*Preprint, Institute of Mathematics, AS CR, Prague. 2008-7-14*

# DUAL SPACES OF LOCAL MORREY-TYPE SPACES

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Abstract. In this paper we have shown that associated and dual spaces of local Morrey-type spaces are "so called" complementary local Morrey-type spaces. Our method is based on characterization of multidimensional reverse Hardy inequalities.

## 1. Introduction

If $E$ is a nonempty measurable subset on $\mathbb{R}^n$ and $f$ is a measurable function on $E$, then we put

$$\|g\|_{L_p(E)} := \left(\int_E |f(y)|^p dy\right)^{\frac{1}{p}},\ 0 < p < +\infty,$$

$$\|f\|_{L_\infty(E)} := \sup\{\alpha : |\{y \in E : |f(y)| \geq \alpha\}| > 0\}.$$

If $I$ a nonempty measurable subset on $(0, +\infty)$ and $g$ is a measurable function on $I$, then we define $\|g\|_{L_p(I)}$ and $\|g\|_{L_\infty(I)}$ correspondingly.

By $A \lesssim B$ we mean that $A \leq cB$ with some positive constant $c$ independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ *are equivalent*.

We put

$$p' := \begin{cases} \frac{p}{1-p} & \text{if} \quad 0 < p < 1, \\ +\infty & \text{if} \quad p = 1, \\ \frac{p}{p-1} & \text{if} \quad 1 < p < +\infty, \\ 1 & \text{if} \quad p = +\infty, \end{cases}$$

and $1/(+\infty) = 0$, $0/0 = 0$, $0 \cdot (\pm\infty) = 0$.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x, r)$ be the open ball centered at $x$ of radius $r$ and ${}^{\complement}B(x, r) := \mathbb{R}^n \backslash B(x, r)$.

We recall definitions of local Morrey-type space and complementary local Morrey-type space.

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1991 *Mathematics Subject Classification.* 26D15, 46E30 .

*Key words and phrases.* Local Morrey-type spaces, complementary local Morrey-type spaces, associated spaces, dual spaces, multidimensional Hardy inequalities.

The research of A. Gogatishvili was partialy supported by the grant no. 201/05/2033 and 201/08/0383 of the Grant Agency of the Czech Republic, by the Institutional Research Plan no. AV0Z10190503 of AS CR. The research of R. Mustafayev was supported by a Post Doctoral Fellowship of INTAS (Grant 06-1000015-6385).

**Definition 1.1.** ([1]) Let $0 < p, \theta \leq \infty$ and let $w$ be a non-negative measurable function on $(0, \infty)$. We denote by $LM_{p\theta,\omega}$ the local Morrey-type spaces, the spaces of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{LM_{p\theta,\omega}} \equiv \|f\|_{LM_{p\theta,\omega}(\mathbb{R}^n)} = \big\|w(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\theta(0,\infty)}.$$

**Definition 1.2.** ([2]) Let $0 < p, \theta \leq \infty$ and let $w$ be a non-negative measurable function on $(0, \infty)$. We denote by ${}^{\complement}LM_{p\theta,\omega}$ the complementary local Morrey-type spaces, the spaces of all functions $f \in L_p({}^{\complement}B(0,t))$ for all $t > 0$ with finite quasinorm

$$\|f\|_{{}^{\complement}LM_{p\theta,\omega}} \equiv \|f\|_{{}^{\complement}LM_{p\theta,\omega}(\mathbb{R}^n)} = \Big\|w(r)\|f\|_{L_p({}^{\complement}B(0,r))}\Big\|_{L_\theta(0,\infty)}.$$

**Definition 1.3.** Let $0 < p, \theta \leq \infty$. We denote by $\Omega_\theta$ the set all non-negative measurable functions $\omega$ on $(0, \infty)$ such that

$$\|\omega\|_{L_\theta(t,\infty)} < \infty, \ t > 0,$$

and by ${}^{\complement}\Omega_\theta$ the set all non-negative measurable functions $\omega$ on $(0, \infty)$ such that

$$\|\omega\|_{L_\theta(0,t)} < \infty, \ t > 0.$$

We calculated the associated spaces of local Morrey-type spaces. More precisely, we show that associated spaces of local Morrey-type spaces are complementary local Morrey-type spaces. Moreover, for some values of parameters these associated spaces are dual of local Morrey-type spaces.

## 2. Completeness of local Morrey-type spaces

The following Theorem is true.

**Theorem 2.1.** *Let $1 \leq p, \theta < \infty$, $\omega \in \Omega_\theta$. Suppose $f_n \in LM_{p\theta,\omega}$, $(n = 1, 2, \ldots)$ and*

$$\sum_{n=1}^{\infty} \|f_n\|_{LM_{p\theta,\omega}} < \infty. \tag{2.1}$$

*Then $\sum_{n=1}^{\infty} f_n$ converges in $LM_{p\theta,\omega}$ to a function $f$ in $LM_{p\theta,\omega}$ and*

$$\|f\|_{LM_{p\theta,\omega}} \leq \sum_{n=1}^{\infty} \|f_n\|_{LM_{p\theta,\omega}}. \tag{2.2}$$

*In particular, $LM_{p\theta,\omega}$ is complete.*

*Proof.* It is easy to see that for any $R > 0$

$$\|\omega\|_{L_\theta(R,\infty)} \|f\|_{L_p(B(0,R))} \leq \|f\|_{LM_{p\theta,\omega}}.$$

Thus

$$\sum_{n=1}^{\infty} \|f_n\|_{L_p(B(0,R))} \leq c \sum_{n=1}^{\infty} \|f_n\|_{LM_{p\theta,\omega}}.$$

Since $L_p(B(0,R))$ is complete, then $\sum_{n=1}^{\infty} f_n$ converges a.e. to some $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$

$$\sum_{n=1}^{\infty} f_n = f$$

and

$$\|f\|_{L_p(B(0,R))} \le \sum_{n=1}^{\infty} \|f_n\|_{L_p(B(0,R))}. \tag{2.3}$$

But then

$$\|f\|_{LM_{p\theta,\omega}} = \|\omega(r)\|f\|_{L_p(B(0,R))}\|_{L_\theta(0,\infty)} \le \|\omega(r)\sum_{n=1}^{\infty} \|f_n\|_{L_p(B(0,R))}\|_{L_\theta(0,\infty)}$$
$$\le \sum_{n=1}^{\infty} \|\omega(r)\|f_n\|_{L_p(B(0,R))}\|_{L_\theta(0,\infty)} = \sum_{n=1}^{\infty} \|f_n\|_{LM_{p\theta,\omega}}.$$

□

The following Theorem can be proved in analogous way.

**Theorem 2.2.** *Let* $1 \le p,\ \theta < \infty$, $\omega \in {}^{\complement}\Omega_\theta$. *Suppose* $f_n \in {}^{\complement}LM_{p\theta,\omega}$, $(n = 1, 2, \ldots)$ *and*

$$\sum_{n=1}^{\infty} \|f_n\|_{{}^{\complement}LM_{p\theta,\omega}} < \infty. \tag{2.4}$$

*Then* $\sum_{n=1}^{\infty} f_n$ *converges in* ${}^{\complement}LM_{p\theta,\omega}$ *to a function* $f$ *in* ${}^{\complement}LM_{p\theta,\omega}$ *and*

$$\|f\|_{{}^{\complement}LM_{p\theta,\omega}} \le \sum_{n=1}^{\infty} \|f_n\|_{{}^{\complement}LM_{p\theta,\omega}}. \tag{2.5}$$

*In particular,* ${}^{\complement}LM_{p\theta,\omega}$ *is complete.*

## 3. The multidimensional reverse Hardy inequality

Let us recall some results from [5].

**Theorem 3.1.** *Assume that* $0 < q \le p \le 1$. *Let* $\omega$ *and* $u$ *be a weight functions on* $\mathbb{R}^n$ *and* $(0,\infty)$ *respectively. Let* $\|u\|_{L_q(0,t)} < +\infty$ *for all* $t \in (0,\infty)$. *Then the inequality*

$$\|gw\|_{L_p(\mathbb{R}^n)} \le c \left\| u(t) \int_{{}^{\complement}B(0,t)} g(y)\, dy \right\|_{L_q(0,\infty)}. \tag{3.1}$$

*holds for all non-negative measurable* $g$ *if and only if*

$$A_1 := \sup_{t\in(0,\infty)} \|w\|_{L_{p'}(B(0,t))} \|u\|^{-1}_{L_q(0,t)} < +\infty.$$

*The best possible constant* $c$ *in* (3.1) *satisfies* $c \approx A_1$.

Consider now the inequality (3.1) in the case when $0 < p \le 1$, $p < q \le +\infty$ and define $r$ by

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q}. \tag{3.2}$$

In such a case we shall write a condition characterizing the validity of inequality (3.1) in a compact form involving $\int_{(0,\infty)} f\,dh$, where $f(t) = \|w\|^r_{L_{p'}(B(0,t))}$ and $h(t) = -\|u\|^{-r}_{L_q(0,t+)}$, $t \in (0,\infty)$. ($\|u\|_{L_q(0,t+)} := \lim_{s\to t+} \|u\|_{L_q(0,s)}$) (Hence, the Lebesgue-Stieltjes integral $\int_{(0,\infty)} f\,dh$ is defined by the non-decreasing and right-continuous function $h$ on $(0,\infty)$). However, it can happen that $\|u\|_{L_q(0,t+)} = 0$ for all $t \in (0,c)$ with a convenient $c \in (0,\infty)$ (provided that we omit the trivial case when $u = 0$ a.e. on $(0,\infty)$). Then we have to explain what is the meaning of the Lebesgue-Stieltjes integral since in such a case the function $h = -\infty$ on $(0,c)$. To this end, we adopt the following convention.

**Convention 3.2.** Let $I = (a,b) \subseteq \mathbb{R}$, $f : I \to [0,+\infty]$ and $h : I \to [-\infty,0]$. Assume that $h$ is non-decreasing and right-continuous on $I$. If $h : I \to (-\infty,0]$, then the symbol $\int_I f\,dh$ means the usual Lebesgue-Stieltjes integral. However, if $h = -\infty$ on some subinterval $(a,c)$ with $c \in I$, then we define $\int_I f\,dh$ only if $f = 0$ on $(a,c]$ and we put

$$\int_I f\,dh = \int_{(c,b)} f\,dh.$$

**Theorem 3.3.** *Assume that $0 < p \le 1$, $p < q \le +\infty$ and $r$ is given by (3.2). Let $\omega$ and $u$ be a weight functions on $\mathbb{R}^n$ and $(0,\infty)$ respectively. Let $u$ satisfy $\|u\|_{L_q(0,t)} < +\infty$ for all $t \in (0,\infty)$ and $u \ne 0$ a.e. on $(0,\infty)$. Then the inequality (3.1) holds for all non-negative measurable $g$ on $\mathbb{R}^n$ if and only if*

$$A_2 := \left( \int_{(0,\infty)} \|w\|^r_{L_{p'}(B(0,t))}\, d\left(-\|u\|^{-r}_{L_q(0,t+)}\right)\right)^{\frac{1}{r}} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|u\|_{L_q(0,\infty)}} < +\infty.$$

*The best possible constant $c$ in (3.1) satisfies $c \approx A_2$.*

**Remark 3.4.** Let $q < +\infty$ in Theorem 3.3. Then

$$\|u\|_{L_q(0,t+)} = \|u\|_{L_q(0,t)} \quad \text{for all} \quad t \in (0,\infty),$$

which implies that

$$A_2 = \left( \int_{(0,\infty)} \|w\|^r_{L_{p'}(B(0,t))}\, d\left(-\|u\|^{-r}_{L_q(0,t)}\right)\right)^{\frac{1}{r}} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|u\|_{L_q(0,\infty)}}.$$

Our next assertion is a counterpart of Theorem 3.1.

**Theorem 3.5.** *Assume that $0 < q \le p \le 1$. Let $\omega$ and $u$ be a weight functions on $\mathbb{R}^n$ and $(0,\infty)$ respectively. Let $\|u\|_{L_q(t,\infty)} < +\infty$ for all $t \in (0,\infty)$. Then the* *inequality*

$$\|gw\|_{L_p(\mathbb{R}^n)} \le c \left\| u(t) \int_{B(0,t)} g(y)\,dy \right\|_{L_q(0,\infty)} \tag{3.3}$$

*holds for all non-negative measurable $g$ on $\mathbb{R}^n$ if and only if*

$$B_1 := \sup_{t\in(0,\infty)} \|w\|_{L_{p'}({}^{\complement}B(0,t))} \|u\|^{-1}_{L_q(t,\infty)} < +\infty. \tag{3.4}$$

*The best possible constant $c$ in (3.3) satisfies $c \approx B_1$.*

Let us denote by $\|u\|^{-r}_{q,(t-,b),\nu} := \lim_{s\to t-} \|u\|^{-r}_{L_q[s,\infty)}$, $t \in (0,\infty)$. The following Theorem is true.

**Theorem 3.6.** *Assume that $0 < p \le 1$, $p < q \le +\infty$ and $r$ is given by (3.2). Let $\omega$ and $u$ be a weight functions on $\mathbb{R}^n$ and $(0,\infty)$ respectively. Let $u$ satisfy $\|u\|_{L_q(t,\infty)} < +\infty$ for all $t \in (0,\infty)$ and $u \ne 0$ a.e. on $(0,\infty)$. Then the inequality (3.3) holds for all non-negative measurable if and only if*

$$B_2 := \left( \int_{(0,\infty)} \|w\|^r_{L_{p'}({}^{\complement}B(0,t))}\, d\left( \|u\|^{-r}_{L_q(t-,\infty)} \right) \right)^{\frac{1}{r}} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|u\|_{L_q(0,\infty)}} < +\infty.$$

*The best possible constant $c$ in (3.3) satisfies $c \approx B_2$.*

**Remark 3.7.** Let $q < +\infty$ in Theorem 3.6. Then

$$\|u\|_{L_q(t-,\infty)} = \|u\|_{L_q(t,\infty)} \quad \text{for all} \quad t \in (0,\infty),$$

which implies that

$$B_2 = \left( \int_{(a,b)} \|w\|^r_{L_{p'}({}^{\complement}B(0,t))}\, d\left( \|u\|^{-r}_{L_q(t,\infty)} \right) \right)^{\frac{1}{r}} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|u\|_{L_q(0,\infty)}}.$$

## 4. Associated spaces of local Morrey-type and complementary local Morrey-type spaces

In this section by using results of previous section we calculate the associated spaces of local Morrey-type and complementary local Morrey-type spaces.

**Corollary 4.1.** *Assume $1 \le p < \infty$, $\theta \le 1$. Let $\omega \in {}^{\complement}\Omega_\theta$. Then*

$$\sup_{g \in {}^{\complement}LM_{p\theta,\omega}} \frac{\int_{\mathbb{R}^n} f(x)g(x)dx}{\|g\|_{{}^{\complement}LM_{p\theta,\omega}}} \approx \sup_{t\in(0,\infty)} \frac{\|f\|_{L_{p'}(B(0,t))}}{\|\omega\|_{L_\theta(0,t)}}$$

*Proof.* Since $\frac{\theta}{p} \le \frac{1}{p} \le 1$, then by Theorem 3.1 the inequality

$$\|g^p f^p\|_{L_{\frac{1}{p}}(\mathbb{R}^n)} \le c \left\| \omega^p(t) \int_{{}^{\complement}B(0,t)} g^p(y)\,dy \right\|_{L_{\frac{\theta}{p}}(0,\infty)} \tag{4.1}$$

holds for all non-negative measurable $g$ on $\mathbb{R}^n$ if and only if

$$C_1 := \sup_{t\in(0,\infty)} \|f^p\|_{L_{\left(\frac{1}{p}\right)'}(B(0,t))} \|\omega^p\|^{-1}_{L_{\frac{\theta}{p}}(0,t)} < +\infty$$

The best possible constant $c$ in (4.1) satisfies $c \approx C_1$. □

**Corollary 4.2.** *Assume* $1 \le p < \infty$, $1 < \theta \le \infty$. *Let* $\omega \in {}^{\complement}\Omega_\theta$. *Then*

$$\sup_{g\in {}^{\complement}LM_{p\theta,\omega}} \frac{\int_{\mathbb{R}^n} f(x)g(x)dx}{\|g\|_{{}^{\complement}LM_{p\theta,\omega}}}$$
$$\approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}(B(0,t))} d\left(-\|\omega\|^{-\theta'}_{L_\theta(0,t+)}\right)\right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}}.$$

From Theorem 3.5 and Theorem 3.6 we conclude next statements

**Corollary 4.3.** *Assume* $1 \le p < \infty$, $\theta \le 1$. *Let* $\omega \in \Omega_\theta$. *Then*

$$\sup_{g\in LM_{p\theta,\omega}} \frac{\int_{\mathbb{R}^n} f(x)g(x)dx}{\|g\|_{LM_{p\theta,\omega}}} \approx \sup_{t\in(0,\infty)} \frac{\|f\|_{L_{p'}({}^{\complement}B(0,t))}}{\|\omega\|_{L_\theta(t,\infty)}}.$$

**Corollary 4.4.** *Assume* $1 \le p < \infty$, $1 < \theta \le \infty$. *Let* $\omega \in \Omega_\theta$. *Then*

$$\sup_{g\in LM_{p\theta,\omega}} \frac{\int_{\mathbb{R}^n} f(x)g(x)dx}{\|g\|_{LM_{p\theta,\omega}}}$$
$$\approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}({}^{\complement}B(0,t))} d\|\omega\|^{-\theta'}_{L_\theta(t-,\infty)}\right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}} < +\infty.$$

Let $X$ be Banach space. Denote by $X'$ its associated space, that is,

$$\|f\|_{X'} = \sup\left\{\int_0^\infty f(t)g(t)dt : \|g\|_X \le 1\right\}.$$

Now we can characterize the associated spaces of local Morrey-type and complementary local Morrey-type spaces.

**Theorem 4.5.** *Assume* $1 \le p < \infty$, $0 < \theta \le \infty$. *Let* $\omega \in {}^{\complement}\Omega_\theta$. *Set* $X = {}^{\complement}LM_{p\theta,\omega}$.

(i) Let $0 < \theta \le 1$. Then

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}(B(0,t))} \|\omega\|^{-1}_{L_\theta(0,t)}.$$

(ii) Let $1 < \theta \le \infty$. Then

$$\|f\|_{X'} \approx \left(\int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}(B(0,t))} d\left(-\|\omega\|^{-\theta'}_{L_\theta(0,t+)}\right)\right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}}.$$

*Proof.* This is just a simple application of Corollary 4.1 and 4.2. □

**Theorem 4.6.** *Assume $1 \le p < \infty$, $0 < \theta \le \infty$. Let $\omega \in \Omega_\theta$. Set $X = LM_{p\theta,\omega}$.*

(i) Let $0 < \theta \le 1$. Then

$$\|f\|_{X'} \approx \sup_{t\in(0,\infty)} \|f\|_{L_{p'}(^{\complement}B(0,t))} \|\omega\|^{-1}_{L_\theta(t,\infty)}.$$

(ii) Let $1 < \theta \le \infty$. Then

$$\|f\|_{X'} \approx \left( \int_{(0,\infty)} \|f\|^{\theta'}_{L_{p'}(^{\complement}B(0,t))} d\|\omega\|^{-\theta'}_{L_\theta(t-,\infty)} \right)^{\frac{1}{\theta'}} + \frac{\|f\|_{L_{p'}(\mathbb{R}^n)}}{\|\omega\|_{L_\theta(0,\infty)}}.$$

*Proof.* This is just a simple application of Corollary 4.3 and 4.4. □

## 5. Dual spaces of local Morrey-type and complementary local Morrey-type spaces

In this section we calculate dual spaces of local Morrey-type and complementary local Morrey-type spaces. More precisely, we show that for some values of parameters the dual spaces coincide with the asssociated spaces.

The following theorem is true.

**Theorem 5.1.** *Assume $1 \le p < \infty$, $1 < \theta < \infty$. Let $\omega \in \Omega_\theta$ and $\|\omega\|_{L_\theta(0,\infty)} = \infty$. Then*

$$(LM_{p\theta,\omega})^* = {^{\complement}LM_{p'\theta',\widetilde{\omega}}}\,, \tag{5.1}$$

*where $\widetilde{\omega}(t) = \omega^{\theta-1}(t)\left(\int_t^\infty \omega^\theta(s)ds\right)^{-1}$, under the following pairing:*

$$< f, g >= \int_{\mathbb{R}^n} fg.$$

*Moreover $\|f\|_{^{\complement}LM_{p'\theta',\widetilde{\omega}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, where the supremum is taken over all functions $g \in LM_{p\theta,\omega}$ with $\|g\|_{LM_{p\theta,\omega}} \le 1$.*

*Proof.* If $f \in {^{\complement}LM_{p'\theta',\widetilde{\omega}}}$ and $g \in LM_{p\theta,\omega}$, then by Corollary 4.4 we have

$$| < f, g > | \le \int_{\mathbb{R}^n} |fg| \le \|f\|_{^{\complement}LM_{p'\theta',\widetilde{\omega}}} \|g\|_{LM_{p\theta,\omega}}.$$

In particular, every function $f \in {^{\complement}LM_{p'\theta',\widetilde{\omega}}}$ induces a bounded linear functional on $LM_{p\theta,\omega}$.

Conversely, suppose $L$ is a bounded linear functional on $LM_{p\theta,\omega}$ with the norm $\|L\| < \infty$. If $g$ is supported in $D_0 = {^{\complement}B(0, r_0)}$ for some $r_0 > 0$, then

$$\|g\|_{LM_{p\theta,\omega}} \le \|\omega\|_{L_\theta(r_0,\infty)} \|g\|_{L_p(^{\complement}B(0,r_0))},$$

and

$$|L(g)| \le \|L\| \|\omega\|_{L_\theta(r_0,\infty)} \|g\|_{L_p(^{\complement}B(0,r_0))}.$$

Hence $L$ induces a bounded linear functional on $L_p(^{\complement}B(0, r_0))$ and acts with some function $f^0 \in L_{p'}(^{\complement}B(0, r_0))$. By taking $D_j = {^{\complement}B(0, r_0/j)}$, $j = 1, 2, 3, \ldots$, we have

$f^j = f^{j+1}$ on $D_j$, so we get a single function $f$ on $\mathbb{R}^n$ that $f \in L_{p'}({}^{\complement}B(0,r))$ for any $r > 0$, and such that $L(g) = \int_{\mathbb{R}^n} fg$ when $g \in L_p({}^{\complement}B(0,t))$ with support in ${}^{\complement}B(0,t)$ for any $t > 0$.

For arbitrary $r > 0$, take $g = \chi_{{}^{\complement}B(0,r)} |f|^{p'} f^{-1}$, then

$$\int_{{}^{\complement}B(0,r)} |f|^{p'} = |L(g)| \le \|L\| \|\omega\|_{L_\theta(r,\infty)} \left( \int_{{}^{\complement}B(0,r)} |f|^{p'} \right)^{\frac{1}{p}},$$

thus

$$\left( \int_{{}^{\complement}B(0,r)} |f|^{p'} \right)^{\frac{1}{p'}} \le \|L\| \|\omega\|_{L_\theta(r,\infty)},$$

hence

$$\left( \int_r^\infty \widetilde{\omega}(t)^{\theta'} \left( \int_{{}^{\complement}B(0,t)} |f|^{p'} \right)^{\frac{\theta'}{p'}} dt \right)^{\frac{1}{\theta'}} \le \left( \int_r^\infty \widetilde{\omega}(t)^{\theta'} dt \right)^{\frac{1}{\theta'}} \|L\| \|\omega\|_{L_\theta(r,\infty)} \le \|L\|.$$

Therefore, $f \in {}^{\complement}LM_{p'\theta',\widetilde{\omega}}$ with

$$\|f\|_{{}^{\complement}LM_{p'\theta',\widetilde{\omega}}} \le \|L\|.$$

For $g \in LM_{p\theta,\omega}$ and any $n \in \mathbb{N}$, denote by $g_n(x) = g(x)\chi_{B(0,n)\setminus B(0,\frac{1}{n})}$. It is evident that $g_n \to g$, $n \to \infty$ a.e in $\mathbb{R}^n$. By Lebesgue's Dominated Convergence Theorem, we get that $\|g - g_n\|_{LM_{p\theta,\omega}} \to 0$, $n \to \infty$. Therefore

$$L(g_n) \to L(g), \ n \to \infty. \tag{5.2}$$

On the other hand, by Corollary 4.4

$$\left| \int fg - \int fg_n \right| \le \int |f(g - g_n)| \le \|f\|_{{}^{\complement}LM_{p\theta,\omega}} \|g - g_n\|_{LM_{p\theta,\omega}} \to 0. \tag{5.3}$$

Since $g_n \in L_p({}^{\complement}B(0,\frac{1}{n}))$, then

$$L(g_n) = \int fg_n.$$

Consequently, from (5.2) and (5.3) we obtain

$$L(g) = \int fg.$$

□

In a similar manner the following theorem is proved.

**Theorem 5.2.** *Assume* $1 \le p < \infty$, $1 < \theta < \infty$. *Let* $\omega \in {}^{\complement}\Omega_\theta$ *and* $\|\omega\|_{L_\theta(0,\infty)} = \infty$. *Then*

$$\left( {}^{\complement}LM_{p\theta,\omega} \right)^* = LM_{p'\theta',\overline{\omega}}, \tag{5.4}$$

*where* $\overline{\omega}(t) = \omega^{\theta-1}(t)\left(\int_0^t \omega^\theta(s)ds\right)^{-1}$, *under the following pairing:*

$$< f, g >= \int_{\mathbb{R}^n} fg.$$

*Moreover* $\|f\|_{LM_{p'\theta',\overline{\omega}}} = \sup_g \left|\int_{\mathbb{R}^n} fg\right|$, *where the supremum is taken over all functions* $g \in {}^{\complement}LM_{p\theta,\omega} : \|g\|_{{}^{\complement}LM_{p\theta,\omega}} \le 1$.

**Acknowledgement.** Rza Mustafayev would like to express his gratitude to INTAS for the possibility to visit the Institute of Mathematics of Academy of Sciences of Czech Republic during 2007/8, under his INTAS YSF Post Doctoral programme and also the Institute of Mathematics of Academy of Sciences of Czech Republic for its hospitality.

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