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Quotients of Boolean algebras and regular subalgebras

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Abstract

Let $\mathbb{B}, \mathbb{C}$ be Boolean algebras and $e : \mathbb{B} \to \mathbb{C}$ an embedding. We examine hierarchy of ideals on $\mathbb{C}$ for which $\overline{e} : \mathbb{B} \to \mathbb{C}/\mathcal{I}$ is a regular (i.e. complete) embedding and as an application we deal with interrelationship among $\mathcal{P}(\omega)/fin$ in ZFC groundmodel and in its extension. If M is an extension of V adding new subset of ω, then in M there is almost disjoint refinement of the family $([\omega]^\omega)^V$. Moreover, there is exactly one ideal $\mathcal{I}$ on ω in M such that $(\mathcal{P}(\omega)/fin)^V$ is dense subalgebra of $(\mathcal{P}(\omega)/\mathcal{I})^M$ if and only if M does not add independent (splitting) real.

We show that for a generic extension $V[G]$, the canonical embedding

$$\mathcal{P}^V(\omega)/fin \hookrightarrow \mathcal{P}(\omega)/(U(Os)(\mathbb{B}))^G$$

is a regular one, where $U(Os)(\mathbb{B})$ is the Urysohn closure of zero - convergent structure on $\mathbb{B}$.

1 Introduction

Let V be a model of ZFC and M its extension. Then $(\mathcal{P}(\omega)/\mathrm{fin})^V$ is a subalgebra of the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ in M.

It is natural to ask whether $(\mathcal{P}(\omega)/\mathrm{fin})^V$ is a regular subalgebra of $\mathcal{P}(\omega)/\mathrm{fin}$.

This question makes sense only in cases when there are new reals in the extension M, otherwise these algebras coincide. Hence in what follows we suppose that M is an arbitrary ZFC extension of ground model V adding new reals.

L. Soukup posed the following question:

Does the family $([\omega]^\omega)^V$ have an almost disjoint refinement in any generic extension, which adds a new real?

It was known that this holds true in different types of generic extension, e.g. adding one Cohen real [Hec78].

We shall consider a little more general situation, when we take into account arbitrary ZFC extension M of V. Clearly to have a chance for the refinement, the extension M has to add a new real, i.e.

$$(\mathcal{P}(\omega))^V \subsetneq (\mathcal{P}(\omega))^M,$$

in this generalised setting we show in paragraph 3 the following theorem This result was achieved independently by J. Brendle, his proof is rather different and can be found in L. Soukup's paper [Sou07].

Theorem 1. *In any ZFC extension M of V adding a new real there is an almost disjoint refinement of $([\omega]^\omega)^V$.*

In the following $A, B \subset \omega$; $A \subset^* B$ will denote the fact that $A \setminus B$ is finite. Note that, in fact Almost Disjoint (AD) family is a pairwise disjoint family in the Boolean algebra $\mathcal{P}(\omega)/fin$ and a Maximal Almost Disjoint (MAD) family is a partition of unity in the same algebra.

Definition. Family $\mathcal{S} \subset [\omega]^\omega$ has an *almost disjoint refinement (ADR)* if there is an almost disjoint family $\{A_X : X \in \mathcal{S}\}$ such that $A_X \in [X]^\omega$ for every $X \in \mathcal{S}$.

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Instead of this 'indexed' refinement we will benefit from [BPS80] and use any of these equivalents without further mentioning.

Proposition. *For a family $\mathcal{S} \subset [\omega]^\omega$ the following are equivalent:*

(i) The family S has ADR, i.e. there is an almost disjoint family $\{A_X : X \in \mathcal{S}\}$ such that $A_X \in [X]^\omega$ for every $X \in \mathcal{S}$.

(ii) There is an almost disjoint family $\mathcal{A}$ such that for any $X \in \mathcal{S}$ there is $A \in \mathcal{A}$ such that $A \subset^* X$.

(iii) There is an almost disjoint family $\mathcal{A}$ such that for any $X \in \mathcal{S}$

$$|\{A \in \mathcal{A} : |X \cap A| = \omega\}| = 2^\omega.$$

Proof. *(i) → (ii)* This implication is trivial since the almost disjoint family from *(i)* satisfies also *(ii)*.
(ii) → (iii) Let $\mathcal{A}$ be an almost disjoint family as in *(ii)*. In $[\omega]^\omega$ there is a maximal almost disjoint family $\langle B_i^A : i \in 2^\omega\rangle$ of a size 2^ω below any $A \in \mathcal{A}$. Hence $\langle B_i^A : i \in 2^\omega, \quad A \in \mathcal{A}\rangle$ satisfies *(iii)*.
(iii) → (i) First enumerate $\mathcal{S} = \{X_\alpha : \alpha \in 2^\omega\}$ and for any $X \in \mathcal{S}$ denote $\mathcal{A}_X = \{A \in \mathcal{A} : |X \cap A| = \omega\}$, $|\mathcal{A}_X| = 2^\omega$. Now proceed by induction and for each $X_\alpha \in \mathcal{S}$ choose an $A_\alpha \in \mathcal{A}_{X_\alpha} - \bigcup\{A_\beta : \beta < \alpha\}$. Clearly the family $\{A_\alpha \cap X_\alpha : \alpha \in 2^\omega\}$ gives an almost disjoint refinement for $\mathcal{S}$. □

Our approach to Theorem 1 strongly benefit from results of [BPS80] or see [BS89]; let us quickly summarize results we use. For other notions concerning Boolean algebras see [Kop89].

Note that an algebra $\mathbb{B}$ is $(\kappa, \cdot, 2)$ distributive if and only if any κ-many partitions of unity have a common refinement, or equivalently if and only if the intersection $\cap_{\alpha<\kappa} D_\alpha$ of κ-many open dense sets is dense.

Cardinal invariant $\mathfrak{h}$ (*non-distributivity number*) is characterised through distributivity properties of the algebra $\mathcal{P}(\omega)/\text{fin}$ as follows:

Definition.

$$\mathfrak{h} = \min\,\{\kappa : \mathcal{P}(\omega)/\text{fin is not } (\kappa, \cdot, 2) \text{ distributive }\},$$

In the proof of Theorem 1 we use the techniques of base tree. Base tree is a special kind of a dense subset of $\mathcal{P}(\omega)/fin$; see e.g. [BS89].

Theorem. [BPS80]) *There is a* base tree $(T, \supset^*)$ *for* $[\omega]^\omega$*, i.e.*

(i) $(T, \supseteq^*) \subset [\omega]^\omega$ is a tree,

(ii) if $B \in T$ then the family of immediate successors of B in T is a maximal almost disjoint family below B of a full (2^ω) size,

(iii) for each $A \in [\omega]^\omega$ there is $B \in T$ such that $B \subset A$,

(iv) the height of T is $\mathfrak{h}$.

It is well known that if new real is added, then $(\mathcal{P}(\omega)/fin)^V$ is not a regular subalgebra of $(\mathcal{P}(\omega)/fin)^M$. There is a natural question wheather there is an ideal $\mathcal{I}$ such that the canonical embedding

$$(\mathcal{P}(\omega)/fin)^V \hookrightarrow (\mathcal{P}(\omega)/fin)^M/\mathcal{I},$$

becomes regular. We show in paragraph 2 more general theorem

Theorem 2. *Let $\mathbb{B}$ be a subalgebra of a Boolean algebra $\mathbb{C}$. There is an ideal $\mathcal{I}$ on $\mathbb{C}$ such that the canonical homomorphism*

$$\begin{aligned} i : \mathbb{B} &\longrightarrow \mathbb{C}/\mathcal{I} \\ b &\longmapsto [b]_{\mathcal{I}}, \end{aligned}$$

is a regular embedding of $\mathbb{B}$ into $\mathbb{C}/\mathcal{I}$.

Finally in paragraphs 4 and 5 we compute the minimal *regularization* ideal for embeddings $(\mathcal{P}(\omega)/fin)^V \hookrightarrow (\mathcal{P}(\omega)/fin)^M/\mathcal{I}$ and $\mathbb{B} \hookrightarrow \mathbb{B}^\omega/Fin$. Latter and former regularisation ideals are closely connected with order sequential topology on Boolean algebras, which we briefly introduce here in the Topological intermezzo.

2 Regularisation ideals

We start with Theorem 2. First, let us recall the definition of regular subalgebra $\mathbb{B}$ of a Boolean algebra $\mathbb{C}$ and its equivalents.

A subalgebra $\mathbb{B}$ of a Boolean algebra $\mathbb{C}$ is called *regular* if any $X \subset \mathbb{B}$ which has a supremum $\bigvee^{\mathbb{B}} X$ in $\mathbb{B}$, has the same element as a supremum of X in $\mathbb{C}$, i.e. $\bigvee^{\mathbb{B}} X = \bigvee^{\mathbb{C}} X$. An embedding $i : \mathbb{B} \to \mathbb{C}$ is *regular* if the image $i[\mathbb{B}]$ is the regular subalgebra of algebra $\mathbb{C}$.

Proposition 3. *For a subalgebra $\mathbb{B} \subset \mathbb{C}$ the following are equivalent*

(i) $\mathbb{B}$ *is a regular subalgebra of* $\mathbb{C}$,

(ii) every maximal pairwise disjoint family in $\mathbb{B}$ is maximal in $\mathbb{C}$,

(iii) for each $c \in \mathbb{C}^+$ there is a 'pseudoprojection' *$b_c \in \mathbb{B}^+$; i.e. for every $a \leq b_c$, $a \in \mathbb{B}^+$*

$$a \wedge c \neq \mathbf{0},$$

(iv) for every generic filter F on $\mathbb{C}$, $F \cap \mathbb{B}$ is a generic filter on $\mathbb{B}$.

Proof. The proofs of implications *(i)* $\leftrightarrow$ *(ii)* $\leftrightarrow$ *(iii)* $\leftrightarrow$ *(v)* and *(vi)*$\rightarrow$ *(ii)* are straight forward.

To show that *(ii)*$\rightarrow$ *(vi)* let $c \in \mathbb{C}^+$. Take arbitrary maximal pairwise disjoint family $B_c \subset \{b \in \mathbb{B} \ : \ b \wedge c = \mathbf{0}\}$. From *(ii)* it follows that B_c is not maximal in $\mathbb{B}$, hence there is some b_c disjoint with B_c and we are done. □

Let $\mathbb{B}, \mathbb{C}$ be Boolean algebras and $e : \mathbb{B} \to \mathbb{C}$ an embedding. We are looking for ideals on $\mathbb{C}$ for which the factor embedding $\bar{i}$ is regular.

Theorem 2. *Let $\mathbb{B}$ be a subalgebra of a Boolean algebra $\mathbb{C}$. There is a minimal ideal $\mathcal{I}_{min}$ on $\mathbb{C}$ such that the canonical homomorphism*

$$\begin{aligned} i : \mathbb{B} &\longrightarrow \mathbb{C}/\mathcal{I}_{min} \\ b &\longmapsto [b]_{\mathcal{I}_{min}}, \end{aligned}$$

is a regular embedding of $\mathbb{B}$ into $\mathbb{C}/\mathcal{I}_{min}$.

Proof. Let

$$\mathcal{I} = \{u \in \mathbb{C} : \exists \text{ max. pairwise disjoint family } X \subset \mathbb{B} \text{ such that } u \wedge x = \mathbf{0} \text{ for any } x \in X\}.$$

We check that $\mathcal{I}$ is an ideal. The set $\mathcal{I}$ is downward closed. Let $u, v \in \mathcal{I}$. Take maximal pairwise disjoint families X and Y that guarantee that u respectively v belongs to $\mathcal{I}$. Then $z = \{x \wedge y \neq \mathbf{0} : x \in X \ \&\ y \in Y\}$ is a maximal pairwise disjoint family of elements of $\mathbb{B}$ and $u \vee v$ is disjoint from every element of z. Therefore $u \vee v \in \mathcal{I}$, hence $\mathcal{I}$ is an ideal.

No $b \in \mathbb{B}^+$ belongs to $\mathcal{I}$, so the mapping $i : \mathbb{B} \to \mathbb{C}/\mathcal{I}$ is an embedding. We show that i is a regular embedding. Let $\{c_i : i \in I\}$ be a maximal pairwise disjoint family in $i[\mathbb{B}]$, the family $\{[c_i] : i \in I\}$ is a maximal pairwise disjoint family in $\mathbb{C}/\mathcal{I}$. Assume that there is $[u]$, disjoint with every $[c_i]$ in $\mathbb{C}/\mathcal{I}$, i.e. $c_i \wedge u \in \mathcal{I}$, hence there is a maximal pairwise disjoint set $X_i \subset \mathbb{B} \restriction c_i$ such that u is disjoint from every element of X_i. The set $\bigcup\{X_i : i \in I\}$ is maximal in $\mathbb{B}$ and so $u \in \mathcal{I}$, i.e. $[u] = \mathbf{0} \in \mathbb{C}/\mathcal{I}$.

Such obtained ideal I is minimal. □

Proposition 4. *Let $\mathbb{B}$ be a subalgebra of a Boolean algebra $\mathbb{C}$ and let $\mathcal{J} \subset \mathbb{C}$ be a maximal ideal such that $\mathbb{B} \cap \mathcal{J} = \{\mathbf{0}\}$. Then canonical embedding*

$$i : \mathbb{B} \longrightarrow \mathbb{C}/\mathcal{J},$$

is a regular one. In this case $i[\mathbb{B}]$ is even dense in $\mathbb{C}/\mathcal{J}$.

Proof. Suppose that $i[\mathbb{B}]$ is not dense in $\mathbb{C}/\mathcal{J}$. Then there is a $c \in \mathbb{C}$, $c \notin \mathcal{J}$ such that for any $b \in \mathbb{B}^+$ $b \not\leq_{\mathcal{J}} c$. Since $\mathcal{J}$ is maximal and $c \notin \mathcal{J}$ there is a $j \in \mathcal{J}$ such that there is a $b \in \mathbb{B}^+$ so that $b \leq c \vee j$ i.e. $b \leq_{\mathcal{J}} c$; contradiction. □

Corollary 5. Let $\mathbb{B}$ be a subalgebra of a Boolean algebra $\mathbb{C}$ and let $\mathcal{J} \subset \mathbb{C}$ be a maximal regularising ideal. Then

(i) if $\mathbb{B}$ is complete, then $\mathbb{B} \simeq \mathbb{C}/\mathcal{J}$;

(ii) if $\mathbb{C}$ is complete, then $\mathrm{RO}(\mathbb{B}) \simeq \mathbb{C}/\mathcal{J}$.

Proposition 6. *Let $\mathbb{B}$ be a subalgebra of a Boolean algebra $\mathbb{C}$ and let*

$$\mathcal{K} = \{\mathcal{J} \ : \ \mathcal{J} \text{ is an ideal on } \mathbb{C} \text{ maximal with respect to } \mathcal{J} \cap \mathbb{B}^+ = \emptyset\}$$

then

(i) $\bigcap \mathcal{K} = \mathcal{I}_{min}$ *and*

(ii) $\bigcup \mathcal{K} = \{c \in \mathbb{C} \ : \ \neg(\exists b \in \mathbb{B}^+)\ b \leq c\}$.

Proof. Suppose that $\mathcal{I} \setminus \mathcal{J} \neq \emptyset$ and $a \in \mathcal{I} \setminus \mathcal{J}$. Since $\mathcal{J}$ is maximal then there is a $j \in \mathcal{J}$ for which there is a $b \in \mathbb{B}^+$ such that $b \leq j \vee a$. Since $a \in \mathcal{I}$, there is a maximal antichain M in $\mathbb{B}$ such that $m \wedge a = \mathbf{0}$, for each $m \in M$. Every $b \in \mathbb{B}$ has to intersect some $m \in M$, so $\mathbf{0} \neq m \wedge b \leq j \vee a$, but the m and a are disjoint hence $m \wedge b \leq j$, which is contradiction with the assumption that $\mathcal{J}$ does not intersect $\mathbb{B}$.

Clearly, $\bigcap \mathcal{K} \supset \mathcal{I}$. Take arbitrary $c \in \mathbb{C}^+ \setminus \mathcal{I}$, the set $\{b \in \mathbb{B} \ : \ b \leq -c\}$ is not dense in $\mathbb{B}$ as $c \notin \mathcal{I}$. It means that there is a $b_0 \in \mathbb{B}$ such that

$$\forall b \in \mathbb{B}^+ \quad (b \leq -c) \ \rightarrow \ b - b_0 \neq \mathbf{0}.$$

That is, $b_0 \wedge -c \notin \mathbb{B}$ and one can take a maximal ideal $\mathcal{J}$ containing this element, which shows that $c \notin \bigcap \mathcal{K}$; and we are done. □

3 Almost disjoint refinement of ground model reals

Let M be a ZFC extension of V. We ask about the existence of almost disjoint refinement of $(\mathcal{P}(\omega))^V$ in M Clearly to have a chance for the refinement, the extension M has to add a new real, i.e.

$$(\mathcal{P}(\omega))^V \subsetneq (\mathcal{P}(\omega))^M.$$

Hence, from now on we will assume, that the extension M adds new reals. In fact we ask about the existence (of course in M) of a mapping

$$\varphi : ([\omega]^\omega)^V \to [\omega]^\omega$$

such that for each $x \neq y,\ x, y \in ([\omega]^\omega)^V$

(i) $\varphi(x) \subset x$ and

(ii) $\varphi(x) \cap \varphi(y) =^* \emptyset$.

First we show, that the embedding $(\mathcal{P}(\omega))^V \subsetneq (\mathcal{P}(\omega))^M$ is far from regular.

Lemma 7. *There is $\sigma \subset \omega$, $\sigma \in M$ such that for each $X \in [\omega]^\omega \cap V$ there is a $Y \in [X]^\omega \cap V$ such that $Y \cap \sigma = \emptyset$.*

Proof. Instead of ω one can consider a countable set

$$A = \bigcup\{\ {}^n\{0,1\}\ :\ n \in \omega\}.$$

Let χ be the characteristic function of a new real. Define $\sigma = \{\chi \restriction n : n \in \omega\}$, note that σ is set of compatible functions. Then σ has desired properties:

Let $X \subset A$, $X \in V$ be infinite. From the Ramsey theorem it follows that X contains either infinite subset Y of compatible functions or it contains infinite subset Y of pairwise disjoint functions. In the latter case clearly $|Y \cap \sigma| \leq 1$. Now suppose that Y is set of compatible functions and $Y \cap \sigma$ is infinite. Then $\bigcup Y = \chi$, but $\bigcup Y \in V$ and $\chi \notin V$, a contradiction. Hence $Y \cap \sigma =^* \emptyset$ and we are done. □

This yields a list of straight forward corollaries.

Corollary 8. (i) Let V be a model of ZFC and M its extension that adds new reals. Then $(\mathcal{P}(\omega)/\mathrm{fin})^V$ is not a regular subalgebra of the Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ in M; σ from the Lemma 7 has no pseudoprojection.

(ii) In any ZFC extension M of V adding a new real there is $\sigma \subset \omega$, $\sigma \in M$ such that σ does not contain infinite ground model set.

(iii) In any ZFC extension M of V adding a new real $(\mathcal{P}(\omega)/fin)^V$ is not a regular subalgebra of $\mathcal{P}(\omega)/fin$, i.e. there is a MAD family in $(\mathcal{P}(\omega)/fin)^V$ which is no longer MAD in M; cf. Lemma 3.

(iv) If there is a $H \subset [\omega]^\omega$ dense in $(\mathcal{P}(\omega)/fin)^M$ such that $H \subset V$. Then $\mathcal{P}(\omega) = \mathcal{P}^V(\omega)$.

The following theorem gives an affirmative answer to L. Soukup's question.

Theorem 1. *In any ZFC extension M of V adding a new real there is an almost disjoint refinement of $([\omega]^\omega)^V$.*

Proof. From Corollary 8 we already know, that the Boolean algebra $(\mathcal{P}(\omega)/fin)^V$ is not regular in $\mathcal{P}(\omega)/fin$. Hence by the definition, there is some MAD family in $(\mathcal{P}(\omega)/fin)^V$, which is no longer MAD in the extension.

Let $(T, \supseteq^*) \subset [\omega]^\omega$ be a base tree for $[\omega]^\omega$, in groundmodel V; and let $\mathcal{A} \in V$ be a destructible MAD family with its 'destructor' $\sigma \in [\omega]^\omega$, $\sigma \in M$. We denote T_α the α-level of the tree T.

By recursion we construct a base tree $T^* \in V$ for $[\omega]^\omega \cap V$. We start with the root $t \in T_0$ of the tree T and leave it untouched. The set t is an infinite subset of ω, take arbitrary bijection $b : t \to \omega$ in V. So $b^{-1}[\mathcal{A}]$ is a destructible MAD family on t with destructor $b^{-1}(\sigma) \in M$. There is a common refinement of the MAD families $b^{-1}[\mathcal{A}]$ and T_1. This common refinement will be the next level T_1^* of the constructed tree T^*.

Let T_α^* level be constructed. For every $t \in T_\alpha^*$ pick a bijection $b_t : t \to \omega$. The $T_{\alpha+1}^*$ level will be the common refinement of $T_{\alpha+1}$ and the maximal almost disjoint family

$$\{b_t^{-1}[A] \ : \ t \in T_\alpha^*, \ A \in \mathcal{A}\}.$$

On the limit stages $\gamma < \mathfrak{h}$. Take T_γ^* common refinement of the T_α^* for each $\alpha \leq \gamma$. Such refinement exists by the definition of $\mathfrak{h}$.

The tree $T^* \in V$ is clearly a base tree for $([\omega]^\omega)^V$. Moreover, for each $t \in T^*$ we found a subset $b_t^{-1}(\sigma) \in M$. Note that each $b_t^{-1}(\sigma)$ is almost disjoint with every $s \in T_\beta^*$ for each $\beta > \alpha$. Hence, for each $t \neq s$, $b_s^{-1}(\sigma)$ is almost disjoint from $b_t^{-1}(\sigma)$ and

$$\{b_t^{-1}(\sigma) \ : \ t \in T^*\}$$

is an almost disjoint refinement of $([\omega]^\omega)^V$, which completes the proof. □

4 Regularisation ideal for $\mathcal{P}(\omega)/fin$

From the previous paragraphs we know, that for arbitrary ZFC extension M, there is a minimal ideal such that the embedding $(\mathcal{P}(\omega)/fin) \hookrightarrow (\mathcal{P}(\omega)/fin)^M/\mathcal{I}$ is regular. We are able to describe a regularisation ideal only in the case of generic extension rather then an arbitrary one. i.e. the minimal ideal $\mathcal{I}_{\min}$ such that the embedding

$$(\mathcal{P}(\omega)/fin)^V \hookrightarrow (\mathcal{P}(\omega)/fin)^{V(\mathbb{B})}/\mathcal{I}_{\min}$$

is regular. To describe $I_{\min}$ we introduce Order sequential topology on Boolean algebras.

Topological Intermezzo

In order to equip a Boolean algebra with a topological structure that agrees with the Boolean operations we start with a *convergence structure*. It is enough to determine which sequences converge to $\mathbf{0}$ because using the symmetrical difference operation we can move convergent sequences to an arbitrary element $a \in \mathbb{B}$. It is natural to use the usual notion of a limit; i.e. $\lim a_n = \mathbf{0}$ if and only if

$$\limsup a_n = \bigwedge_n \bigvee_{k \geq n} a_k = \mathbf{0} = \bigvee_n \bigwedge_{k \geq n} a_k = \liminf a_n.$$

It is clear that right-hand side of the previous formula is redundant and one can define *the order convergence structure* on Boolean algebra $\mathbb{B}$ as the following ideal

$$Os(\mathbb{B}) = \{f \in \mathbb{B}^\omega \ : \ \limsup f = \mathbf{0}\}.$$

Note that it follows directly from the definition that $f \in Os(\mathbb{B})$ if and only if there is $g \in \mathbb{B}^\omega$ so that $g \searrow \mathbf{0}$ and $f \leq g$.

The order convergence structure $Os(\mathbb{B})$ determines the *Order sequential topology* τ_s on the Boolean algebra: The set $A \subset \mathbb{B}$ is *τ_s-closed* if and only if

$$\forall f \in A^\omega \ f \text{ convergent sequence}, \ \lim f \in A.$$

$(\mathbb{B}, \tau_s)$ is generally a T_1 topological space. The τ_s topology allows us to define an ideal; *Urysohn closure* of $Os(\mathbb{B})$

$$U(Os(\mathbb{B})) = \{f \in \mathbb{B}^\omega : f \xrightarrow{\tau_s} \mathbf{0}\}.$$

There is an obvious relation between algebraic and topological convergence.

Proposition 9. *A sequence $\langle x_n \rangle$ converges to x in the topology τ_s, $x_n \xrightarrow{\tau_s} \mathbf{0}$, if and only if any subsequence of $\langle x_n \rangle$ has a subsequence that converges to $\mathbf{0}$ algebraically.*

The definition of the topological structure sketched here works well only in case the Boolean algebra in question is complete (or at least σ-complete). In general, the assumption on σ-completeness of $\mathbb{B}$ is not necessary. We give a general definition here; for more details see [Vla69], [BFH99], [BJP05] or [Paz07].

Definition 10. Let $\mathbb{B}$ be an arbitrary Boolean algebra,

$$Os(\mathbb{B}) \ = \ \{f \in \mathbb{B}^\omega : \exists \mathcal{A} \subset \mathbb{B} \text{ a maximal countable antichain such that } f \perp \mathcal{A}\},$$

where $f \perp \mathcal{A}$ means that the set $\{n \in \omega : f(n) \wedge a \neq \mathbf{0}\}$ is finite for every $a \in \mathcal{A}$.

The structure with piece-wise Boolean operation is again Boolean algebra; one also can look at $\mathbb{B}^\omega$ as a set of $\mathbb{B}$-names for subsets of ω in the forcing extension by $\mathbb{B}$. From this point of view, the ideal $Os(\mathbb{B})$ consist of names for finite subsets of ω.

Proposition 11. *Let $\mathbb{B}$ be a complete Boolean algebra. Then for any generic G on $\mathbb{B}$*

$$Os^G(\mathbb{B}) \ = \ \{f_G \ : \ f \in Os(\mathbb{B})\} \ = \ \mathit{fin} \ = \ [\omega]^{<\omega},$$

where $f_G = \{n \in \omega : f(n) \in G\}$.

Proof. Let $f \in Os$ and suppose on contrary that f_G is an infinite set for some generic G. Since $f \in Os$, there exists $g \searrow \mathbf{0}$ such that $f \leq g$. Clearly if $f(n) \in G$ then $g(n) \in G$. Since g is monotone and f_G is infinite, we have $g(n) \in G$ for every $n \in \omega$. This is a contradiction since $\mathbf{0} = \bigwedge\{g(n) : n \in \omega\} \in G$.

On the other hand, suppose that $f \notin Os$ and set $d = \overline{\lim} f > \mathbf{0}$. Choose a generic filter G such that $d \in G$. Clearly, $\forall k \in \omega \ d \leq \bigvee\{f(n) : n > k\}$, which means that $\forall k \in \omega \ \exists m > k \ f(m) \in G$; hence the set f_G is infinite. □

Computing a regularization ideal for $\mathcal{P}(\omega)/fin$

Now we are ready to show that the minimal regularisation ideal $\mathcal{I}_{\min}$ for the canonical embedding of Boolean algebra $(\mathcal{P}(\omega)/\mathit{fin})^V$ into $(\mathcal{P}(\omega)/\mathit{fin})^{V(\mathbb{B})}$ is given by the evaluation of names from $U(Os(\mathbb{B}))$.

Theorem 12. *Let $\mathbb{B}$ be a complete Boolean algebra and let G be a generic in $\mathbb{B}$ over V. Then*

$$\mathcal{I}_{\min} \ = \ U(Os)^G.$$

Proof. Let $f \in V$ be such that $f_G = \rho \subset \omega$ destroys a MAD $\mathcal{A} \in V$. Find a name $g \in U(Os)$ for the set ρ. Suppose $f \notin U(Os)$; i.e. there is a $X \subset \omega$ infinite such that $f \restriction Y \notin Os$ for each $Y \in [X]^\omega$. Let

$$\mathfrak{X} \ = \ \{ \ X \in [\omega]^\omega \ : \ \forall Y \in [X]^\omega \ f \restriction Y \notin Os \ \}.$$

For $X \in \mathfrak{X}$ there is an $A \in \mathcal{A}$ such that $X \cap A$ is infinite; denote this infinite intersection $Y_X = X \cap A$. Since $X \in \mathfrak{X}$, $f \restriction Y_X \notin Os$; i.e. $\overline{\lim}_{Y_X} f \notin G$. Otherwise if

$$\bigwedge_{k \in \omega} \ \bigvee_{k \leq n \in Y_X} \ f(n) \ \in \ G,$$

then $\bigvee_{k \leq n \in Y_X} \ f(n) \ \in \ G$ for each $k \in \omega$ and the set $f_G \cap (A \cap X)$ is infinite, which contradicts the fact that f_G destroys $\mathcal{A}$. Now, put

$$c \ = \ \bigvee_{X \in \mathfrak{X}} \ \overline{\lim}_{n \in Y_X} \ f(n) \ \notin G,$$

and $g(n) = f(n) - c$; clearly $g_G = f_G = \rho$ and $g \in U(Os)$.

Let $f \in U(Os) \setminus Os$ i.e. for every infinite X there is a $Y_X \in [X]^\omega$ such that $f \restriction Y_X \in Os$. The family

$$\mathcal{F} \ = \ \{ \ Y_X \ : \ X \in [\omega]^\omega \ \}$$

is then dense in $\mathcal{P}(\omega)/fin$. Now, pick an arbitrary MAD family $\mathcal{A} \subset \mathcal{F}$. Clearly, f_G is an infinite set ($f \notin Os$) and destroys the MAD $\mathcal{A}$. □

This result together with Corollary 8 yields the following equivalence. This equivalence was achieved independently by M. S. Kurilić and A. Pavlović.

Corollary. [KP07] For a complete Boolean algebra $\mathbb{B}$ the following are equivalent

(i) $U(Os(\mathbb{B})) \ = \ Os(\mathbb{B})$,

(ii) there are no $V^{\mathbb{B}}$-destructible MAD in V,

(iii) the algebra $\mathbb{B}$ as a forcing notion does not add new reals.

In a special case when there are no independent reals in the extension M there is even a unique *largest* regularisation ideal (cf. Proposition 4) with simple and straightforward description. We say that $A \subset \mathcal{P}^M(\omega)$ is independent real if for every $X \in [\omega]^\omega \cap V$ are both sets $A \cap X$ and $X \setminus A$ infinite.

Definition 13. Let H be the family of subsets of ω that do not contain infinite sets from the ground model

$$H = \{\sigma \in M : \sigma \subset \omega \quad \& \quad \neg \, \exists \, a \in ([\omega]^\omega)^V \ a \subset \sigma\}.$$

Theorem 14. *The following holds in* M.

(i) H *is an open dense subset of* $([\omega]^\omega, \subseteq)$.

(ii) H *is an ideal if and only if* M *does not add independent reals.*

Proof. First note that if M adds a new real $\chi \subset \omega$, $\chi \notin V$, then H contains infinite set. It is easy to see, that σ given by lemma 7 is an infinite set belonging to H.

To prove *(i)*, let $A \in ([\omega]^\omega)^V$. Then there is a bijection f in V between ω and A and by the previous proposition 3 there is a subset $\sigma \subset \omega$ in M which does not contain an infinite ground model set, so $f[\sigma] \in H$ is subset of A. Generally, if $A \in [\omega]^\omega$ then $A \in H$ or there is an $A' \in ([\omega]^\omega)^V$, $A' \subset A$ and we can use the same reasoning.

(ii) Suppose that M adds an independent real σ. Clearly $\sigma \in H$ and $-\sigma \in H$, hence H is not an ideal.

On the other hand if H is not an ideal, then there are $a, b \in H$ so that there is an $X \in ([\omega]^\omega)^V$ and $X \subset a \cup b$. Again, we can identify X and ω in ground model and then $X \cap a$ is an independent real in M. □

It is clear, that whenever H is an ideal, then it is the unique regularisation ideal; cf. Proposition 6.

Proposition 15. *Let M be a ZFC extension of V adding new reals, then M does not add independent reals if and only if there is unique ideal H such that the canonical embedding $\mathcal{P}(\omega)/fin \hookrightarrow \mathcal{P}(\omega)/H$ is regular.*

Proof. Direct consequence of Propositions 4 and 6. □

5 Regularisation ideal for $\mathbb{B}^\omega/Fin$

In this final part we assume that Boolean algebras are at least σ-complete. This assumption is necessary but since our motivation comes from forcing it is not too restrictive.

The canonical embedding

$$\begin{array}{rcl} e:\mathbb{B} & \longrightarrow & \mathbb{B}^\omega \\ b & \longmapsto & \langle b : n \in \omega\rangle \end{array}$$

is obviously regular. The more interesting situation is the derived embedding $\hat{e} : \mathbb{B} \hookrightarrow \mathbb{B}/\mathrm{Fin}$, where $\mathrm{Fin} = \{f \in \mathbb{B}^\omega : |\{n : f(n) \neq \mathbf{0}\}| < \omega\}$. This embedding is not regular since image of maximal countable antichain $\langle a_n : n \in \omega\rangle \subset \mathbb{B}$ is not maximal in $\mathbb{B}^\omega/\mathrm{Fin}$. It is enough to put $f = \langle a_n : n \in \omega\rangle \in (\mathbb{B}^\omega \setminus \mathrm{Fin})$ and we get $f \wedge e(a_n) \in \mathrm{Fin}$ for every $n \in \omega$. Note that by our assumption that $\mathbb{B}$ is σ-complete, there are countable maximal antichains in $\mathbb{B}$.

It is natural to ask what is the minimal regularisation ideal $\mathcal{I}_{\min}$ for this situation and how algebra $\mathbb{B}^\omega/\mathcal{I}_{\min}$ behaves from the forcing point of view.

Proposition 16. *The canonical embedding of σ-complete Boolean algebra $\mathbb{B}$ into $\mathbb{B}^\omega/Os(\mathbb{B})$ is regular. Moreover, whenever the canonical embedding $\mathbb{B} \hookrightarrow \mathbb{B}^\omega/\mathcal{I}$ is regular for some ideal $\mathcal{I} \supset Fin$, then $Os(\mathbb{B}) \subset \mathcal{I}$.*

Proof. Let $f \in \mathbb{B}^\omega - Os$ then $d = \overline{\lim} f > \mathbf{0}$ is the required pseudoprojection witnessing the fact that the embedding $\mathbb{B} \hookrightarrow \mathbb{B}^\omega/Os$ is regular.

Computing I_{min} from the Theorem 2 we obtain that

$$I_{min} = \{f \in \mathbb{B}^\omega : \exists \text{ max. antichain } \mathcal{A} \text{ in } \mathbb{B} \text{ such that } f \perp \mathcal{A}\}.$$

It is clear from the definition that $Os \subset I_{min}$, which completes the proof. □

We conclude with the forcing description of algebra $\mathbb{B}^\omega/\mathcal{I}$, where $\mathcal{I}$ is regularisation ideal.

Theorem 17. *Let $\mathbb{B}$ be a complete Boolean algebra and Fin $\subset \mathcal{I} \subset \mathbb{B}^\omega$ an ideal for which the canonical embedding $\mathbb{B} \hookrightarrow \mathbb{B}^\omega/\mathcal{I}$ is regular, then $\mathbb{B}^\omega/\mathcal{I})$ is isomorphic with an iteration of $\mathbb{B}$ and $P(\omega)/\mathcal{I}^G$, where G is the generic filter on $\mathbb{B}$; i.e.*

$$\mathbb{B}^\omega/\mathcal{I} \quad \cong \quad \mathbb{B} \star (\mathcal{P}(\omega)/\mathcal{I}^G).$$

Proof. We define

$$\begin{aligned} \varphi : \mathbb{B} \star \mathcal{P}(\omega)/\mathcal{I}^G &\longrightarrow \mathbb{B}^\omega/\mathcal{I} \\ (b, f) &\longmapsto e(b) \wedge f, \end{aligned}$$

where f is a $\mathbb{B}$-name for a subset of ω. Let us remind the ordering i.e.

$$(b, f) \leq (c, g) \text{ if and only if } b \leq c \;\&\; b \Vdash \text{“}[f]_\mathcal{I} \leq [g]_\mathcal{I}\text{”},$$

where $b \Vdash$ “$[f]_\mathcal{I} \leq [g]_\mathcal{I}$” means that $e(b) \wedge f \leq_\mathcal{I} e(b) \wedge g$.

It is a routine check to verify that φ preserves ordering, disjoint relation and that $\varphi[\mathbb{B}\star\mathcal{P}(\omega)/\mathcal{I}^G]$ is dense in $\mathbb{B}^\omega/\mathcal{I}$. □

The following result was originally proved by A. Kamburelis.

Corollary 18. If $\mathbb{B}$ is a complete Boolean algebra, then

$$\mathbb{B}^\omega/Os(\mathbb{B}) \;\cong\; \mathbb{B} \star (\mathcal{P}(\omega)^{V(\mathbb{B})}/\text{Fin}).$$

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