n\kappa_n|h|^n)}(s)\|_{\overline{X}} \\
&\le 2\| \left(f^{**}(s) - f^{*}(3^n\kappa_n|h|^n)\right) \chi_{(0,3^n\kappa_n|h|^n)}(s)\|_{\overline{X}} \\
&\lesssim \| \left(f^{*}(s) - f^{*}(3^n\kappa_n|h|^n)\right) \chi_{(0,3^n\kappa_n|h|^n)}(s)\|_{\overline{X}} \\
&\lesssim \| \left(f^{**}(s) - f^{*}(s)\right) \chi_{(0,3^n\kappa_n|h|^n)}(s)\|_{\overline{X}}.
\end{aligned}
$$

Here, the last but one inequality is the boundedness of $P$ on $\overline{X}$ (recall that $i_X > 1$) and the last one is (2.8).

Similarly,

$$
\begin{aligned}
J &= \| \left(g(\kappa_n|x+h|^n) - g(\kappa_n|x|^n\right) \chi_{\{|x|>2h\}}(x)\|_X \\
&= \left\| \left( \int_{\kappa_n \min\{|x|^n,|x+h|^n\}}^{\kappa_n \max\{|x|^n,|x+h|^n\}} |g'(u)|\,du \right) \chi_{\{|x|>2h\}}(x) \right\|_X \\
&\le \kappa_n \left\| \left( ||x+h|^n - |x|^n| \operatorname*{ess\,sup}_{\kappa_n\left(\frac{|x|}{2}\right)^n \le u \le \kappa_n\left(\frac{3|x|}{2}\right)^n} |g'(u)|\,du \right) \chi_{\{|x|>2h\}}(x) \right\|_X \\
&\lesssim \left\| \left( |x|^{n-1}|h| \operatorname*{ess\,sup}_{\kappa_n\left(\frac{|x|}{2}\right)^n \le u \le \kappa_n\left(\frac{3|x|}{2}\right)^n} |g'(u)|\,du \right) \chi_{\{|x|>2h\}}(x) \right\|_X \\
&\lesssim |h| \left\| \left( s^{1-\frac{1}{n}} \operatorname*{ess\,sup}_{\left(\frac{1}{2}\right)^n s \le u \le \left(\frac{3}{2}\right)^n s} |g'(u)|\,du \right) \chi_{(2^n\kappa_n|h|^n,\infty)}(s) \right\|_{\overline{X}}.
\end{aligned}
$$

Since

$$
(f^{**})'(u) = -\frac{1}{u^2} \int_0^u \left(f^*(y) - f^*(u)\right)\,dy,
$$

we get

$$
\begin{aligned}
II &\lesssim |h| \left\| \left( s^{1-\frac{1}{n}} s^{-2} \int_0^{\left(\frac{3}{2}\right)^n s} \left( f^*(y) - f^*\left(\left(\frac{3}{2}\right)^n s\right)\right)\,dy \right) \chi_{(2^n\kappa_n|h|^n,\infty)}(s) \right\|_{\overline{X}} \\
&\lesssim |h| \left\| s^{-\frac{1}{n}} \left(f^{**}(s) - f^*(s)\right) \chi_{(3^n\kappa_n|h|^n,\infty)}(s) \right\|_{\overline{X}}.
\end{aligned}
$$

Altogether, we have

$$\begin{aligned}
\omega_X(F,t) &= \sup_{|h|\le t} \|\Delta_h F\|_X \\
&\lesssim \sup_{|h|\le t} \| \left(f^{**}(s)-f^*(s)\right) \chi_{(0,3^n\kappa_n|h|^n)}(s) \|_{\overline{X}} \\
&\qquad + \sup_{|h|\le t} |h| \left\| s^{-\frac{1}{n}} \left(f^{**}(s)-f^*(s)\right) \chi_{(3^n\kappa_n|h|^n,\infty)}(s) \right\|_{\overline{X}} \\
&= K+L,
\end{aligned}$$

say. Now, the expression behind the supremum in $K$ is obviously increasing in $|h|$. We have to estimate $L$. So,

$$\begin{aligned}
L &\le \sup_{|h|\le t} |h| \left\| s^{-\frac{1}{n}} \left(f^{**}(s)-f^*(s)\right) \chi_{(3^n\kappa_n|h|^n,3^n\kappa_n t^n)}(s) \right\|_{\overline{X}} \\
&\quad + \sup_{|h|\le t} |h| \left\| s^{-\frac{1}{n}} \left(f^{**}(s)-f^*(s)\right) \chi_{(3^n\kappa_n t^n,\infty)}(s) \right\|_{\overline{X}} \\
&= L_1 + L_2,
\end{aligned}$$

say. Obviously,

$$L_2 = t \left\| s^{-\frac{1}{n}} \left(f^{**}(s)-f^*(s)\right) \chi_{(3^n\kappa_n t^n,\infty)}(s) \right\|_{\overline{X}},$$

while

$$\begin{aligned}
L_1 &\lesssim \sup_{|h|\le t} |h| (3^n\kappa_n|h|^n)^{-\frac{1}{n}} \left\| \left(f^{**}(s)-f^*(s)\right) \chi_{(0,3^n\kappa_n t^n)}(s) \right\|_{\overline{X}} \\
&\lesssim K.
\end{aligned}$$

Summarizing all the estimates obtained, we finally get

$$\begin{aligned}
\omega_X(F,t) &\lesssim \| \left(f^{**}(s)-f^*(s)\right) \chi_{(0,3^n\kappa_n t^n)}(s) \|_{\overline{X}} \\
&\qquad + t \left\| s^{-\frac{1}{n}} \left(f^{**}(s)-f^*(s)\right) \chi_{(3^n\kappa_n t^n,\infty)}(s) \right\|_{\overline{X}}.
\end{aligned}$$

Using the boundedness of the dilation operator on $\overline{X}$, we get the assertion. □

*Proof of* Corollary 1.6. Using the monotonicity of $t(f^{**}(t)-f^*(t)$, we have

$$\begin{aligned}
F^{**}(t)-F^*(t) &= \frac{1}{t}\int_0^t \left(f^{**}(s)-f^*(s)\right)\, ds \\
&\ge \frac{1}{t}\int_{\frac{t}{2}}^t \frac{ds}{s}\frac{t}{2}\left(f^{**}\left(\frac{t}{2}\right) - f^*\left(\frac{t}{2}\right)\right) \\
&\approx \left(f^{**}\left(\frac{t}{2}\right) - f^*\left(\frac{t}{2}\right)\right).
\end{aligned}$$

We thus have

$$f^{**}(t)-f^*(t) \lesssim \left(F^{**}(2t)-F^*(2t)\right),$$

and therefore he assertion follows from Theorem 1.5 combined with the boundedness of the dilation operator on $\overline{X}$. □

*Proof of* Theorem 1.8. First assume that $B(X,Y) \hookrightarrow V$. Let $\|f\|_Z < \infty$. We define the function $F$ as in Theorem 1.5. We then get, thanks to the boundedness of $P$ on $\overline{X}$,

$$\|F\|_V \lesssim \|F\|_{B(X,Y)} = \|F\|_X + \|\omega_X(F,t)\|_Y \lesssim \|f\|_Z.$$

Observing that

$$\|F\|_V = \|F^*\|_V = \|f^{**}\|_V \geq \|f^*\|_V,$$

we get $Z \hookrightarrow V$ as desired.

Conversely, assume that $Z \hookrightarrow V$. Then, by Theorem 1.3, we have, for any measurable $f$,

$$\|f\|_V \lesssim \|f\|_Z \lesssim \|f\|_X + \|\omega_X(f,t)\|_Y \leq \|f\|_{B(X,Y)},$$

that is, $B(X,Y) \hookrightarrow V$, finishing the proof. □

*Proof of* Theorem 1.10. By the Hölder inequality, (1.12), (2.8) and (1.7), applied to $t$ in place of $t^n$, we obtain

$$\begin{aligned} f^{**}(t) - f^*(t) &= \frac{1}{t}\int_0^t (f^*(s) - f^*(t))\, ds \\ &\leq \left\|\chi_{(0,t)}(s)\,(f^*(s) - f^*(t))\right\|_{\overline{X}} \frac{\varphi_{X'}(t)}{t} \\ &\lesssim \left\|\chi_{(0,t)}(s)\,(f^{**}(s) - f^*(s))\right\|_{\overline{X}} \frac{1}{\varphi_X(t)} \\ &\lesssim \frac{\omega_X(f, t^{\frac{1}{n}})}{\varphi_X(t)}, \qquad t \in (0,\infty). \end{aligned} \tag{2.13}$$

This shows (1.13). Using (2.1) and (2.13), we get

$$f^{**}(t) = \int_t^\infty \frac{f^{**}(s) - f^*(s)}{s}\, ds \lesssim \int_t^\infty \frac{\omega_X(f, s^{\frac{1}{n}})}{\varphi_X(s)} \frac{ds}{s}, \qquad t \in (0,\infty),$$

proving (1.14). □

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