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AN E_∞-EXTENSION OF THE ASSOCIAHEDRA AND THE TAMARKIN CELL MYSTERY

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ABSTRACT. In this note based on the author's communication with **M. Batanin**, we study a cofibrant E_∞-operad generated by the Fox-Neuwirth cells of the configuration space of points in the Euclidean space. We show that, below the 'critical dimensions' in which 'bad cells' exist, this operad is modeled by the geometry of the Fulton-MacPherson compactification of this configuration space. We analyze the Tamarkin bad cell and calculate the differential of the corresponding generator. We also describe a simpler, four-dimensional bad cell. We finish the paper by proving an auxiliary result giving a characterization, over integers, of free Lie algebras.

CONTENTS

1. INTRODUCTION

All algebraic objects will be considered over the ring of integers $\mathbb{Z}$. Terminology used in this introduction follows [19].

We describe an operad J that can be viewed as an E_∞-analog of the minimal model $\mathcal{A}_\infty$ of the operad $\mathcal{A}ss$ for associative algebras, governing Stasheff's A_∞-algebras [16]. Our J lives in the monoidal category of differential graded abelian groups. It is of the form $J = (\mathbb{F}(G), \partial)$ where

- $\mathbb{F}(G)$ is the free operad generated by the graded Σ-module $G = \{G_*(n)\}_{n\geq 2}$ specified in Definition 3, and

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- the differential ∂ is the sum $\partial = \partial_{\rm lin} + \partial_{\rm prt}$ of the 'linear' part $\partial_{\rm lin}$ induced from a differential (denoted by the same symbol) $\partial_{\rm lin}$ on the Σ-module G introduced in Definition 5, and the 'perturbed' part $\partial_{\rm prt}$ that maps G into the decomposables of the free operad $\mathbb{F}(G)$. Moreover,

- the operad J is equipped with a dg-operad homomorphism $\rho : J \to \mathcal{C}om$ that makes J a cofibrant resolution of the operad $\mathcal{C}om = \mathcal{E}nd_{\mathbb{Z}}$ for commutative associative algebras.

The nth piece $(G_*(n), \partial_{\rm lin})$ of the generating Σ-module G is, for each $n \geq 2$, the colimit of the (shifted) cellular chain complexes of the one-point compactifications of the configuration spaces $\mathrm{Cnf}(\mathbb{R}^H, o)n$ of n distinct labeled points in the h-dimensional Euclidean space $\mathbb{R}^h$, with the Fox-Neuwirth cell structure. Alternative descriptions of the right Σ_n-dg-abelian group $(G_*(n), \partial_{\rm lin})$ are given in Section 2. The existence of the operad $J = (\mathbb{F}(G), \partial)$ with the properties formulated above follows from

- Theorem 34 of Section 6 which states that $(G_*(n), \partial_{\rm lin})$ is, for each $n \geq 2$, a $\mathbb{Z}[\Sigma_n]$-free resolution of the (shifted) Σ_n-module $\mathrm{sgn}_n \otimes \mathcal{L}ie(n)'$, where sgn_n is the signum representation and $\mathcal{L}ie(n)'$ the linear dual of the nth piece of the operad $\mathcal{L}ie$ for Lie algebras,

- Theorem 6.7, Fact 6.2 and Proposition 5.2.12 of [9] which imply that the cobar construction on the linear dual of the Lie operad is quasi-isomorphic, over $\mathbb{Z}$, to the operad $\mathcal{C}om$ and, finally,

- Propositions 5.2.13 and 3.2.6 of [9] resp. Lemma 6 below that imply, given the results mentioned in the above two items, the existence of the perturbation $\partial_{\rm prt}$,

see Subsection 2.1 for details. Theorem 34 seems to be generally assumed, but we were unable to find a suitable reference. All results of the standard citations [6, 7] related to Lie algebras require coefficients in a field.

In Section 3 we show that, below the 'critical dimension' in which the 'bad' cells exist (see Definition 7), J is determined by the cell structure of the configuration operad F induced from the Fox-Neuwirth decomposition of the configuration space. In Section 4 we analyze two particular bad cells and calculate the differential of the corresponding generators of G. The first one is the famous 6-dimensional Tamarkin cell, the second is a simpler 4-dimensional bad whose existence was a surprise for us. Properties of general bad cells are analyzed in Section 5.

Since the image of the canonical embedding $\mathsf{K} \hookrightarrow \mathsf{F}$ of the Stasheff's associahedron K into the configuration operad F does not contain bad cells (see Figure 5), the operad J is an extension of the A_∞-operad $\mathcal{A}_\infty$ of cellular chains of K. This explains the title of the paper.

Let $\underline{G}_*(n) \subset G_*(n)$ be a graded abelian group[1] generating $G_*(n)$ as a free graded Σ_n-module – one such a specific generating space will be described on page 5. It follows from standard facts that $J(n)$ is the free Σ_n-module generated by the nth piece $\underline{\mathbb{F}}(\underline{G})(n)$ of the free non-Σ operad $\underline{\mathbb{F}}(\underline{G})$ generated by $\underline{G}$. In particular, $(J(n), \partial)$ is, for each $n \geq 1$, a Σ_n-free resolution of the trivial Σ_n-module $\mathbb{Z} = \mathcal{C}om(n)$.

[1]As in [19], underlining indicates the non-Σ version of an object.

Since J is cofibrant, for an arbitrary dg E_∞-operad $\mathcal{E}$ (as the Barratt-Eccles operad, surjection operad, Eilenberg-Zilber operad, see [4, 18], etc.), there exist an operadic morphism $J \to \mathcal{E}$ that lifts the identity endomorphism of the operad $\mathcal{C}om$. If the ground ring is a field of characteristic zero, J contains as its deformation retract the minimal model $\mathcal{C}_\infty$ of the operad $\mathcal{C}om$ [16] (operad $\mathcal{C}_\infty$ describes C_∞, also called commutative or balanced A_∞, algebras).

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Glossary of notation is given on page 35.

2. Trees, barcodes and the E_∞-operad $J = (\mathbb{F}(G), \partial)$

In this section we describe the graded Σ-module $G = \{G_*(n)\}_{n\geq 2}$ generating the operad J, together with the linear part ∂_{lin} of the differential, and prove the existence of a perturbation $\partial = \partial_{\mathrm{lin}} + \partial_{\mathrm{prt}}$ with the properties stated in the introduction. Let us start by recalling some definitions of [3]. We denote, as usual, by $[k]$ the ordered set $1 < 2 < \cdots < k$.

Definition 1. *Let $h \geq 1$. A* tree of height h *(or* tree with h levels*, or h*-tree*) is a sequence of order preserving maps*

$$T = [k_h] \xrightarrow{\rho_{h-1}} [k_{h-1}] \xrightarrow{\rho_{h-2}} \cdots \xrightarrow{\rho_0} [1]. \tag{1}$$

We are not going to consider degenerate trees, so we assume that all $k_m \geq 1$, for $0 \leq m \leq h$.

A *vertex* of height m, $0 \leq m \leq h$, is an element of $[k_m]$. One may imagine that each vertex $i \in [k_m]$, $m \geq 1$, determines the oriented *edge* that starts at i and ends at the vertex $\rho_{m-1}(i) \in [k_{m-1}]$. With this intuition, one may indeed interpret objects of Definition 1 as planar directed trees with vertices arranged at $h+1$ horizontal lines shown in Figure 1. A *leaf of height* m is a vertex $i \in [k_m]$ which is not in the image of ρ_m. A *tip* is a leaf of maximal height h. The *arity* of T is then the number of tips. The tree is *pruned* if all its leaves are tips. These definitions should be clear from Figure 1.

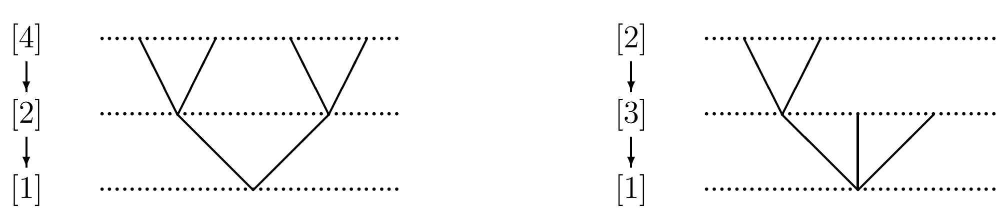


Figure 1. Example of trees of height two. The left tree is pruned, the right one is not. Arity of the left tree is 4, arity of the right one is 2.

We say that a tree T as in (1) has a *trunk* if $k_m = 1$ for some $m \geq 1$. A *trunk* of T is then everything 'below' k_m; see Figure 2.

FIGURE 2. The left tree has a trunk (bold edge) and is not pruned. The right tree is its maximal reduced subtree.

We say that a tree is *reduced*[2] if it is pruned and if it has no trunk. Obviously, for each T there exists a unique maximal reduced subtree $r(T)$ of maximal height. See again Figure 2 – the right tree is obtained from the left one by first cutting off the trunk and then pruning the remaining shrub. So pruning is for us cutting out branches that do not end in tips as opposed to what one does in garden, namely cutting of those that stick up too high. Finally, for a tree T as in (1) define its *dimension* $\dim(T)$ as

$$\dim(T) := e(T) - h - 1, \tag{2}$$

where $e(T)$ is the number of edges and h the height.

The *terminal tree* U_h is the tree with all $k_m = 1$. Terminal trees play a very special rôle and, unless stated otherwise, we will not consider them. If necessary, we set $\dim(U_h) := 0$ (formula (2) would give $\dim(U_h) = -1$). We also define $r(U_h) := U_h$.

Notation 2. Denote by $\underline{\mathrm{Tree}}^h(n) = \bigcup_{i \geq 0} \underline{\mathrm{Tree}}^h_i(n)$ the graded set whose ith component consists of pruned h-trees of dimension i, with n tips, $h, n \geq 1$. We also denote $\mathrm{Tree}^h(n) = \bigcup_{i \geq 0} \mathrm{Tree}^h_i(n)$ the graded set of *labeled* pruned trees of height h. Elements of $\mathrm{Tree}^h_i(n)$ are couples $\mathbf{T} = (T, \ell)$, where $T \in \underline{\mathrm{Tree}}^h_i(n)$ is as in (1) and ℓ an isomorphism (labeling) $\ell : [k_h] \xrightarrow{\cong} [n]$. The symmetric group Σ_n freely acts on $\mathrm{Tree}^h(n)$ by relabeling the tips.

One has the inclusion $\underline{\mathrm{Tree}}^h(n) \subset \mathrm{Tree}^h(n)$ given by numbering the legs of an unlabeled planar tree from the left to the right. The subset $\underline{\mathrm{Tree}}^h(n)$ freely generates $\mathrm{Tree}^h(n)$ as a right graded Σ_n-set.

There are the *suspensions* $\underline{s} : \underline{\mathrm{Tree}}^h(n) \hookrightarrow \underline{\mathrm{Tree}}^{h+1}(n)$ resp. $s : \mathrm{Tree}^h(n) \hookrightarrow \mathrm{Tree}^{h+1}(n)$ that adjoin to a (labeled) tree a trunk of hight one. The graded sets $\underline{\mathrm{Tree}}(n) := \varinjlim \underline{\mathrm{Tree}}^h(n)$ resp. $\mathrm{Tree}(n) := \varinjlim \mathrm{Tree}^h(n)$ clearly consist of (labeled) reduced trees of an arbitrary height.

The first step towards our definition of the operad J is:

Definition 3. *The nth component $G_*(n)$ of the graded Σ-module $G_* = \{G_*(n)\}_{n \geq 2}$ generating the operad J is the free graded abelian group* $\mathrm{Span}(\mathrm{Tree}_*(n))$ *spanned by the graded Σ_n-set* $\mathrm{Tree}(n)$ *of labeled reduced trees with n tips.*

[2]Our terminology is not a standard one – reduced usually means no vertices of arity 1.

Each $G_*(n)$ is clearly a free Σ_n-module Σ_n-generated by the graded abelian group $\underline{G}_*(n) =$ Span$(\underline{\text{Tree}}_*(n))$ spanned by (unlabeled) reduced trees. A complete list of unlabeled reduced trees up to dimension 3 is given in Figure 3; there are exactly 2^d reduced trees of dimension d.

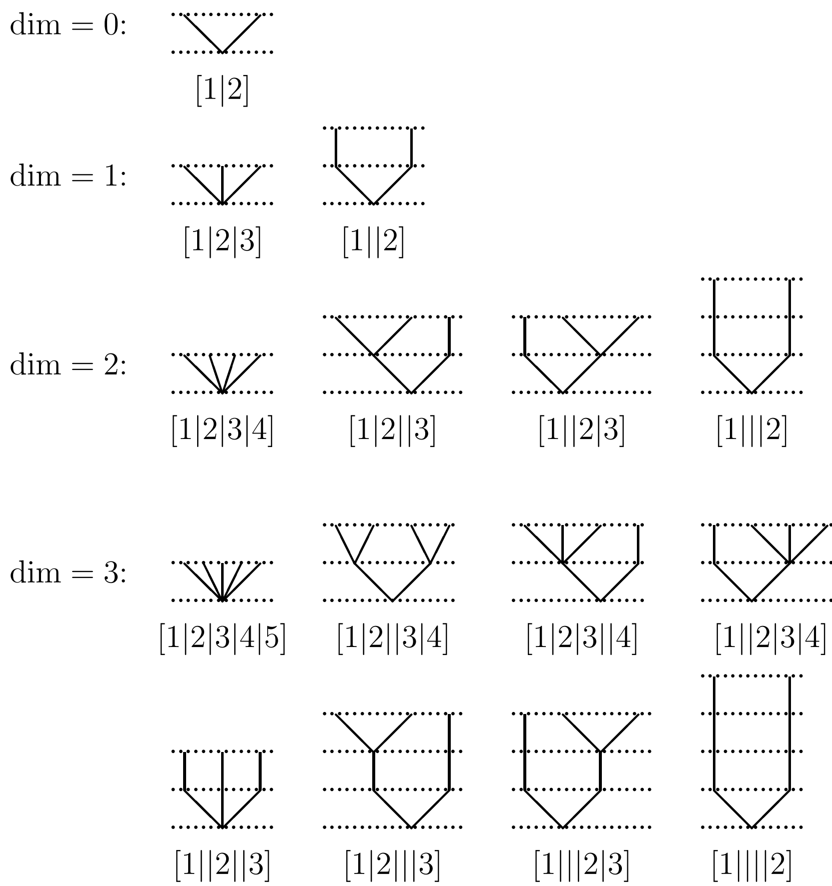


FIGURE 3. Complete list of reduced trees up to dimension 3 and their barcodes.

Observe that $G_*(n) = \varinjlim G^h_*(n)$, where $G^h_*(n) := \text{Span}(\text{Tree}^h_*(n))$. There is a convenient "barcode" notation for the reduced labeled trees (and therefore also the Fox-Neuwirth cells recalled in Section 3) introduced in [12]:

Definition 4. *The* barcode *of a reduced labeled tree is the list of labels of its tips, separated by the vertical bars whose number equals the depth of the gaps between the tips.*

Since the tips of an unlabeled tree can be labeled by $1, 2, \ldots$ in the increasing order from the left to the right, the barcodes can be used for unlabeled trees as well. See again Figure 3. The height of the corresponding reduced tree is the maximal number of the adjacent bars, and the dimension is the number of vertical bars minus 1.

The shortest way to describe the differential ∂_{lin} on the Σ-module $G = \{G(n)\}_{n\geq 2}$ is to identify this Σ-module to a suitable dg-submodule of an iterated bar construction. Let $\mathsf{B}(A)$ denote the bar construction of an associative algebra A, i.e. the tensor algebra $\mathbb{T}(\uparrow A)$ generated by the suspension $\uparrow A$ of the abelian group A, with the degree -1 differential ∂_{B} induced by the multiplication of A. It is classical [13, X.12] that, if A is commutative, the shuffle product of the tensor algebra makes $\mathsf{B}(A) = (\mathbb{T}(\uparrow A), \partial_{\mathsf{B}})$ a commutative associative algebra, thus the bar construction can be iterated.

Let us denote, for $h \geq 1$, by $\mathsf{B}^h(A)$ the h-th iterate of $\mathsf{B}(-)$ applied to A. Since the natural inclusion $\iota^h : \mathsf{B}^h_*(A) \hookrightarrow \mathsf{B}^{h+1}_*(A)$ is a degree $+1$ map, to have a natural grading on the direct limit we need to regrade by putting $\widehat{\mathsf{B}}^h_*(A) :=\downarrow^{h+1} \mathsf{B}^h_*(A)$. The induced inclusion $\widehat{\mathsf{B}}^h_*(A) \hookrightarrow \widehat{\mathsf{B}}^{h+1}_*(A)$ is degree-preserving so one may take $\widehat{\mathsf{B}}^\infty(A) = (\widehat{\mathsf{B}}^\infty(A), \partial^\infty_{\mathsf{B}})$, the direct limit $\varinjlim \widehat{\mathsf{B}}^h(A)$ with the induced differential.

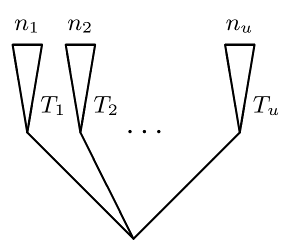
Consider the free abelian group V spanned by $x_1, \ldots, x_n$, interpreted as a commutative algebra with the trivial multiplication. Denote by $\widehat{\mathsf{B}}^h_{1,\ldots,1}(V)$ the sub-dg abelian group of $\widehat{\mathsf{B}}^h(V)$ spanned by monomials that contain each basic element $x_1, \ldots, x_n$ exactly once, with the obvious right Σ_n-action given by relabeling. Finally, let $\widehat{\mathsf{B}}^\infty_{1,\ldots,1}(V) := \varinjlim \widehat{\mathsf{B}}^h_{1,\ldots,1}(V)$.

As observed in [12] and, in more general setting, also in [10], the graded abelian group $\widehat{\mathsf{B}}^\infty_{1,\ldots,1}(V)$ is isomorphic to the graded abelian group $G_*(n)$ of Definition 3. The isomorphism $\omega : G(n) \xrightarrow{\cong} \widehat{\mathsf{B}}^\infty_{1,\ldots,1}(V)$ has the following inductive description.

Let $T \in \underline{\mathrm{Tree}}^h(n)$ and g_T the corresponding generator of $G(n)$. If $h = 1$, then T is the n-corolla $\star_n$, i.e. the 1-tree with the barcode $[1|\ldots|n]$. In this case we put

$$\omega(g_T) :=\downarrow^2 (\uparrow x_1 \otimes \cdots \otimes \uparrow x_n) \in \widehat{\mathsf{B}}^1_{1,\ldots,1}(V) \subset \widehat{\mathsf{B}}^\infty_{1,\ldots,1}(V).$$

Assume $\omega(g_S)$ has been defined for all $S \in \underline{\mathrm{Tree}}^k(n)$ with $1 \leq k < h$, and has the property that $\omega(g_S)$ actually belongs to the subspace $\widehat{\mathsf{B}}^k_{1,\ldots,1}(V) \subset \widehat{\mathsf{B}}^\infty_{1,\ldots,1}(V)$. Let $T \in \underline{\mathrm{Tree}}^h(n)$. There obviously exists some u, $1 \leq u \leq n$, such that T is obtained by grafting pruned, not necessarily reduced, $(h-1)$-trees $T_1, \ldots, T_u$ at the tips of the u-corolla $\star_u$:

$T =$ 

where $n_1, \ldots, n_u \geq 1$ with $n_1 + \cdots + n_u = n$ are the arities of the trees $T_1, \ldots, T_u$. For i, $1 \leq i \leq u$, we denote

$$V_i := \mathrm{Span}\{x_j;\ n_1 + \cdots + n_{i-1} + 1 \leq j \leq n_1 + \cdots + n_i\}.$$

We distinguish two cases.

(a) $n_i \geq 2$. Then let $R_i \in \underline{\mathrm{Tree}}^{k_i}(n_i)$ be the maximal reduced subtree of T_i. By induction, $\omega(g_{R_i}) \in \widehat{\mathsf{B}}^{k_i}_{1,\ldots,1}(V_i)$ is defined and we put $\omega_i \in \widehat{\mathsf{B}}^{h-1}_{1,\ldots,1}(V_i)$ the image of $\omega(g_{R_i})$ under the natural inclusion $\widehat{\mathsf{B}}^{k_i}_{1,\ldots,1}(V_i) \hookrightarrow \widehat{\mathsf{B}}^{h-1}_{1,\ldots,1}(V_i)$.

(b) $n_i = 1$. In this case, let $j := n_1 + \cdots + n_{i-1} + 1$ and define $\omega_i \in \widehat{\mathsf{B}}^{h-1}_{1,\ldots,1}(V_i)$ the image of $\downarrow^2 (\uparrow x_j) \in \widehat{\mathsf{B}}^{1}_{1,\ldots,1}(V_i)$ under the natural inclusion $\widehat{\mathsf{B}}^{1}_{1,\ldots,1}(V_i) \hookrightarrow \widehat{\mathsf{B}}^{h-1}_{1,\ldots,1}(V_i)$.

Observe that, in both cases, $\uparrow^{h+1} \omega_i \in \widehat{\mathsf{B}}^{h}_{1,\ldots,1}(V_i)$. Finally, let

$$\omega(g_T) := \downarrow^{h+1} \left(\uparrow^{h+1} \omega_1 \otimes \cdots \otimes \uparrow^{h+1} \omega_u\right) \in \widehat{\mathsf{B}}^{h}_{1,\ldots,1}(V) \subset \widehat{\mathsf{B}}^{\infty}_{1,\ldots,1}(V).$$

For example,

$$\omega(g_{\vee}) = \downarrow^2 (\uparrow x_1 \otimes \uparrow x_2 \otimes \uparrow x_3) \in \widehat{\mathsf{B}}^{1}_{1,\ldots,1}(V) \subset \widehat{\mathsf{B}}^{\infty}_{1,\ldots,1}(V),$$
$$\omega(g_{\vee}) = \downarrow^3 (\uparrow (\uparrow x_1) \otimes \uparrow (\uparrow x_2 \otimes \uparrow x_3)) \in \widehat{\mathsf{B}}^{2}_{1,\ldots,1}(V) \subset \widehat{\mathsf{B}}^{\infty}_{1,\ldots,1}(V), \text{ \&c.}$$

Definition 5. *The differential $\partial_{\rm lin}$ on $G_*(n)$ is defined by $\partial_{\rm lin} := \omega^{-1} \circ \partial_{\mathsf{B}} \circ \omega$. Thus $\partial_{\rm lin}$ is the unique differential such that $\omega : (G_*(n), \partial_{\rm lin}) \to (\widehat{\mathsf{B}}^{\infty}_{1,\ldots,1}(V), \partial_{\mathsf{B}})$ is an isomorphism of dg-abelian groups.*

It is easy to see that $(G_*(2), \partial_{\rm lin})$ is the cellular chain complex of the 'globular' decomposition of the infinite sphere $\mathbb{S}^\infty$. The piece $G_*(3)$ contains Stasheff's associator, two Mac Lane's hexagons (right and left), etc.

2.1. **The perturbed differential ∂.** We finish this section by proving the existence of the perturbed differential $\partial = \partial_{\rm lin} + \partial_{\rm prt}$, the arguments were already indicated in the introduction. Recall that the *cobar construction* on a coaugmented cooperad $\mathcal{C}$ is the dg-operad $\Omega(\mathcal{C})$ of the form $\Omega(\mathcal{C}) = (\mathbb{F}(\mathbf{s}\!\downarrow\!\overline{\mathcal{C}}), \partial_\Omega)$, where $\overline{\mathcal{C}}$ is the co-augmentation co-ideal of $\mathcal{C}$ and $\mathbf{s}\!\downarrow\!\overline{\mathcal{C}}$ the Σ-module defined by

$$\mathbf{s}\!\downarrow\!\overline{\mathcal{C}}(n) := \mathrm{sgn}_n \otimes \uparrow^{n-2} \overline{\mathcal{C}}(n), \ n \geq 2,$$

the product of the signum representation and the suspension of $\overline{\mathcal{C}}$ iterated $(n-2)$-times. The differential ∂_Ω is induced in the standard manner from the structure operations of the cooperad $\mathcal{C}$, see [19, Definition II.3.9].

Let $\mathcal{L}ie = \{\mathcal{L}ie(n)\}_{n\geq 1}$ be the operad describing Lie algebras. It is well-known that each component $\mathcal{L}ie(n)$ of this operad is a finite-dimensional free abelian group. Denote by $\mathcal{L}ie' = \{\mathcal{L}ie(n)'\}_{n\geq 1}$ the component-wise linear dual of $\mathcal{L}ie$ with the induced cooperad structure. It follows from Theorem 6.7, Fact 6.2 and Proposition 5.2.12 of [9] that the natural morphism

$$\alpha : \Omega(\mathcal{L}ie') \to \mathcal{C}om. \tag{3}$$

of dg-operads is a homology isomorphism over $\mathbb{Z}$. This can also be expressed by saying that the operads $\mathcal{C}om$ and $\mathcal{L}ie$ are Koszul dual to each other, and Koszul over $\mathbb{Z}$. The last step is based on Lemma 6 below, compare also Propositions 5.2.13 and 3.2.6 of [9].

Suppose we are given a dg-operad of the form $(\mathbb{F}(E), \vartheta)$, for some Σ-module $E = \{E(n)\}_{n\geq 2}$, such that $\vartheta(E)$ consist of decomposable elements in $\mathbb{F}(E)$. Suppose we are also given a dg-Σ-module $M = \{(M(n), \partial_{\rm lin})\}_{n\geq 2}$ such that $(M(n), \partial_{\rm lin})$ is a Σ_n-projective resolution of $E(n)$, for each $n \geq 2$. We use the same symbol $\partial_{\rm lin}$ both for the differential on the Σ-module M and the differential induced by $\partial_{\rm lin}$ on the free operad $\mathbb{F}(M)$. The following important lemma follows from standard homological algebra.

Lemma 6. *In the above situation, there exists a perturbation $\partial = \partial_{\rm lin} + \partial_{\rm prt}$ of the differential $\partial_{\rm lin}$ on $\mathbb{F}(M)$ and a homomorphism of dg-operads $(\mathbb{F}(M), \partial) \to (\mathbb{F}(E), \vartheta)$ inducing an isomorphism of homology.*

The existence of the operad $J = (\mathbb{F}(G), \partial = \partial_{\rm lin} + \partial_{\rm prt})$ now follows from Lemma 6 by taking E the Σ-module whose nth component equals $\uparrow^{n-2} {\rm sgn}_n \otimes \mathcal{L}ie(n)'$ and $(M, \partial_{\rm lin})$ the Σ-module $(G, \partial_{\rm lin})$ which is, by Theorem 34, a component-wise Σ-free resolution of the collection E defined in this way. The operad J resolves $\mathcal{C}om$ via the composition

$$J = (\mathbb{F}(G), \partial) \to \Omega(\mathcal{L}ie') \stackrel{\alpha}{\to} \mathcal{C}om,$$

of the map of Lemma 6 and α in (3).

3. Relation to the compactification of the configuration space

In Section 2 we studied the operad $J = (\mathbb{F}(G), \partial = \partial_{\rm lin} + \partial_{\rm prt})$ with the properties advertised in the introduction. While the generating Σ-module $G = \{G(n)\}_{n\geq 2}$, as well as the linear part $\partial_{\rm lin}$ of the differential was explicitly described in Definitions 3 and 5, we gave only an existence proof of the perturbed part $\partial_{\rm prt}$.

Let $\mathring{\mathsf{F}}_h(n) := {\rm Cnf}(\mathbb{R}^h, n)/{\rm Aff}(\mathbb{R}^h)$ be the moduli space of configurations of n distinct points in the h-dimensional Euclidean plane $\mathbb{R}^h$, modulo the action of the affine group of dilatations and translations. Getzler and Jones in [12] described a compactification $\mathsf{F}_h(n)$ of $\mathring{\mathsf{F}}_h(n)$ such that the Σ-space $\mathsf{F}_h := \{\mathsf{F}_h(n)\}_{n\geq 1}$ is an operad in the category of manifolds with corners, see also [17].

Will will in fact be interested in the colimit $\mathsf{F} := \varinjlim \mathsf{F}_h$ which inherits an operad structure from its constituents F_h. As observed in [12], the Fox-Neuwirth cells (recalled below) of the open part $\mathring{\mathsf{F}}_h(n)$ induce a decomposition of the colimit $\mathring{\mathsf{F}} := \varinjlim \mathring{\mathsf{F}}_h$ which in turn induces a cell decomposition of the compactification F compatible with the operad structure.

There was a belief that the operad J can be easily read off from the combinatorics of this cell decomposition, including a formula for the differential. This turned out not to be the case because, due to the existence of 'bad' cells, the cell structure of F is not regular. We identify, in Propositions 15 and 16, the dimensions in which bad cells exist and analyze them further in Section 5. We show that the differential of the operad J is still explicitly determined by the combinatorics of the Fox-Neuwirth cells in dimensions less than the critical dimension specified in:

Definition 7. *The* critical dimension $d_{\rm crit}(n)$ *is defined as*

$$d_{\rm crit}(n) = \begin{cases} \infty, & \text{if } n = 2,3 \text{ and} \\ n, & \text{if } n \geq 4. \end{cases}$$

The results of this section are summarized in the following statement which is a combination of Propositions 15, 17 and the results of Subsection 3.1.

Proposition 8. *Below the critical dimension, the differential ∂ of the operad $J = (\mathbb{F}(G), \partial)$ is determined by the combinatorics of the Fox-Neuwirth decomposition of the compactification F. On the component $G_i(n)$ with $i < d_{\rm crit}(n)$ it is given by formula (10) on page 16.*

We start by recalling, following closely [12], a correspondence between pruned trees and flags of pre-orders. This point of view will be useful in describing decompositions of configuration spaces.

Definition 9. *A* pre-order *π on a non-empty set S is a reflexive transitive relation $\leq$ such that if $a, b \in S$, either $a \leq b$ or $b \leq a$.*

A pre-order defines an equivalence $\sim$ on S by $a \sim b$ if and only if $a \leq b$ and $b \leq a$, and induces a total order on the quotient $S/\!\sim$. We denote $|\pi|$ the number of equivalence classes. A pre-order π is *trivial* if $a \leq b$ of all $a, b \in S$ or, equivalently, if $|\pi| = 1$. Pre-orders on S form a poset: $\pi_1 \prec \pi_2$ if $a \leq_2 b$ implies $a \leq_1 b$ for all $a, b \in S$. The maximal elements of this poset are the total orders of S, the unique minimal element is the trivial pre-order.

Definition 10. *A* flag of pre-orders *on the set S of height $h \geq 1$ is a sequence $(\pi_1 \prec \cdots \prec \pi_h)$ of pre-orders on S such that π_h is a total order of S. Such a flag is* reduced *if π_1 is not the trivial preorder.*

Let $\mathrm{Flag}^h(n) = \bigcup_{i \geq 0} \mathrm{Flag}^h_i(n)$ denote the graded set whose ith component is formed by flags of preorders of height h on the set $\{1, \ldots, n\}$ satisfying $i = \sum_{s=1}^h |\pi_s| - h - 1$. We also denote $\underline{\mathrm{Flag}}^h(n) = \bigcup_{i \geq 0} \underline{\mathrm{Flag}}^h_i(n)$ the graded subset of flags of preorders $(\pi_1 \prec \cdots \prec \pi_h)$ such that π_h is the standard linear order of $\{1, \ldots, n\}$.

The *suspension* $s : \mathrm{Flag}^h(n) \hookrightarrow \mathrm{Flag}^{h+1}(n)$ resp. $\underline{s} : \underline{\mathrm{Flag}}^h(n) \hookrightarrow \underline{\mathrm{Flag}}^{h+1}(n)$ extends a given flag from the left by the trivial preorder. The graded sets $\mathrm{Flag}(n) := \varinjlim \mathrm{Flag}^h(n)$ resp. $\underline{\mathrm{Flag}}(n) := \varinjlim \underline{\mathrm{Flag}}^h(n)$ consist of reduced flags of an arbitrary height. The next proposition relies on Notation 2.

Proposition 11. *For each $n, h \geq 1$, there are natural isomorphisms of graded sets $\mathrm{Tree}^h(n) \cong \mathrm{Flag}^h(n)$ and $\underline{\mathrm{Tree}}^h(n) \cong \underline{\mathrm{Flag}}^h(n)$ which induce isomorphisms of the colimits $\mathrm{Tree}(n) \cong \mathrm{Flag}(n)$ and $\underline{\mathrm{Tree}}(n) \cong \underline{\mathrm{Flag}}(n)$.*

Proof. Let $\mathbf{T} = (T, \ell) \in \mathrm{Tree}^h(n)$ be a labeled tree, i.e. $T \in \underline{\mathrm{Tree}}^h(n)$ is as in (1) and $\ell : [k_h] \stackrel{\cong}{\to} [n]$ a labeling. Such a tree $\mathbf{T}$ defines a flag of preorders $(\pi_1 \prec \cdots \prec \pi_h)$ as follows.

For $i, j \in \{1, \ldots, n\}$ we put $i \leq_h j$ if and only if $\ell^{-1}(i) \leq \ell^{-1}(j)$. In other words, π_h is the image of the natural order on $\{1, \ldots, k_h\}$ under the isomorphism ℓ. For $1 \leq s < h$ we write $i \leq_s j$ if and only if

$$\rho_s \rho_{s+1} \cdots \rho_{h-1}(\ell^{-1}(i)) \leq \rho_s \rho_{s+1} \cdots \rho_{h-1}(\ell^{-1}(j)).$$

It is easy to prove that the above correspondence is one to one, induces an isomorphism of the colimits and restricts to isomorphisms of the 'underlined' versions $\underline{\mathrm{Tree}}^h(n) \cong \underline{\mathrm{Flag}}^h(n)$ and $\underline{\mathrm{Tree}}(n) \cong \underline{\mathrm{Flag}}(n)$. It is also clear that for the flag $(\pi_1 \prec \cdots \prec \pi_h)$ corresponding to a tree $\mathbf{T} = (T, \ell)$ one has $e(T) = \sum_{s=1}^h |\pi_s|$, therefore, by (2),

$$\dim(T) = \textstyle\sum_{s=1}^h |\pi_s| - h - 1,$$

so the above isomorphism are compatible with the gradings. □

Convention 12. Given Proposition 11, we will make no difference between pruned trees and the corresponding flags of preorders. Thus, for instance, a boldfaced $\mathbf{T}$ will denote both a labeled tree (T, ℓ) and the corresponding flag $(\pi_1 \prec \cdots \prec \pi_h)$.

Recall that $\mathrm{Cnf}(\mathbb{R}^h, n)$ denotes the configuration space of n distinct labeled points $p_1, \ldots, p_n$ in the h-dimensional affine space $\mathbb{R}^h$, $n, h \geq 1$. It is an hn-dimensional oriented smooth non-compact manifold whose points are monomorphisms $f : \{1, \ldots, n\} \to \mathbb{R}^h$ given as $f(k) := p_k$, $1 \leq k \leq n$. For such an f and $1 \leq s \leq h$, denote by f_s the composition of f with the projection $\mathbb{R}^h \twoheadrightarrow \mathbb{R}^s$, $(x_1, \ldots, x_h) \mapsto (x_1, \ldots, x_s)$, to the first s coordinates. We finally denote π_s^f the preorder on the set $\{1, \ldots, n\}$ given by the pullback of the lexicographic order of $\mathbb{R}^s$ via f_s. In this way, each monomorphism $f : \{1, \ldots, n\} \to \mathbb{R}^h \in \mathrm{Cnf}(\mathbb{R}^h, n)$ determines a flag of preorders $\mathbf{T}^f = (\pi_1^f \prec \cdots \prec \pi_h^f)$.

Conversely, for a given tree[3] $\mathbf{T} = (T, \ell) \in \mathrm{Tree}^h(n)$ define $[\mathbf{T}] := \{f \in \mathrm{Cnf}(\mathbb{R}^h, n);\ \mathbf{T}^f = \mathbf{T}\}$. It is clear that $\mathrm{Cnf}(\mathbb{R}^h, n)$ is the disjoint union

$$\mathrm{Cnf}(\mathbb{R}^h, n) = \textstyle\bigcup_{\mathbf{T} \in \mathrm{Tree}^h(n)} [\mathbf{T}]. \tag{4}$$

Each $[\mathbf{T}]$ is an open ball of dimension $e(T) = \sum_{s=1}^h |\pi_s|$, therefore a tree $\mathbf{T} \in \mathrm{Tree}_i^h(n)$ determines a cell of dimension $i+h+1$. For $h = 2$, (4) describes the classical *Fox-Neuwirth* decomposition [8] generalized in [12] to arbitrary $h \geq 2$. One may assign an orientation to $[\mathbf{T}]$ taking first the coordinates x_1 of the equivalence class π_1 in the increasing order, next the coordinates x_2 of the equivalence class π_2, also in the increasing order, &c. An example can be found on page 33 of Section 6.

As we already indicated, we will need the *moduli space* $\mathring{\mathsf{F}}_h(n) := \mathrm{Cnf}(\mathbb{R}^h, n)/\mathrm{Aff}(\mathbb{R}^h)$, where the affine group $\mathrm{Aff}(\mathbb{R}^h) = \mathbb{R}^h \ltimes \mathbb{R}_+^\times$ acts by translations and dilatations in the obvious manner. We denote the quotient of the cell $[\mathbf{T}]$ modulo $\mathrm{Aff}(\mathbb{R}^h)$ by $\mu[\mathbf{T}]$. It is clear that $\mu[\mathbf{T}]$ is an open ball of dimension $e(T) - h - 1$. This explains formula (2) for the dimension of a tree. One has the disjoint decomposition

$$\mathring{\mathsf{F}}_h(n) = \textstyle\bigcup_{\mathbf{T} \in \mathrm{Tree}^h(n)} \mu[\mathbf{T}]. \tag{5}$$

Let us denote $\mathring{\mathsf{F}}_h$ the Σ-space $\mathring{\mathsf{F}}_h = \{\mathring{\mathsf{F}}(n)\}_{n \geq 2}$. Fulton and MacPherson construct in [11] a compactification $\mathsf{F}_h = \{\mathsf{F}_h(n)\}_{n \geq 1}$ of $\mathring{\mathsf{F}}_h$ such that $\mathsf{F}_h(n)$ is, for $n \geq 2$, a smooth manifold with

[3] We are already using Convention 12.

corners containing $\overset{\circ}{\mathsf{F}}_h(n)$ as its unique open stratum. The Σ-space F_h is obtained by gluing the free operad $\mathbb{F}(\overset{\circ}{\mathsf{F}}_h)$. In particular, decomposition (5) of $\overset{\circ}{\mathsf{F}}_h$ induces a decomposition of the free operad $\mathbb{F}(\overset{\circ}{\mathsf{F}}_h)$ which in turn induces a CW-structure of F_h via the gluing map $\mathbb{F}(\overset{\circ}{\mathsf{F}}_h) \twoheadrightarrow \mathsf{F}_h$.

This implies that the cells of F_h are indexed by the free set-operad $\mathbb{F}(\mathrm{Tree}^h)$. Since the pieces $\mathrm{Tree}^h(n)$ of the generating Σ-set $\mathrm{Tree}^h = \{\mathrm{Tree}^h(n)\}_{n\geq 2}$ are freely Σ_n-generated by the subset $\underline{\mathrm{Tree}}^h(n) \subset \mathrm{Tree}^h(n)$, the natural inclusion $\underline{\mathbb{F}}(\underline{\mathrm{Tree}}^h) \hookrightarrow \mathbb{F}(\mathrm{Tree}^h)$ induces, for each $n \geq 2$, the isomorphism of right Σ_n-sets

$$\mathbb{F}(\mathrm{Tree}^h)(n) \cong \mathrm{Ind}^{\Sigma_n}_{\mathbf{1}_n} \underline{\mathbb{F}}(\underline{\mathrm{Tree}}^h)(n) = \underline{\mathbb{F}}(\underline{\mathrm{Tree}}^h)(n) \times \Sigma_n,$$

where $\mathbf{1}_n$ denotes the trivial representation of Σ_n and $\underline{\mathbb{F}}(-)$ the free non-Σ operad functor. We will abbreviate the above display by

$$\mathbb{F}(\mathrm{Tree}^h) \cong \underline{\mathbb{F}}(\underline{\mathrm{Tree}}^h) \times \Sigma. \tag{6}$$

It follows from (6) and the structure theorem for free operads [19, Section II.1.9] that

$$\mathbb{F}(\mathrm{Tree}^h)(n) = \textstyle\bigcup_{\tau \in \mathrm{PTree}(n)} \tau(\underline{\mathrm{Tree}}^h),\ \ n \geq 2, \tag{7}$$

where $\mathrm{PTree}(n)$ denotes the set of *planar rooted* trees whose each vertex has at least two input edges[4], with leaves labeled by an isomorphism $\omega : \mathit{Leaf}(\tau) \to \{n\}$, see [19, Sect. II.1.5] for the terminology and notation. The trees in $\mathrm{PTree}(n)$ are different than the trees in Section 2 in that they do not have levels. The set $\tau(\underline{\mathrm{Tree}}^h)$ in (7) is the cartesian product

$$\tau(\underline{\mathrm{Tree}}^h) := \textstyle\bigtimes_{v \in \mathit{Vert}(\tau)} \underline{\mathrm{Tree}}^h(ar(v)),$$

where $\mathit{Vert}(\tau)$ is the set of vertices of τ and $ar(v)$ the number of input edges (= the arity) of a vertex v. Informally, (7) means that the cells of F_h are indexed by planar leaf-labeled rooted trees whose vertices are decorated by the graded set $\underline{\mathrm{Tree}}^h$ of pruned non-labeled h-trees.

The inclusion $\mathbb{R}^h \hookrightarrow \mathbb{R}^{h+1}$, $(x_1, \ldots, x_h) \mapsto (0, x_1, \ldots, x_h)$, induces an inclusion of Σ-spaces $\overset{\circ}{\mathsf{F}}_h \hookrightarrow \overset{\circ}{\mathsf{F}}_{h+1}$ so one can take the colimit $\overset{\circ}{\mathsf{F}} := \varinjlim \overset{\circ}{\mathsf{F}}_h$. Decomposition (5) induces the decomposition

$$\overset{\circ}{\mathsf{F}}(n) = \textstyle\bigcup_{\mathbf{T} \in \mathrm{Tree}(n)} \mu[\mathbf{T}] \tag{8}$$

with the cells indexed by *reduced* trees. The colimit $\mathsf{F} = \varinjlim \mathsf{F}_h$ is again obtained by gluing the free operad $\mathbb{F}(\overset{\circ}{\mathsf{F}})$, so (8) gives a decomposition of F with cells indexed by the free set-operad $\mathbb{F}(\mathrm{Tree})$.

At this stage we need to extend Definition 7 of the critical dimension for finite h by

$$d^h_{\mathrm{crit}}(n) := \begin{cases} \infty, & \text{if } n = 2, 3, \text{ or } n = 4, 5 \text{ and } h \leq 2, \text{ or } n \geq 6 \text{ and } h = 1, \\ n, & \text{in the remaining cases.} \end{cases} \tag{9}$$

Clearly, $d_{\mathrm{crit}}(n) = \lim_{h\to\infty} d^h_{\mathrm{crit}}(n)$. Let $\mathrm{Tree}^{\mathrm{reg},h}_*$ be, for $h \geq 1$, the graded Σ-subset of the graded Σ-set Tree^h_* consisting of reduced trees of dimension less that the critical one, i.e. the graded Σ-set such that

$$\mathrm{Tree}^{\mathrm{reg},h}_i(n) := \begin{cases} \mathrm{Tree}^h_i(n), & \text{if } i < d^h_{\mathrm{crit}}(n), \text{ and} \\ \emptyset, & \text{if } i \geq d^h_{\mathrm{crit}}(n). \end{cases}$$

[4] This is the usual meaning of being reduced, compare the footnote on page 4.

Observe that $\mathrm{Tree}_*^{\mathrm{reg},1} = \mathrm{Tree}_*^1$. We will also need the direct limit $\mathrm{Tree}_*^{\mathrm{reg}} := \varinjlim \mathrm{Tree}_*^{\mathrm{reg},h}$. Clearly $\mathrm{Tree}_*^{\mathrm{reg}}(n) = \mathrm{Tree}_*(n)$ if $n = 2,3$ while, for $n \geq 4$,

$$\mathrm{Tree}_i^{\mathrm{reg}}(n) = \begin{cases} \mathrm{Tree}_i(n), & \text{if } i < n, \text{ and} \\ \emptyset, & \text{if } i \geq n. \end{cases}$$

We will call, just for the purposes of this section, the trees in $\mathrm{Tree}^{\mathrm{reg},h}$ or $\mathrm{Tree}^{\mathrm{reg}}$ the *regular* trees. We also denote by $\underline{\mathrm{Tree}}^{\mathrm{reg},h}$ (resp. $\underline{\mathrm{Tree}}^{\mathrm{reg}}$) the Σ-subset of $\mathrm{Tree}^{\mathrm{reg},h}$ (resp. $\mathrm{Tree}^{\mathrm{reg}}$) of unlabeled regular trees, i.e. $\underline{\mathrm{Tree}}^{\mathrm{reg},h} := \mathrm{Tree}^{\mathrm{reg},h} \cap \underline{\mathrm{Tree}}$ (resp. $\underline{\mathrm{Tree}}^{\mathrm{reg}} := \mathrm{Tree}^{\mathrm{reg}} \cap \underline{\mathrm{Tree}}$).

Definition 13. *The* regular skeleton $\mathsf{F}^{\mathrm{reg}}$ *of* F *is the union of the cells of the CW-complex* F *indexed by the suboperad* $\mathbb{F}(\mathrm{Tree}^{\mathrm{reg}}) \subset \mathbb{F}(\mathrm{Tree})$. *The* regular skeleton $\mathsf{F}_h^{\mathrm{reg}}$ *of* F_h *is the intersection* $\mathsf{F}^{\mathrm{reg}} \cap \mathsf{F}_h$.

For $\mathbb{F}(\mathrm{Tree}^{\mathrm{reg}})$ we have a formula similar to (7), i.e.

$$\mathbb{F}(\mathrm{Tree}^{\mathrm{reg}})(n) = \bigcup\nolimits_{\tau \in \mathrm{PTree}(n)} \tau(\underline{\mathrm{Tree}}^{\mathrm{reg}}),\ n \geq 2,$$

thus the cells of $\mathsf{F}^{\mathrm{reg}}$ are indexed by planar rooted labeled trees with vertices decorated by unlabeled reduced regular trees. It is clear that $\mathsf{F}_h^{\mathrm{reg}}$ is the union of the cells indexed by the suboperad $\mathbb{F}(\mathrm{Tree}^{\mathrm{reg},h})$. It is equally obvious that the sub-Σ-modules $\mathsf{F}_h^{\mathrm{reg}} \subset \mathsf{F}_h$ and $\mathsf{F}^{\mathrm{reg}} \subset \mathsf{F}$ are suboperads and that $\mathsf{F}_1^{\mathrm{reg}} = \mathsf{F}_1$. Let us recall the following standard

Definition 14. *A CW-complex is* regular *if (i) the attaching maps are homeomorphisms and (ii) the boundary of each cell is a union of cells.*

We have the following correction to [12, Lemma 5.11] whose proof we postpone to Section 5.

Proposition 15. *The CW-structures of* F_h, $h \geq 1$, *and of* F *are compatible with the operad structures and the symmetric group acts freely on the cells. The spaces* $\mathsf{F}(n)$ *are regular cell complexes if and only if* $n = 2$ *or* 3. *The complexes* $\mathsf{F}_h(n)$ *are regular if and only if*

(i) $n = 2,3$ *and* h *arbitrary, or*
(ii) $n \leq 5$ *and* $h \leq 2$, *or*
(iii) n *arbitrary and* $h = 1$.

The spaces $\mathsf{F}_h(n)$ *and* $\mathsf{F}(n)$ *satisfy condition (i) of Definition 14 for arbitrary* n *and* h. *The CW-subcomplexes* $\mathsf{F}_h^{\mathrm{reg}} \subset \mathsf{F}_h$ *and* $\mathsf{F}^{\mathrm{reg}} \subset \mathsf{F}$ *are regular, with the cell structures compatible with the operad structures and the symmetric group acting freely on the cells.*

Observe that, by the definition of the critical dimension, Proposition 15 says that the spaces $\mathsf{F}_h(n)$ (resp. $\mathsf{F}(n)$) are regular if and only if $d^h_{\mathrm{crit}}(n) = \infty$ (resp. $d_{\mathrm{crit}}(n) = \infty$). We call cells violating (ii) of Definition 14 the *bad* cells. The following statement provides a 'coordinate-free' definition of the regular skeleta.

Proposition 16. *For each $n \geq 4$, $h \geq 3$ or $n \geq 6$, $h \geq 2$, there exist a bad cell e_n^h in the open stratum $\mathring{\mathsf{F}}_h(n)$ whose dimension $\dim(e_n^h)$ equals $d_{\rm crit}^h(n)$. Likewise, for each $n \geq 4$ there exists a bad cell $e_n \subset \mathring{\mathsf{F}}(n)$ such that $\dim(e_n) = d_{\rm crit}(n)$. The regular skeleta are therefore the maximal regular subcomplexes closed under the operad structure.*

The first bad cell was found by D. Tamarkin. We will call this particular bad cell the Tamarkin cell and recall its definition in Section 4 in which we also prove Propositions 15 and 16. The case $h = 1$ is special; F_1 is the *Stasheff's operad* of the associahedra [22] which indeed forms a regular cell complex. The dimensions/arities in which the bad cells of the open strata sit are shown in Figures 4 and 5.

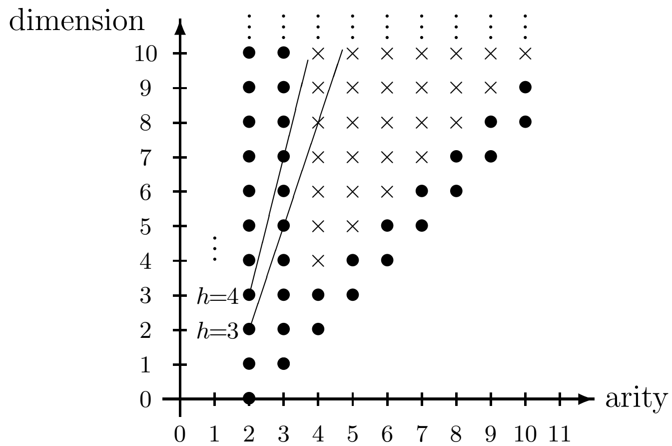


FIGURE 4. The bad (marked $\times$) and regular (marked $\bullet$) cells of $\mathring{\mathsf{F}}$, and of $\mathring{\mathsf{F}}_h$ for $h \geq 3$. The lines marked $h = 3, 4$ show the dimension of $\mathsf{F}_h(n)$.

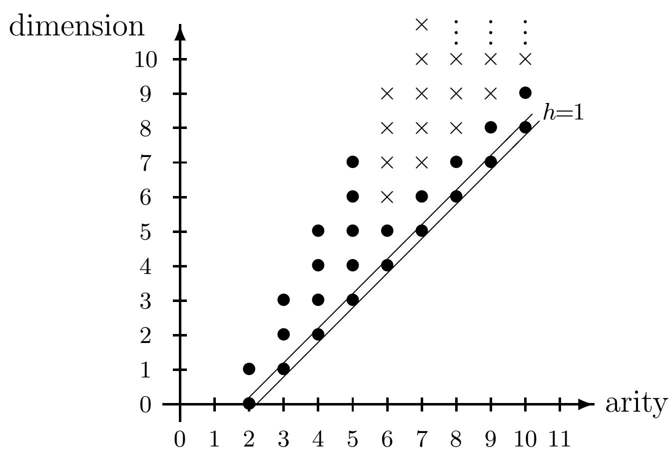


FIGURE 5. The bad (marked $\times$) and regular (marked $\bullet$) cells of $\mathring{\mathsf{F}}_2$. The double line represents the embedding of the top-dimensional cell of the associahedron K.

Let us define the increasing filtration $\mathscr{F}_*(n) = \cdots \mathscr{F}_0(n) \subset \mathscr{F}_1(n) \subset \mathscr{F}_2(n) \cdots$ of $\mathsf{F}^{\rm reg}(n)$ by

$$\mathscr{F}_p(n) := \bigcup\{e \text{ a cell of } \mathsf{F}^{\rm reg}(n);\ \dim(e) \leq p\}.$$

Since the cells of $\mathsf{F}^{\rm reg}$ are indexed by the free operad $\mathbb{F}(\mathrm{Tree}^{\rm reg})$, this filtration is manifestly operadic, i.e. each $\mathscr{F}_p(n)$ is Σ_n-invariant and $\mathscr{F}_p(m) \circ_i \mathscr{F}_q(n) \subset \mathscr{F}_{p+q}(m+n-1)$ for $m, n \geq 1$, $1 \leq i \leq m$. Each layer $(\mathscr{E}^r_{**}(n), \partial^r)$ of the induced spectral sequence determines a dg-operad $\mathscr{E}^r := \{(\mathscr{E}^r_*(n), \partial^r)\}_{n \geq 1}$ with $\mathscr{E}^r_*(n) := \bigoplus_{*=p+q} \mathscr{E}^r_{pq}(n)$.

The dg-operad $\mathscr{E}^2$ will be of a particular importance for us. As usual, the abelian group $\mathscr{E}^2_{pq}(n)$ equals the reduced homology $\overline{H}_{p+q}(\mathscr{F}_p(n)/\mathscr{F}_{p-1}(n))$, so the regularity of the CW-structure established in Proposition 15 implies that

$$\mathscr{E}^2_{pq}(n) = \begin{cases} \mathrm{Span}(\text{set of } p\text{-dimensional cells of } \mathsf{F}(n)), & \text{if } q = 0 \text{ and} \\ 0, & \text{if } q \neq 0. \end{cases}$$

In particular, $\mathscr{E}^2_{p0}$ contains, for each $\mathbf{T} \in \mathrm{Tree}^{\rm reg}_p(n)$, the generator $c_{\mathbf{T}}$ corresponding to the cell $\mu[\mathbf{T}]$.

Denote finally by $G^{\rm reg}_* = \mathrm{Span}(\mathrm{Tree}^{\rm reg}_*)$ the graded Σ-submodule of the Σ-module G_* from Definition 3 spanned by the generators $g_{\mathbf{T}}$ indexed by regular trees $\mathbf{T} \in \mathrm{Tree}^{\rm reg}$. We have a natural map of graded operads $j : \mathbb{F}(G^{\rm reg}) \to \mathscr{E}^2$ given by $j(g_{\mathbf{T}}) := c_{\mathbf{T}}$ for $\mathbf{T} \in \mathrm{Tree}^{\rm reg}$.

Proposition 17. *There is a unique differential ∂ on the free operad $\mathbb{F}(G^{\rm reg})$ such that the map $j : (\mathbb{F}(G^{\rm reg}), \partial) \to (\mathscr{E}^2, \partial^2)$ is a map of dg-operads. Moreover, ∂ is the sum $\partial_{\rm lin} + \partial_{\rm prt}$ where $\partial_{\rm lin}$ is as in Definition 5.*

In this way, the restriction of the differential ∂ of the dg-operad $J = (\mathbb{F}(G), \partial)$ to the regular Σ-submodule $G^{\rm reg} \subset G$ is determined by combinatorics of the Fox-Neuwirth cells of the configuration space.

Proof. It is clear from the description of the cell structure of $\mathsf{F}^{\rm reg}$ via the free set-operad $\mathbb{F}(\mathrm{Tree}^{\rm reg})$ that the map j is an isomorphism of graded operads, which implies the existence and uniqueness of the differential ∂. The fact that ∂ constructed in this way is a perturbation of the linear part $\partial_{\rm lin}$ of Definition 5 will follow from explicit calculations given below and Proposition 19. □

3.1. **The differential ∂ in sub-critical dimensions.** Proposition 17 translates the description of the differential of the operad J in sub-critical dimensions into the standard task of calculating the second term of the spectral sequence associated to the regular cell complex $\mathsf{F}^{\rm reg}$. Given an i-dimensional cell e of $\mathsf{F}^{\rm reg}$, one needs first to identify cells forming the boundary of e. The differential of the generator corresponding to e then is then the sum of the generators corresponding to $(i-1)$-dimensional cells in the boundary of e, with the signs determined by the orientations.

In our particular case, the compatibility of the differential with the operad structure implies that it suffices to describe the boundaries of the cells $\mu[T]$ corresponding to the operadic generators in $G^{\rm reg}$, indexed by unlabeled regular reduced trees $T \in \underline{\mathrm{Tree}}^{\rm reg}$. This was in fact already done in [3], so we only need to recall the necessary notions. Let us recollect the notation first.

Notation 18. We introduced the following objects indexed by trees $\mathbf{T} \in$ Tree (resp. the unlabeled versions $T \in \underline{\text{Tree}}$): the corresponding generator $g_\mathbf{T}$ (resp. g_T) of $G = \text{Span(Tree)}$ (resp. of $\underline{G} = \text{Span}(\underline{\text{Tree}})$), the Fox-Neuwirth cell $\mu[\mathbf{T}]$ (resp. $\mu[T]$) of $\mathring{\mathsf{F}}$, and $c_\mathbf{T}$ (resp. c_T) – the corresponding generator of $\mathscr{E}^2$. We will also denote by E_T the corresponding generator of the free set-operad $\mathbb{F}$(Tree) and, for an element $C \in \mathbb{F}$(Tree), by $\mu[C]$ the corresponding cell of F.

Let us return to our task of describing the differential ∂. According to [3, Definition 2.2], a *morphism of h-trees*

$$T = [k_h] \xrightarrow{\rho_{h-1}} [k_{h-1}] \xrightarrow{\rho_{h-2}} \cdots \xrightarrow{\rho_0} [1]$$

and

$$S = [s_h] \xrightarrow{\xi_{h-1}} [s_{h-1}] \xrightarrow{\xi_{h-2}} \cdots \xrightarrow{\xi_0} [1]$$

is given by a sequence $\sigma = (\sigma_h, \ldots, \sigma_0)$ of not necessary order preserving maps $\sigma_m : [k_m] \to [s_m]$, $0 \le m \le h$, with the property that for each m and each $j \in [k_{m-1}]$, the restriction of σ_m to $\rho_{m-1}^{-1}(j)$ preserves the induced order.[5]

Let $T \in \underline{\text{Tree}}$ be a reduced unlabeled h-tree as above. We will consider maps $\sigma : T \to S$ of h-trees such that

(i) the tree S is pruned, but possibly with a trunk, and
(ii) the map σ induces an epimorphism of tips, that is, $\sigma_h : [k_h] \to [s_h]$ is onto.

We will call such a map σ a *face* of the tree T. Observe that σ is determined by the values $\sigma_h(i)$, $i \in [k_h]$. Let us explain how a face σ determines a cell of F in the boundary of $\mu[T]$. We need first to describe, following again M. Batanin's [3], faces σ in terms of fibers.

Let $\sigma : T \to S$ be a face of T as above. For each tip $j \in [s_h]$, let S_j be the path in S connecting $\{j\}$ with the root of S. Then the *jth fiber of* σ is the subtree $F_j := \sigma^{-1}(S_j)$ of T. We believe that Figure 6 elucidates this definition.

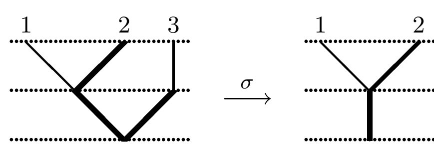


FIGURE 6. Fiber F_2 (shown in bold lines) of the map $\sigma : T \to S$ given by $\sigma_2(1) = \sigma_2(3) = 1$ and $\sigma_2(2) = 2$.

Each such a $\sigma : T \to S$ is characterized by its *fiber diagram*, obtained by drawing fibers F_j over the corresponding tips of S. Some examples of fiber diagrams can be found in Figure 7.

Diagram D_1 is the fiber diagram of face σ from Figure 6. Diagram D_2 is the fiber diagram of the same trees as in Figure 6, but with σ determined by $\sigma_2(1) = \sigma_2(2) = 1$ and $\sigma_2(3) = 2$. Diagram D_3 is the diagram of the map $\sigma :$  $\to$  given by $\sigma_2(1) = 1$, $\sigma_2(2) = 3$ and $\sigma_2(3) = 2$. Diagram D_4 describes the map $\sigma :$  $\to$  given by $\sigma_2(1) = \sigma_2(2) = 1$ and $\sigma_2(3) = 2$.

[5]We believe the same implicit notation for a permutation and a morphism of trees will not confuse the reader.

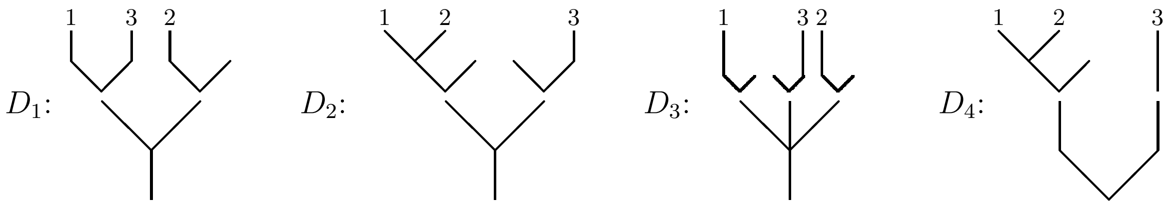


FIGURE 7. Examples of fiber diagrams.

We sometimes decorate the tips of fibers by numbers that indicate to which tip of S they are mapped, see again Figure 7. Other examples of fiber diagrams can be found in Figures XV and XVI of [3].

We are ready to describe the element $C_\sigma \in \mathbb{F}(\mathrm{Tree})$ indexing the cell $\mu[\sigma] := \mu[C_\sigma]$ of F corresponding to the face σ. We take the fiber diagram of σ and replace all trees of this diagram by their maximal reduced subtrees. We obtain a tree-shaped diagram of reduced trees in Tree which are, by definition, the generators of the free operad set operad $\mathbb{F}(\mathrm{Tree})$. The terminal tree represents the identity $1\!\!1 \in \mathbb{F}(\mathrm{Tree})(1)$. We then interpret this *reduced fiber diagram* as the indicated composition of elements in the free operad $\mathbb{F}(\mathrm{Tree})$ using the direct limit of isomorphisms (6)

$$\mathbb{F}(\mathrm{Tree}) \cong \underline{\mathbb{F}}(\underline{\mathrm{Tree}}) \times \Sigma.$$

Let us denote this composition by C_σ. We believe that the construction of C_σ is clear from Figure 8 which relies on Notation 18.

It will be convenient to extend the barcode notation of Definition 4 to elements of the free operad $\mathbb{F}(\mathrm{Tree})$. For example, the *extended barcode* $[[1||3]|2]$ of the element $E_{\vee} \circ (E_{\cup} \times 1\!\!1) \circ (132)$ is obtained by inserting the barcode $[1||2]$ for the tree $E_{\cup}$ into the first position in the barcode $[1|2]$ for $E_{\vee}$ and permuting the labels according to the permutation (132). See Figure 8 for more examples of the extended barcodes.

The last step is counting $\deg(C_\sigma)$ by adding up the degrees of generators that constitute C_σ. For example, in Figure 8 all C_σ's are of degree 1 except the one corresponding to D_2 which is of degree 0. Let $\iota : \mathbb{F}(\mathrm{Tree}) \hookrightarrow \mathbb{F}(G)$ be the monomorphism induced by the inclusion $\mathrm{Tree} \hookrightarrow G = \mathrm{Span}(\mathrm{Tree})$, $\mathbf{T} \mapsto g_{\mathbf{T}}$, of graded Σ-sets. For $T \in \underline{\mathrm{Tree}}$ and g_T the corresponding generator of G, put

$$\partial(g_T) := \textstyle\sum_\sigma \pm\iota(C_\sigma), \tag{10}$$

with the sum taken over faces σ of T such that $\dim(C_\sigma) = \dim(T) - 1$. The signs are determined by the orientation of the cells. While it is possible to determine the signs for each particular g_T, we do not know a reasonable general formula.

Proposition 19. *Formula (10) extends to a differential on $\mathbb{F}(G^{\mathrm{reg}}) \subset \mathbb{F}(G)$ having the form $\partial = \partial_{\mathrm{lin}} + \partial_{\mathrm{prt}}$, where ∂_{lin} is as in Definition 5.*

[October 2, 2009]

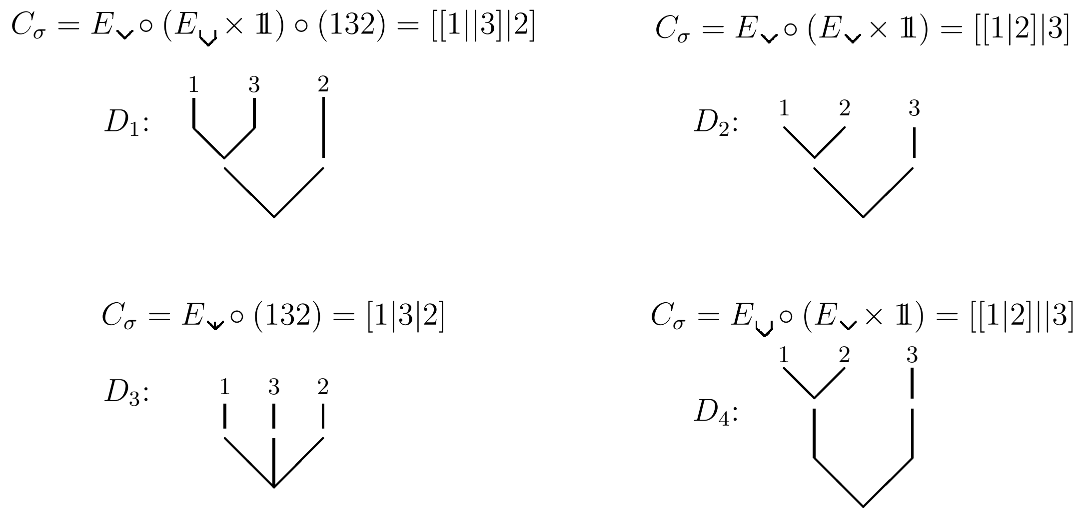


FIGURE 8. Reduced fiber diagrams and elements $C_\sigma \in \mathbb{F}(\mathrm{Tree}) \cong \underline{\mathbb{F}}(\underline{\mathrm{Tree}}) \times \Sigma$ they determine. The symbol $1\!\!1$ denotes the identity and $(132) \in \Sigma_3$ the permutation $(1,2,3) \mapsto (1,3,2)$. For $T \in \underline{\mathrm{Tree}}$, E_T is the corresponding generator of $\underline{\mathbb{F}}(\underline{\mathrm{Tree}})$.

Proof. The first part of the proposition follows from the fact that (10) calculates the cellular differential of the regular cell complex $\mathsf{F}^{\mathrm{reg}}$. Let us prove that the linear part of (10) coincides with ∂_{lin}.

It is obvious that, for $T \in \underline{\mathrm{Tree}}^{\mathrm{reg}}$, $\partial_{\mathrm{lin}}(g_T)$ is given by the sum (10) restricted to the faces $\sigma : T \to S$ with trivial reduced fibers. Equivalently, we restrict to σ's that induce isomorphisms $\sigma_h : [k_h] \cong [s_h]$ of the tips. Batanin calls, in [2], such maps σ *quasibijections*. The reduced fiber diagram of a quasiisomorphism σ is simply S with the tips labeled by the permutation σ_h, see D_3 in Figure 8 for an example. If we denote this labeled tree by $\mathbf{S}_\sigma := (S, \sigma_h)$, then $C_\sigma = \mathbf{S}_\sigma \in \mathrm{Tree} \subset \mathbb{F}(\mathrm{Tree})$. Condition $\dim(S) = \dim(T) - 1$ means that the tree S has one edge less than T. Each such S is obtained from T by the following procedure.

Assume that T is as in (1) and choose $1 \leq m < h$ such that there exists $u \in [k_m]$ satisfying $\rho_{m-1}(u) = \rho_{m-1}(u{+}1)$. Let $b_1, \dots, b_s$ (resp. $b_{s+1}, \dots, b_{s+t}$) be the branches of T over u (resp. $u{+}1$). By a *branch* over u we mean a subtree determined by a vertex $\tilde{u} \in [k_{m+1}]$ satisfying $\rho_m(\tilde{u}) = u$. The corresponding branch is the maximal subtree of T of height $h - m$ whose trunk is the edge connecting $\tilde{u}$ and u. Branches over $u + 1$ are defined analogously. The situation is shown in Figure 9.

Choose finally an (s,t)-unshuffle $\tau \in \Sigma_{s,t}$ and denote by $\mathbf{S}$ the labeled tree obtained from T by identifying the edge e' starting from u with the edge e'' starting at $u + 1$, and permuting the branches $b_1, \dots, b_s, b_{s+1}, \dots, b_{s+t}$ according the shuffle τ, see again Figure 9. Let $\sigma : T \to \mathbf{S}$ be the projection. It is clear that all codimension-one faces σ of T are of this form and that $g_{\mathbf{S}} \in G$ corresponds, under the isomorphism $\omega : G \to \mathsf{B}^{\infty}_{1,\dots,1}(V)$ defined on page 6, to a component of the top-level differential in $\mathsf{B}^{m+1}(V) \subset \mathsf{B}^{\infty}(V)$ applied to $\omega(g_T)$. $\square$

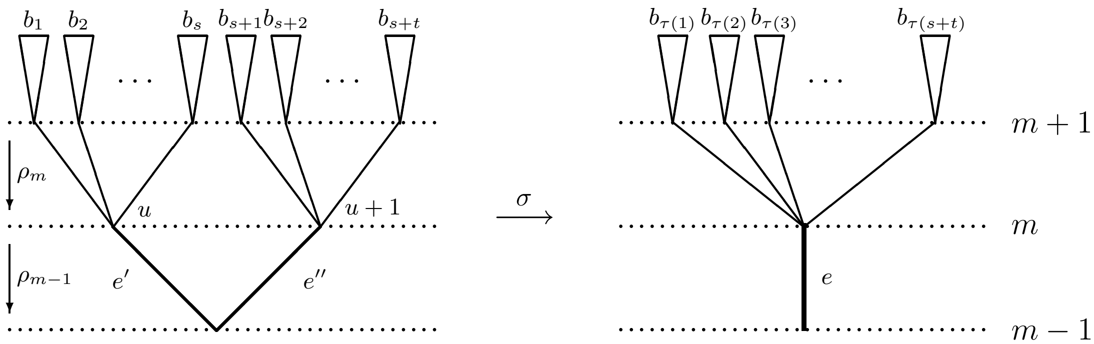


FIGURE 9. The relevant parts of the trees T (left) and S (right). The 'fat' edge e is obtained by identifying e' and e''.

3.2. **Some formulas.** In this subsection we calculate the differential of some low-dimensional generators of the operad $J = \mathbb{F}(G)$. Recall that, for a tree $T \in \underline{\mathrm{Tree}}(n)$, we denoted by g_T the corresponding generator of $G(n)$. We will denote a permutation $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\} \in \Sigma_n$ by the n-tuple $(\sigma_1,\ldots,\sigma_n)$, where $\sigma_i := \sigma^{-1}(i)$ for $1 \leq i \leq n$.

The degree 0 generator $g_{\vee}$ is mapped to the commutative associative multiplication in $\mathcal{C}om$. Of course, $\partial(g_{\vee}) = 0$. The degree one generator $g_{\curlyvee}$ is the 'associator' and

$$\partial(g_{\curlyvee}) = g_{\vee} \circ (g_{\vee} \otimes 1\!\!1 - 1\!\!1 \otimes g_{\vee}).$$

The second degree one generator $g_{\cup}$ represents the homotopy for the commutativity of $g_{\vee}$:

$$\partial(g_{\cup}) = g_{\vee}(1\!\!1 - (21)).$$

The degree two generator g_{Ψ} is the Stasheff/Mac Lane pentagon and we all know the formula

$$\partial(g_{\Psi}) = g_{\curlyvee}(g_{\vee} \otimes 1\!\!1^{\otimes 2}) - g_{\curlyvee}(1\!\!1 \otimes g_{\vee} \otimes 1\!\!1) + g_{\curlyvee}(1\!\!1^{\otimes 2} \otimes g_{\vee}) - g_{\vee}(g_{\curlyvee} \otimes 1\!\!1) - g_{\vee}(1\!\!1 \otimes g_{\curlyvee})$$

from kindergarten, see Figure 10. The degree two generator $g_{\curlyvee\!\cup}$ is the left hexagon whose differential is given by

$$\partial(g_{\curlyvee\!\cup}) = g_{\curlyvee} \circ (1\!\!1 - (132) + (312)) - g_{\cup} \circ (g_{\vee} \otimes 1\!\!1) + g_{\vee} \circ (1\!\!1 \otimes g_{\cup}) + g_{\vee} \circ (g_{\cup} \otimes 1\!\!1)(132),$$

see Figure 10. The formula for the right hexagon $g_{\cup\!\curlyvee}$ is a similar:

$$\partial(g_{\cup\!\curlyvee}) = g_{\curlyvee} \circ (1\!\!1 - (213) + (231)) + g_{\cup} \circ (1\!\!1 \otimes g_{\vee}) - g_{\vee} \circ (1\!\!1 \otimes g_{\cup})(213) - g_{\vee} \circ (g_{\cup} \otimes 1\!\!1).$$

The last degree two generator $g_{\bigcup}$ is the homotopy for the anticommutativity of $g_{\cup}$:

$$\partial(g_{\bigcup}) = g_{\cup}(1\!\!1 + (21)),$$

see again Figure 10. The differential of the degree three generator $g_{\Psi\!\cup}$ is

$$\partial(g_{\Psi\!\cup}) = g_{\curlyvee\!\cup} \circ (1\!\!1 - (213)) - g_{\cup\!\curlyvee} \circ (1\!\!1 - (132)) - g_{\cup} \circ (g_{\cup} \otimes 1\!\!1 - 1\!\!1 \otimes g_{\cup}).$$

Let us give also a formula for $\partial(g_{\curlyvee\!\bigcup})$:

$$\partial(g_{\curlyvee\!\bigcup}) = g_{\curlyvee\!\cup} - g_{\cup\!\curlyvee} \circ (312) + g_{\bigcup} \circ (g_{\vee} \otimes 1\!\!1) - g_{\vee} \circ (1\!\!1 \otimes g_{\bigcup}) - g_{\vee} \circ (g_{\bigcup} \otimes 1\!\!1)(132).$$

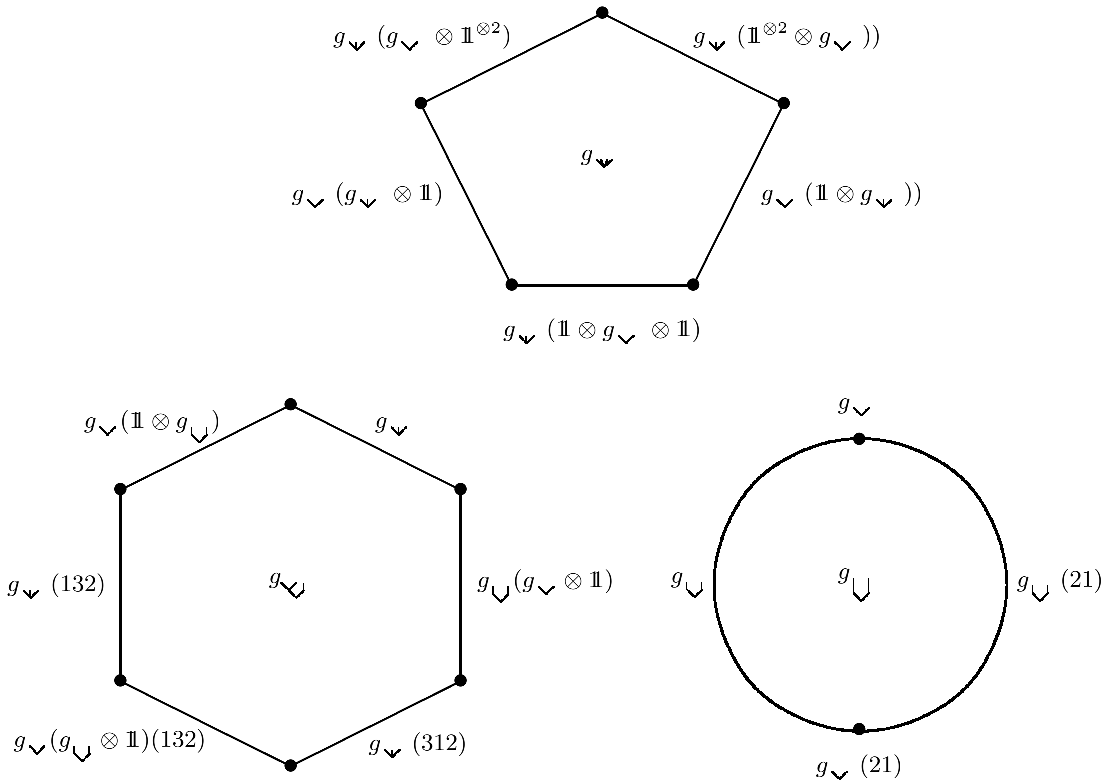


FIGURE 10. Pentagon, left hexagon and disk.

3.3. **E_∞-algebras.** As explained in the Introduction, algebras over the differential graded operad $J = (J, \partial)$ are particular realizations of E_∞-*algebras*. A structure of this E_∞-algebra on a dg-abelian group $V = (V, d)$ is given by multilinear maps $\mu_T : V^{\otimes n} \to V$ indexed by reduced trees. The degree of μ_T equals $\dim(T)$ and the arity n equals the number of the tips of T. The axioms could be read off from the formulas for differential ∂ given in Subsection 3.2. One gets

$$\text{(11a)} \qquad \delta\mu_{\vee}(a,b) = 0,$$

$$\text{(11b)} \qquad \delta\mu_{\curlyvee}(a,b,c) = \mu_{\vee}(\mu_{\vee}(a,b),c) - \mu_{\vee}(a,\mu_{\vee}(b,c)),$$

$$\text{(11c)} \qquad \delta\mu_{\cup}(a,b) = \mu_{\vee}(a,b) - \mu_{\vee}(b,a),$$

$$\text{(11d)} \qquad \delta\mu_{\cup\vee}(a,b,c) = \mu_{\curlyvee}(a,b,c) - \mu_{\curlyvee}(a,c,b) + \mu_{\curlyvee}(c,a,b) - \mu_{\cup}(\mu_{\vee}(a,b),c) + (-1)^{\deg(a)}\mu_{\vee}(a,\mu_{\cup}(b,c)) + \mu_{\vee}(\mu_{\cup}(a,c),b),$$

$$\text{(11e)} \qquad \delta\mu_{\vee\cup}(a,b,c) = \mu_{\curlyvee}(a,b,c) - \mu_{\curlyvee}(b,a,c) + \mu_{\curlyvee}(b,c,a) - \mu_{\vee}(\mu_{\cup}(a,b),c), (-1)^{\deg(a)}\mu_{\cup}(a,\mu_{\vee}(b,c)) - (-1)^{\deg(b)}\mu_{\vee}(b,\mu_{\cup}(a,c))$$

$$\text{(11f)} \qquad \delta\mu_{\Psi}(a,b,c,d) = \mu_{\curlyvee}(\mu_{\vee}(a,b),c,d) - \mu_{\curlyvee}(a,\mu_{\vee}(b,c),d) + \mu_{\curlyvee}(a,b,\mu_{\vee}(c,d)) - \mu_{\vee}(\mu_{\curlyvee}(a,b,c),d) - (-1)^{\deg(a)}\mu_{\vee}(a,\mu_{\curlyvee}(b,c,d)),$$

$$\text{(11g)} \qquad \delta\mu_{\bigcup}(a,b) = \mu_{\cup}(a,b) + \mu_{\cup}(b,a),$$

$$\delta\mu_{\psi}(a,b,c) = \mu_{\cup\vee}(a,b,c) - \mu_{\cup\vee}(b,a,c) - \mu_{\vee\cup}(a,b,c) + \mu_{\vee\cup}(a,c,a) +$$

$$
\begin{aligned}
& -\mu_{\cup}\,(\mu_{\cup}\,(a,b),c) - (-1)^{\deg(a)}\mu_{\cup}\,(a,\mu_{\cup}\,(b,c)),\\
\delta\mu_{\curlyvee\cup}\,(a,b,c) \;=\; & \mu_{\curlyvee\cup}(a,b,c) - \mu_{\cup\curlyvee}(c,a,b) + \mu_{\bigcup}\,(\mu_{\vee}\,(a,b),c) -\\
& -\mu_{\vee}\,(a,\mu_{\bigcup}\,(b,c)) - \mu_{\vee}\,(\mu_{\bigcup}\,(a,c),b),\ \ \&\text{c.}
\end{aligned}
$$

In the above formulas, a, b, c, d are homogeneous elements of V, and δ the induced differential in the endomorphism complex of $V = (V,d)$. For example

$$
\delta\mu_{\vee}\,(a,b) := d\mu_{\vee}\,(a,b) - \mu_{\vee}\,(da,b) - (-1)^{\deg(a)}\mu_{\vee}\,(a,db).
$$

Some of the above axioms have obvious interpretations. Axiom (11b) says that $\mu_{\curlyvee}$ is a (chain) homotopy for the multiplication $\mu_{\vee}$, axiom (11c) means that $\mu_{\cup}$ is $\smile_1$ for $\mu_{\vee}$ and (11g) means that $\mu_{\bigcup}$ is $\smile_2$ for $\mu_{\vee}$. Axioms (11d) and (11e) are algebraic versions of the left and right hexagons. Axiom (11f) is an algebraic version of the pentagon.

More generally, if $\mu_n := \mu_{\star_n}$ with $\star_n \in \underline{\mathrm{Tree}}^1(n)$ the n-corolla with the barcode $[1|\cdots|n]$, then $(V,d,\mu_2,\mu_3,\ldots)$ is an A_∞-algebra, with (11a), (11b) and (11f) Axiom (1) of [14] for $n = 2,3$ and 4. This justifies calling the operad J an extension of Stasheff's operad. Axioms (11a)–(11g) were already obtained in Example 4.8 of the 1996 paper [15].

4. The Tamarkin cell mystery

The first example of a cell violating the regularity of the CW-complex F was found by Dimitri Tamarkin. It is a 6-dimensional cell $\mathscr{T} \subset \mathsf{F}(6)$ which actually lives in the subcomplex $\mathsf{F}_2(6)$ of compactified configurations of six points in $\mathbb{R}^2$. Surprisingly, there exist even a simpler, 4-dimensional 'bad' cell $\mathscr{M} \subset \mathsf{F}(4)$ living in the subcomplex $\mathsf{F}_3(4)$ of compactified configurations of 4 points in $\mathbb{R}^3$. It will be clear from Section 5 that $\mathscr{M}$ has the smallest possible dimension. In this section we analyze the above two cells and show how the differential of the corresponding generators of G looks.

4.1. **The Tamarkin cell.** The Tamarkin cell $\mathscr{T}$ corresponds to the tree T with the barcode $[1|2||3|4||5|6]$ shown in Figure 11 (left). Consider the tree $S := [1|2]$ and the map $\nu : T \to S$ that sends the tips of T labeled 1,3,5 (reps. labeled 2,4,6) to the tip 1 (resp. tip 2) of S, see Figure 11. We easily read off from the fiber diagram of ν that $\mu[\nu] = \mu\left[[1||3||5]\big|[2||4||6]\right]$. Simple degree count shows that $\dim(\mu[\nu]) = 6$ i.e. it is the same as the dimension of the Tamarkin cell $\mathscr{T}$! The explanation is that the face $\mu[\nu]$ is not a subset of the boundary $\partial\mathscr{T}$ of $\mathscr{T}$ but that $\partial\mathscr{T}$ intersect $\mu[\nu]$ in a 5-dimensional subspace which is not a union of 5-dimensional cells. This violates Definition 14(ii). An "ideological" picture of this situation is shown in Figure 12.

Let us analyze this phenomenon in detail. If we denote by $c : \mathbb{R}^2 \to \mathbb{R}$ the projection to the first coordinate, then each point $\mathbf{x} = (x_1,\ldots,x_6)$ in the interior of $\mathscr{T}$ satisfies

$$
\frac{c(x_3)-c(x_1)}{c(x_5)-c(x_3)} = \frac{c(x_4)-c(x_2)}{c(x_6)-c(x_4)}. \tag{12}
$$

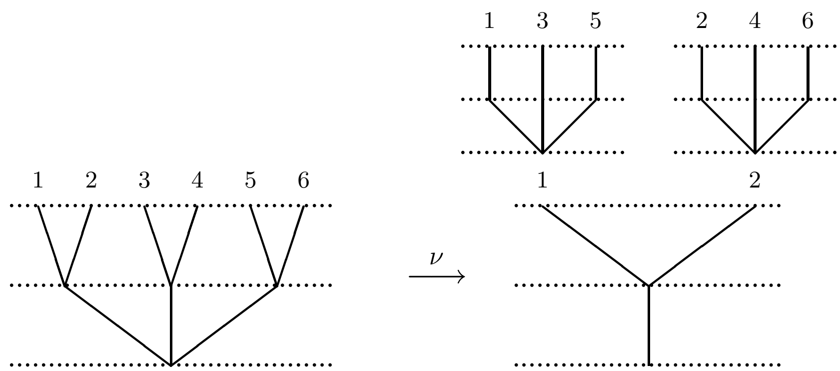


FIGURE 11. The Tamarkin tree T (left), the tree S (right bottom), the map $\nu : T \to S$ and its fiber diagram.

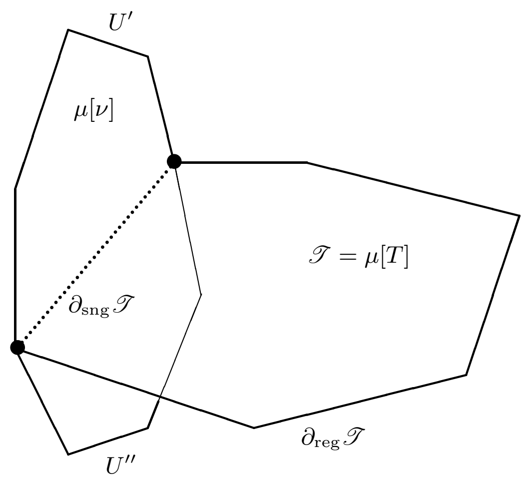


FIGURE 12. The Tamarkin cell, the cell $\mu[\nu]$ and the intersection $\partial_{\rm sng}\mathscr{T} := \partial\mathscr{T} \cap \mu[\nu]$.

Observe that both sides are invariant under the affine group action. The same condition is satisfied also by the points in the intersection $\partial\mathscr{T} \cap \mu[\nu]$. More precisely, $\mu[\nu]$ consists of two 'microscopic' configurations $\mathbf{x}_u$ (resp. $\mathbf{x}_b$) of points (x_2, x_4, x_6) (resp. (x_1, x_2, x_3)) in $\mathsf{F}(3)$ arranged at the vertical line (Figure 13 middle). Since the points in $\partial\mathscr{T} \cap \mu[\nu]$ are the limits of the points in the interior of $\mathscr{T}$, the configurations $\mathbf{x}_u$ and $\mathbf{x}_b$ are still tied by (12), see Figure 13 (right). Therefore the intersection $\partial\mathscr{T} \cap \mu[\nu]$ is a codimension-one subspace of $\mu[\nu]$. Loosely speaking, when the point $\mathbf{x} \in \mathscr{T}$ moves towards the boundary, it still 'remembers' that its coordinates were lined up at tree vertical lines parametrized by a point in $\mathsf{K}(3) = \mathsf{F}_1(3)$. This is a particular instance the 'source-target' condition of in a globular category [1], see also formula (14) in Section 5.

In the rest of this subsection T, S and $\nu : T \to S$ will have the same meaning as above. Let us try to determine the value $\partial(g_T)$ of the differential on the generator $g_T \in G(6)$ corresponding to the Tamarkin tree. Inspired by (10), define

$$\partial_{\rm reg}(g_T) := \textstyle\sum_\sigma \pm\iota(C_\sigma),$$

with the sum taken over all 'regular' faces σ of T, i.e. faces σ such that $\dim(C_\sigma) = \dim(T) - 1$. In Figure 12, the union of these faces is denoted $\partial_{\rm reg}\mathscr{T}$. Since $\partial_{\rm reg}(g_T)$ is of dimension $5 < d_{\rm crit}(6)$, the

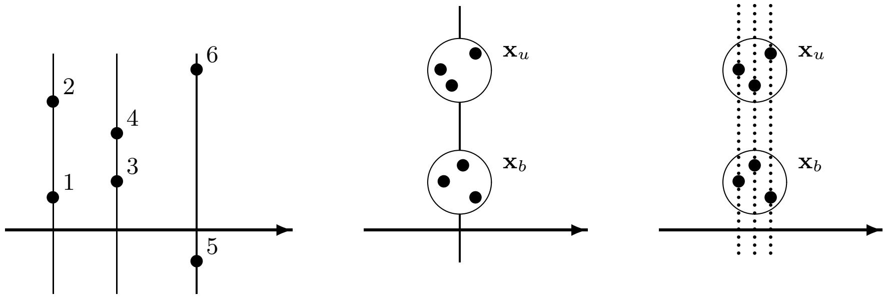


FIGURE 13. A generic point of the Tamarkin cell $\mu[1|2||3|4||5|6]$ (left), of the cell $\nu = \mu\left[[1||3||5]\big|[2||4||6]\right]$ (middle) and of their intersection (right).

value $\partial(\partial_{\rm reg}(g_T))$ is determined by calculations of Subsection 3.1 and equals the sum of elements of $\mathbb{F}(G)$ corresponding to the 4-dimensional cells in the intersection $\partial_{\rm reg}\mathscr{T} \cap \partial_{\rm sng}\mathscr{T}$ marked by two bullets $\bullet$ in Figure 13. In particular, $\partial(\partial_{\rm reg}(g_T)) \neq 0$. We shall find a 'counterterm' $\partial_{\rm sng}(g_T)$ such that $\partial(\partial_{\rm reg}(g_T)) = -\partial(\partial_{\rm sng}(g_T))$ and put

$$\partial(g_T) := \partial_{\rm reg}(g_T) + \partial_{\rm sng}(g_T).$$

The idea of finding such a couterterm is clear from Figure 12; $\partial_{\rm sng}(g_T)$ shall correspond to an union of 5-dimensional cells U such that $\partial U = \partial_{\rm reg}\mathscr{T} \cap \partial_{\rm sng}\mathscr{T}$. In the ideological Figure 13, there are two such unions, the 'upper' U' and the 'lower' U'' which indicates that the choice of U need not be unique.

The first step is to identify 4-dimensional cells in the intersection $\partial_{\rm reg}\mathscr{T} \cap \partial_{\rm sng}\mathscr{T}$. The extended barcodes of these cells will be of the form $[b_1|b_2]$ for some barcodes b_1, b_2. To shorten the formulas, we use an 'additive' notation for the corresponding cells, so that $\mu[b_1' \cup b_1''|b_2] = \mu[b_1'|b_2] \cup \mu[b_1''|b_2]$, &c. With this notation, one easily expresses $\partial_{\rm reg}\mathscr{T} \cap \partial_{\rm sng}$ as the union of 4-dimensional cells:

$$\begin{aligned}
\partial_{\rm reg}\mathscr{T} \cap \partial_{\rm sng}\mathscr{T} \;=\; & \mu\Big[\bigcup_{\tau\in\Sigma_{1,3,5}} [\tau_1|\tau_3|\tau_5] \cup \bigcup_{\tau\in\Sigma'_{1,3,5}} \big[\tau_1\big|[\tau_3||\tau_5]\big] \cup \bigcup_{\tau\in\Sigma''_{1,3,5}} \big[[\tau_1||\tau_3]\big|\tau_5\big]\Big|[2||4||6]\Big] \\
\cup\;& \mu\Big[[1||3||5]\Big|\bigcup_{\tau\in\Sigma_{2,4,6}} [\tau_2|\tau_4|\tau_6] \cup \bigcup_{\tau\in\Sigma'_{2,4,6}} \big[\tau_2\big|[\tau_4||\tau_6]\big] \cup \bigcup_{\tau\in\Sigma''_{2,4,6}} \big[[\tau_2||\tau_4]\big|\tau_6\big]\Big] \\
\cup\;& \mu\Big[\big[[1||3]\big|\big|5\big] \cup \bigcup_{\tau\in\Sigma_{1,3}} [\tau_1|\tau_3||5]\Big|\big[[2||4]\big|\big|6\big] \cup \bigcup_{\tau\in\Sigma_{2,4}} [\tau_2|\tau_4||6]\Big] \\
\cup\;& \mu\Big[\big[1\big|\big|[3||5]\big] \cup \bigcup_{\tau\in\Sigma_{3,5}} [1||\tau_3|\tau_5]\Big|\big[2\big|\big|[4||6]\big] \cup \bigcup_{\tau\in\Sigma_{4,6}} [2\,||\tau_4|\tau_6]\Big].
\end{aligned}$$

In the above display, $\Sigma_{1,3,5}$ is the group of permutations of the set $\{1,3,5\}$, and the symbols $\Sigma_{2,4,6}$, $\Sigma_{1,3}$, $\Sigma_{2,4}$, $\Sigma_{3,5}$ and $\Sigma_{4,6}$ have the obvious similar meanings. Moreover,

$$\begin{aligned}
&\Sigma'_{1,3,5} := \{\tau \in \Sigma_{1,3,5};\ \tau_3 < \tau_5\},\ \Sigma''_{1,3,5} := \{\tau \in \Sigma_{1,3,5};\ \tau_1 < \tau_3\},\\
&\Sigma'_{2,4,6} := \{\tau \in \Sigma_{2,4,6};\ \tau_4 < \tau_6\} \text{ and } \Sigma''_{2,3,6} := \{\tau \in \Sigma_{2,4,6};\ \tau_2 < \tau_4\}.
\end{aligned}$$

The structure of the right hand side should be clear from Figure 14 which shows, without specifying the labels, generic points of the corresponding configurations.

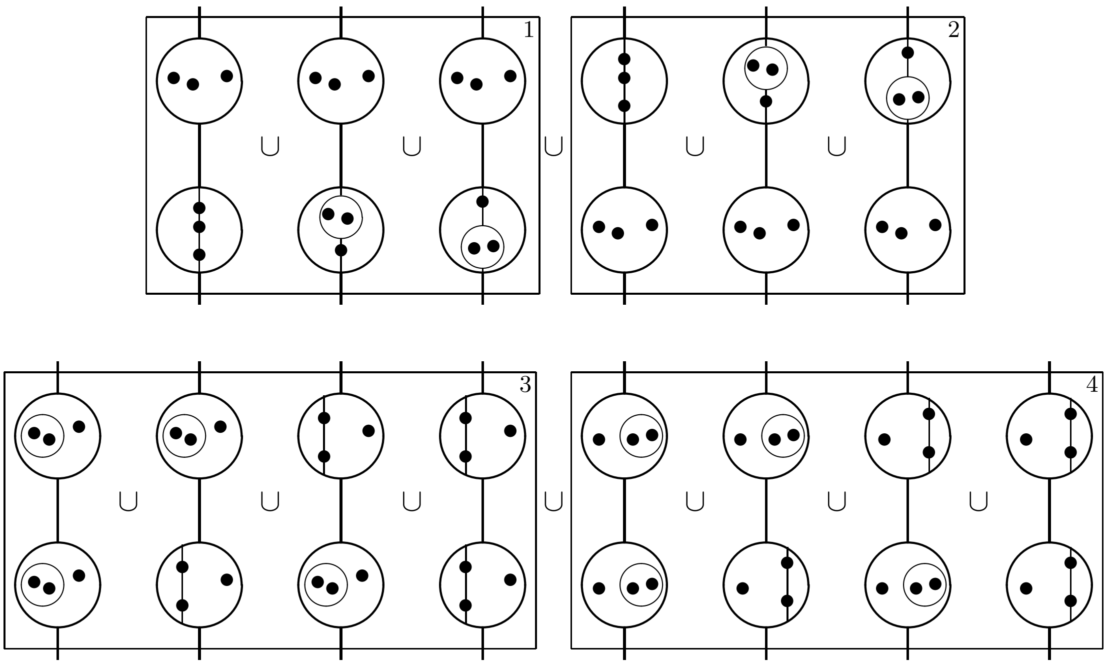


FIGURE 14. 4-dimensional cells in the intersection $\partial_{\rm reg}\mathscr{T} \cap \partial_{\rm sng}\mathscr{T}$.

The four boxes of this figure correspond to the four lines of the display. One of the possible choices for the set U is then

$$U' := \mu\Big[\big[1\big|\big|[3||5]\big] \cup \bigcup_{\tau\in\Sigma_{3,5}} [1||\tau_3|\tau_5]\Big|[2||4||6]\Big] \cup \mu\Big[[1||3||5]\Big|\big[[2||4]\big|\big|6\big] \cup \bigcup_{\tau\in\Sigma_{2,4}} [\tau_2|\tau_4||6]\Big].$$

Another choice is the 'diagonal image'

$$U'' := \mu\Big[\big[[1||3]\big|\big|5\big] \cup \bigcup_{\tau\in\Sigma_{1,3}} [\tau_1|\tau_3||5]\Big|[2||4||6]\Big] \cup \mu\Big[[1||3||5]\Big|\big[2\big|\big|[4||6]\big] \cup \bigcup_{\tau\in\Sigma_{4,6}} [2||\tau_4|\tau_6]\Big].$$

Generic points of the corresponding cells are shown in Figure 15. In both cases, the counterterm $\partial_{\rm sng}(g_T)$ is the sum of 6 terms corresponding to the six 4-cells of U' resp. U''.

4.2. **A 4-dimensional bad cell.** It is the cell $\mathscr{M} := \mu[T]$ indexed by the reduced 3-tree $T := [1|2|||3|4]$ shown in Figure 16. A generic point of this cell is presented in Figure 17. Consider the 3-tree $S := [1|2]$ and the map $\nu : T \to S$ that sends the tips of T labeled 1,3 (reps. 2,4) to the tip 1 (resp. tip 2) of S, see again Figure 16. It is clear that the corresponding face $\mu[\nu]$ of ν equals $\mu\big[[1|||3]\big|[2|||4]\big]$ and that $\dim(\mu[\nu]) = \dim(\mathscr{M}) = 4$.

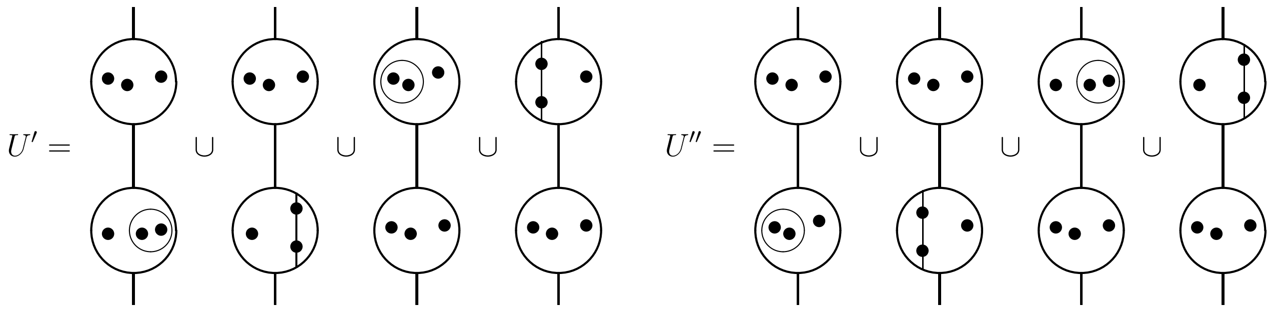


FIGURE 15. Two possible choices of the set U.

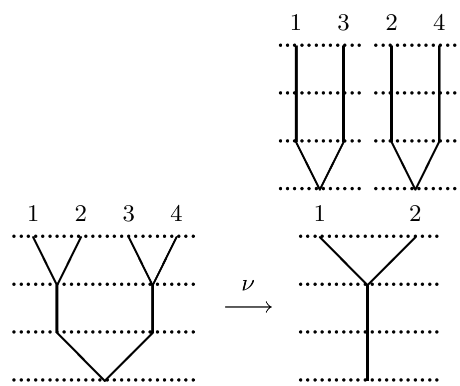


FIGURE 16. The tree T (left), the tree S (right bottom), the map $\nu : T \to S$ and its fiber diagram.

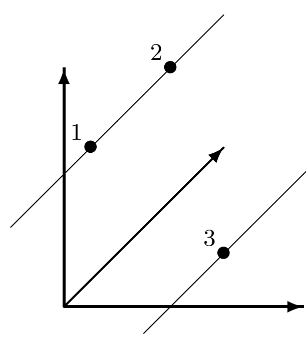


FIGURE 17. A generic point of the cell $\mathscr{M}$. The points in the ambient $\mathbb{R}^3$ lie on two lines parallel to the 3rd coordinate. The 2nd coordinate of the points labeled 1 and 2 is less than the 2nd coordinate of the points labeled 3 and 4.

Let us determine the value $\partial(g_T)$ of the differential on the 4-dimensional generator $g_T \in G_4(4)$ corresponding to T. As in Subsection 4.1, define

$$\partial_{\rm reg}(g_T) := \textstyle\sum_\sigma \pm \iota(C_\sigma),$$

with the sum over all faces σ such that $\dim(C_\sigma) = 3$. One easily sees that these faces form the union

$$\begin{aligned}\partial_{\mathrm{reg}}\mathscr{M} &= \mu[1|2||3|4] \cup \mu[3|4||1|2] \cup \mu\left[1\middle|2\middle|\middle|\middle|[3|4]\right] \cup \mu\left[[1|2]\middle|\middle|\middle|3\middle|4\right] \\ &\cup \bigcup_{\tau\in\Sigma_{1,3}} \mu\left[\tau_1\middle|\tau_3\middle|[2|||4]\right] \cup \mu\left[1\middle|[2|||3]\middle|4\right] \cup \mu\left[3\middle|[1|||4]\middle|2\right] \bigcup_{\tau\in\Sigma_{2,4}} \mu\left[[1|||3]\middle|\tau_2\middle|\tau_4\right].\end{aligned}$$

Observe that the first two terms give the linear part of $\partial_{\mathrm{lin}}(g_T)$. As in Subsection 4.1 we need to describe 2-dimensional cells in the intersection $\partial_{\mathrm{reg}}\mathscr{M} \cap \partial_{\mathrm{sng}}\mathscr{M}$. The result is:

$$\begin{aligned}\partial_{\mathrm{reg}}\mathscr{M} \cap \partial_{\mathrm{sng}}\mathscr{M} &= \mu\left[\bigcup_{\tau\in\Sigma_{1,3}} [\tau_1|\tau_3]\middle|[2|||4]\right] \cup \mu\left[[1|||3]\middle|\bigcup_{\tau\in\Sigma_{2,4}} [\tau_2|\tau_4]\right] \\ &\cup \mu\left[[1||3]\middle|[2||4]\right] \cup \mu\left[[3||1]\middle|[4||2]\right],\end{aligned}$$

where $\Sigma_{1,3}$ (resp. $\Sigma_{2,4}$) is the group of permutations of the set $\{1,3\}$ (resp. $\{1,3\}$). One of the possible choices for the set U of 3-cells generating the counterterm $\partial_{\mathrm{sng}}(g_T)$ is then

$$U' := \mu\left[[1|||3]\middle|[4||2]\right] \cup \mu\left[[1||3]\middle|[2|||4]\right].$$

The second one is the 'diagonal image'

$$U'' := \mu\left[[1|||3]\middle|[2||4]\right] \cup \mu\left[[3||1]\middle|[2|||4]\right].$$

In both cases, the counterterm $\partial_{\mathrm{sng}}(g_T)$ has 2 terms corresponding to two 3-cells of U' resp. U''. The differential $\partial(g_T)$ is the sum of 2 linear terms, 8 regular decomposable terms and 2 singular terms. As in Subsection 4.1, it helps to represent the cells entering the above calculations by depicting their generic points. We leave it to the reader as an exercise.

5. Bad cells

In the first part of this section we analyze the 'source-target' conditions responsible for the existence of bad cells. In the second part we prove Propositions 15 and 16.

5.1. **The source-target conditions.** Assume we are given a pruned unlabeled h-tree $T \in \underline{\mathrm{Tree}}^h(n)$ as in (1). Suppose that there is an $s \geq 2$ and natural numbers

$$1 \leq a_1 < b_1 < a_2 < b_2 \cdots < a_s < b_s \leq k_h = n$$

and some $1 \leq m < h$ such that

$$\rho_m \circ \cdots \circ \rho_{h-1}(a_i) = \rho_m \circ \cdots \circ \rho_{h-1}(b_i), \tag{13}$$

for all $1 \leq i \leq s$. We also assume that the common values of the expression in (13) form a strictly increasing sequence of s elements of $[k_m]$. Suppose there is a h-tree $S \in \underline{\mathrm{Tree}}^h(k)$, $k < n$, and a map $\nu : T \to S$ for which there exist $1 \leq u < v \leq k$ such that $\nu_h(a_i) = u$ and $\nu_h(b_i) = v$ for all $1 \leq i \leq s$.

In this situation, denote A (resp. B) the fiber of ν over u (resp. v) and A' (resp. B') the maximal pruned subtree of A (resp. B) with the tips $a_1, \ldots, a_s$ (resp. $b_1, \ldots, b_s$). Denote finally

$R \in \underline{\mathrm{Tree}}^m(s)$ the pruned unlabeled m-tree obtained from A' by amputating everything above level m. Observe that instead of amputating A' we could have amputated B' with the same result. The situation is visualized in Figure 18.

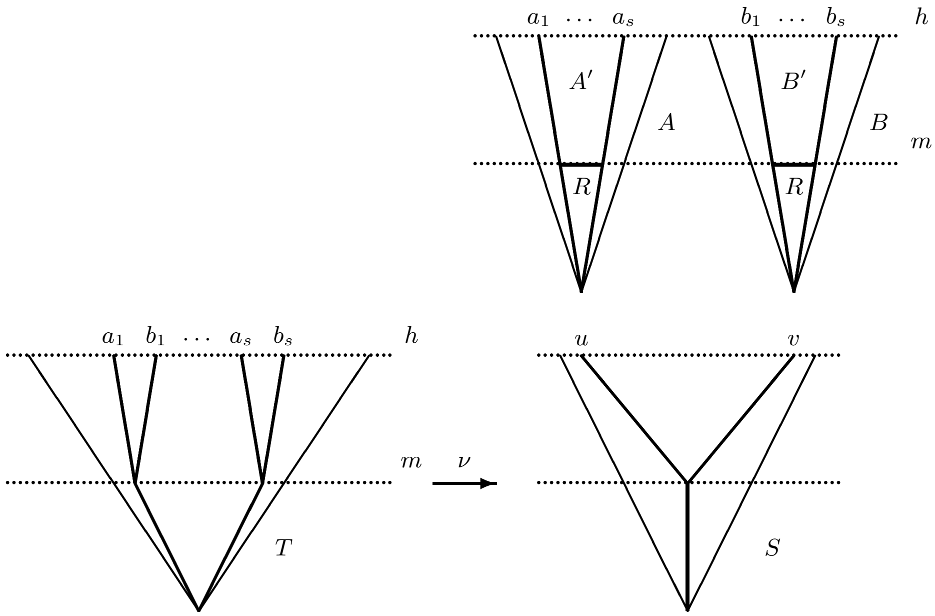


FIGURE 18. The origin of the source-target conditions – schematic picture.

The tree R determines a cell $\mu[R] \subset \mathsf{F}_m(s)$. The *source and target maps* $\pi_s, \pi_t : \mu[C_\nu] \twoheadrightarrow \mu[R]$ are defined as follows. Since $\mu[C_\nu]$ is the cartesian product of the cell $\mu[S]$ with the cells indexed by the fibers of ν, one has the projections $\pi_A : \mu[C] \twoheadrightarrow \mu[A]$ resp. $\pi_B : \mu[C] \twoheadrightarrow \mu[B]$. One also has the 'forgetful' projections $\pi'_A : \mu[A] \twoheadrightarrow \mu[A']$ (resp. $\pi'_B : \mu[B] \twoheadrightarrow \mu[B']$) given by forgetting all points of the configurations in $\mu[A]$ (resp. $\mu[B]$) except those with labels in $\{a_1, \ldots, a_s\}$ (resp. $\{b_1, \ldots, b_s\}$). Let finally $\pi''_A : \mu[A'] \twoheadrightarrow \mu[R]$ (resp. $\pi''_B : \mu[B'] \twoheadrightarrow \mu[R]$) be the projection induced by the projection $\mathbb{R}^h \twoheadrightarrow \mathbb{R}^m$ to the first m coordinates. The maps π_s, π_t are the compositions

$$\pi_s : \mu[C_\nu] \stackrel{\pi_A}{\twoheadrightarrow} \mu[A] \stackrel{\pi'_A}{\twoheadrightarrow} \mu[A'] \stackrel{\pi''_A}{\twoheadrightarrow} \mu[R] \ \text{ and } \ \pi_s : \mu[C_\nu] \stackrel{\pi_B}{\twoheadrightarrow} \mu[B] \stackrel{\pi'_B}{\twoheadrightarrow} \mu[B'] \stackrel{\pi''_B}{\twoheadrightarrow} \mu[R].$$

Definition 20. *The* source-target condition *is the following equality of points of* $\mu[R]$

$$\pi_s(\mathbf{x}) = \pi_t(\mathbf{x}) \tag{14}$$

satisfied by each point $\mathbf{x} \in \mu[C_\nu] \cap \partial\mu[T]$.

Example 21. In the Tamarkin case analyzed in Subsection 4.1, $h = 2$, $n = 6$, $k = 2$, $m = 1$, the trees $T \in \underline{\mathrm{Tree}}^2(6)$, $S \in \underline{\mathrm{Tree}}^2(2)$ and the map $\nu : T \to S$ are as in Figure 11. Moreover, $s = 3$, $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, $b_1 = 2$, $b_2 = 4$, $b_3 = 6$. The amputated tree R equals $[1|2|3]$ so $\mu[R]$ is the open interval $\mathring{\mathsf{F}}_1(3)$.

For the 4-dimensional bad cell of Subsection 4.2, $h = 3$, $n = 4$, $k = 2$, $m = 2$, the trees $T \in \underline{\mathrm{Tree}}^3(4)$, $S \in \underline{\mathrm{Tree}}^3(2)$ and the map $\nu : T \to S$ are as in Figure 16. Moreover, $s = 2$, $a_1 = 1$,

$a_2 = 3$, $b_1 = 2$, and $b_2 = 4$. The amputated tree R equals $[1||2]$ so $\mu[R]$ is the open half-circle in $\mathsf{F}_2(2) = \mathbb{S}^1$.

5.2. **Proofs of Propositions 15 and 16.** Let us introduce the following terminology. We call a cell of F *regular* if its boundary is an union of cells. For subsets $a \subset \mathsf{F}(m)$, $b \subset \mathsf{F}(n)$ and $1 \leq i \leq m$ we write, as expected,

$$a \circ_i b := \{x \circ_i y \in \mathsf{F}(m+n-1);\ x \in a,\ y \in b\}.$$

We call $a \circ_i b$ the $\circ_i$*-composition* of the sets a and b.

Lemma 22. *Let $\sigma \in \Sigma_m$. A cell $e \subset \mathsf{F}(m)$ is a regular if and only if the cell $e\nu := \{x\sigma; x \in e\}$ is regular. The $\circ_i$-composition of cells is regular if and only if and only if both factors are regular.*

Proof. The first part of the lemma is obvious. The second part follows from the equality $\partial(e' \circ_i e'') = \partial e' \circ_i e'' \cup e' \circ_i \partial e''$. □

Theorem 23. *For $n \geq 2$ and $h \geq 1$, each cell of dimension $< d^h_{\rm crit}(n)$ in the open part $\mathring{\mathsf{F}}_h(n)$ is regular and all its faces are regular, too. Similarly, each cell of $\mathring{\mathsf{F}}(n)$ of dimension $< d_{\rm crit}(n)$ is regular along with all its faces.*

Proof. The theorem can be proved by case-studying cells of the indicated dimensions. We, however, prefer a conceptual approach based on the analysis of the source-target conditions given in Subsection 5.1. In particular, we observe that condition (14) is nontrivial only if the dimension of the cell $\mu[R]$ is at least one. This implies that

(i) either $s \geq 2$ and $m \geq 2$, or
(ii) $s \geq 3$ and $m \geq 1$.

In case (i), one has $n \geq 4$ and $h \geq 3$, while in case (ii) one has $n \geq 6$ and $h \geq 2$. Let us prove that each $e \subset \mathring{\mathsf{F}}_h(n)$ with $\dim(e) < d^h_{\rm crit}(n)$ is regular. We again distinguish two cases:

(a) $d^h_{\rm crit}(n) = \infty$, which means $n = 2, 3$, or $n = 4, 5$ and $h \leq 2$, or $n \geq 6$ and $h = 1$,
(b) $d^h_{\rm crit}(n) = n$, which means $n = 4, 5$ and $h \geq 3$, or $n \geq 6$ and $h \geq 2$.

Case (a) is complementary to the cases (i) and (ii) above, therefore the source-target conditions are trivial and e is a regular cell. The faces of the cell e are $\circ_i$-compositions of cells e' from $\mathring{\mathsf{F}}_h(n')$ for some $2 \leq n' \leq n$. Since, by the definition of the critical dimension, $d^h_{\rm crit}(n') \geq d^h_{\rm crit}(n)$, one has $d^h_{\rm crit}(n') = \infty$. As we already established, this implies that each such an e' is regular, therefore, by Lemma 22, each face of e is regular, too.

Let us assume (b), i.e. $d^h_{\rm crit}(n) = n$. Since, by Lemma 22, the symmetric group action preserves the regularity, we may assume that $e = \mu[T]$ for a reduced tree $T \in \underline{\rm Tree}^h(n)$ as in (1) with $\dim(T) < n$. By simple combinatorics,

(α) either $h = 1$ or

(β) $h = 2$ and $k_1 = 2$.

In the first case the cell $\mu[T]$ belongs to the Stasheff polytope $\mathsf{K}(n) = \mathsf{F}_1(n) \subset \mathsf{F}(n)$, so $\mu[T]$ and all its faces are regular cells. In case (β), the amputated tree R must be $[1|2]$, thus the corresponding source-target condition is trivial, so $\mu[T]$ is a regular cell. It is not difficult to verify that if T satisfies (β), S is an arbitrary tree and $\nu : T \to S$ a map, then each reduced fiber of ν also satisfies (α) or (β) above. This implies that all faces of $\mu[T]$ are regular cells, too.

This finishes the proof of the first part of the theorem. Since each cell $e \subset \mathring{\mathsf{F}}(n)$ belongs to some $\mathring{\mathsf{F}}_h(n)$, $h \geq 1$, the second part follows from the first part and the inequality $d_{\mathrm{crit}}(n) \leq d^h_{\mathrm{crit}}(n)$. □

Proof of Proposition 15. The compatibility of the CW-structures of the Σ-modules F_h, $h \geq 1$, and F with the operad structures and the freeness of the symmetric group action on the cells follows from the very definition of the cell structure reviewed on page 11. Recall [2, 21] that there is the canonical embedding

$$\iota : \mathsf{F}_h(n) \hookrightarrow \bigtimes_{\substack{1 \leq i,j \leq n \\ i \neq j}} \mathbb{S}^{h-1} \times \bigtimes_{\substack{1 \leq i,j,k \leq n \\ i \neq j,\ j \neq k,\ k \neq i}} [0, \infty].$$

It follows from the analysis of the images of the cells of $\mathsf{F}_h(n)$ under ι given in Section 6 of [2], namely from Proposition 6.1 of that section, that the spaces $\mathsf{F}_h(n)$ satisfy condition (i) of Definition 14 for arbitrary n and h. The analogous claim for $\mathsf{F}(n)$ stems from the fact that each cell of $\mathsf{F}(n)$ belongs to the subcomplex $\mathsf{F}_h(n)$ for some $h \geq 1$.

Since condition (i) of Definition 14 has already been established, the space $\mathsf{F}_h(n)$ is regular if and only if each its cell is regular in the sense introduced at the beginning of this subsection. Since the cells of $\mathsf{F}_h(n)$ are iterated $\circ_i$-compositions of the cells from $\mathring{\mathsf{F}}_h(n')$ with $n' \leq n$, the complex $\mathsf{F}_h(n)$ is, by Lemma 22, regular if and only if all cells of $\mathring{\mathsf{F}}_h(n')$ are regular, for each $2 \leq n' \leq n$. Since clearly $d^h_{\mathrm{crit}}(n) = \infty$ implies $d^h_{\mathrm{crit}}(n') = \infty$ for each $n' \leq n$, the space $\mathsf{F}_h(n)$ is, by Theorem 23, regular if $d^h_{\mathrm{crit}}(n) = \infty$.

The non-regularity of $\mathsf{F}_h(n)$ if $d^h_{\mathrm{crit}}(n)$ is finite follows from Proposition 16 proved below. This, by the remark following Proposition 15, proves the characterization of the regularity of the spaces $\mathsf{F}_h(n)$. The similar obvious analysis applies to $\mathsf{F}(n)$ as well.

The regularity of the suboperads $\mathsf{F}^{\mathrm{reg}}_h$ and $\mathsf{F}^{\mathrm{reg}}$ follows from the fact that they are generated by the regular cells in $\mathring{\mathsf{F}}_h$ resp. $\mathring{\mathsf{F}}$, from Lemma 22 and the fact that cells in the boundary of regular cells are regular established in Theorem 23. This finishes the proof. □

Proof of Proposition 16. We put $e^3_4 := \mathscr{M} = \mu[1|2|||3|4]$, the cell introduced in Subsection 4.2. The cell e^h_4 for $h \geq 4$ is defined as the image of e^3_4 under the natural inclusion $\mathsf{F}_3(4) \hookrightarrow \mathsf{F}_h(4)$. Likewise, we put $e^3_5 := \mu[1|2|||3|4|5]$ and e^h_5 for $h \geq 4$ defined similarly.

The cell e^2_6 is the Tamarkin cell $\mathscr{T} = \mu[1|2||3|4||5|6]$, and $e^2_n := \mu[1|2||3|4||5|6| \cdots |n]$, for $n \geq 7$. The cells e^h_n for $h \geq 3$ and $n \geq 6$ are the images of e^2_n under the natural inclusions $\mathsf{F}_3(n) \hookrightarrow \mathsf{F}_h(n)$. The cells e_n, $n \geq 4$, are the images of e^3_n under the inclusions $\mathsf{F}_3(n) \hookrightarrow \mathsf{F}(n)$. We leave to the reader to verify that the cells e^h_n defined in this way are bad. □

[October 2, 2009]

6. Free Lie algebras and configuration spaces

In this section we prove integral variants of some results whose characteristic zero versions are known. Therefore, all algebraic objects will be considered over the ring $\mathbb{Z}$ of integers. The results below easily generalize to an arbitrary integral domain with unit.

Let $\mathbb{T}(x_1,\ldots,x_n)$ be the tensor algebra with generators $x_1,\ldots,x_n$, $n\geq 1$, and $\mathbb{L}(x_1,\ldots,x_n)$ the free Lie algebra on the same set of generators, considered in the standard way as a subspace of $\mathbb{T}(x_1,\ldots,x_n)$. We denote by $\mathbb{T}_{1,\ldots,1}(x_1,\ldots,x_n)\subset\mathbb{T}(x_1,\ldots,x_n)$ the linear subspace spanned by words containing each of the generators $x_1,\ldots,x_n$ precisely once, and $\mathbb{L}_{1,\ldots,1}(x_1,\ldots,x_n):=\mathbb{L}(x_1,\ldots,x_n)\cap\mathbb{T}_{1,\ldots,1}(x_1,\ldots,x_n)$. We will sometimes simplify the notation and denote $\mathcal{L}ie(n):=\mathbb{L}_{1,\ldots,1}(x_1,\ldots,x_n)$.

Each space above has a natural right Σ_n-module action permuting the generators. Take another set of generators $\alpha_1,\ldots,\alpha_n$ and denote by

$$\Phi:\mathbb{T}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)\to\mathbb{T}_{1,\ldots,1}(x_1,\ldots,x_n)', \tag{15}$$

where $\mathbb{T}_{1,\ldots,1}(x_1,\ldots,x_n)'$ is the linear dual of $\mathbb{T}_{1,\ldots,1}(x_1,\ldots,x_n)$, the isomorphism defined by

$$\Phi(\alpha_{\rho(1)}\otimes\cdots\otimes\rho_{\sigma(n)})(x_{\omega(1)}\otimes\cdots\otimes x_{\omega(n)}):=\begin{cases}1, & \text{if } \rho=\omega, \text{ and}\\ 0, & \text{otherwise,}\end{cases}$$

where $\rho,\omega\in\Sigma_n$.

For $s,t\geq 1$ denote by $\Sigma_{s,t}$ is the set of all (s,t)-*unshuffles*, i.e. permutations $\tau\in\Sigma_n$, $n=s+t$, such that

$$\tau(1)<\cdots<\tau(s) \text{ and } \tau(s+1)<\cdots<\tau(s+t).$$

Let, finally, $\mathrm{Ush}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)\subset\mathbb{T}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)$ be the linear span of the expressions

$$\sum_{\tau\in\Sigma_{s,t}}\alpha_{\rho\tau(1)}\otimes\cdots\otimes\alpha_{\rho\tau(n)},$$

for $\rho\in\Sigma_n$ and $s,t\geq 1$ such that $s+t=n$.

Theorem 24. *The map (15) induces a Σ_n-equivariant isomorphism*

$$\underline{\Phi}:\mathbb{T}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)/\mathrm{Ush}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)\to\mathbb{L}_{1,\ldots,1}(x_1,\ldots,x_n)'.$$

Theorem 24 will follow from a sequence of claims. The first one is probably known, but we were unable to find a reference.

Claim 25. *For each $n\geq 1$, $\mathcal{L}ie(n)=\mathbb{L}_{1,\ldots,1}(x_1,\ldots,x_n)$ is the free abelian group with basis*

$$b_\lambda:=[x_{\lambda(1)},[x_{\lambda(2)},\ldots,[x_{\lambda(n-1)},x_n]\cdots]],\ \ \lambda\in\Sigma_{n-1}. \tag{16}$$

Proof. It is known that $\mathbb{L}(x_1,\ldots,x_n)$ is torsion free (see, for instance, the overview in [5]), so its subspace $\mathcal{L}ie(n)$ is torsion-free as well. Let us prove, by induction on n, that the elements in (16) span $\mathcal{L}ie(n)$.

This statement is obvious for $n = 1, 2$. Let $n > 2$. Since $\mathcal{L}ie(n)$ is spanned by elements of the form $[F_1, F_2]$, $F_1 \in \mathcal{L}ie(s)$, $F_2 \in \mathcal{L}ie(t)$, $s + t = n$, $s, t \geq 1$, it suffices to prove that each such $[F_1, F_2]$ is a linear combination of elements of the basis (16). We may clearly assume that F_1 contains $x_1, \ldots, x_s$ and F_2 contains $x_{s+1}, \ldots, x_n$. By induction, F_2 is a linear combination of iterated commutators as in (16), with x_n at the extreme right position.

Now we proceed by induction on s. If $s = 1$, $[F_1, F_2]$ is an element of (16). If $s \geq 2$ we may assume that $F_1 = [A, B]$, for some $A \in \mathcal{L}ie(a)$, $B \in \mathcal{L}ie(b)$, with $a + b = s$, $a, b \geq 1$. In that case

$$[F_1, F_2] = [A, [B, F_2]] + [B, [F_2, A]].$$

By induction on n, both $[B, F_2]$ and $[F_2, A]$ are combinations of the commutators as in (16) with x_n at the rightmost place, therefore both $[A, [B, F_2]]$ and $[B, [F_2, A]]$ are combination of basic elements (16), by induction on s.

The linear independence of the elements (16) follows from the well-known fact that the dimension of $\mathcal{L}ie(n)$ is $(n-1)!$ [5], which is the number of elements (16). □

There is a straightforward way to verify the linear independence of the elements (16) based on Claim 26 below which will be useful also for other purposes. Recall that $\mathbb{T}_{1,\ldots,1}(x_1, \ldots, x_n)$ is the free abelian group with basis

$$e_\sigma := x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}, \ \sigma \in \Sigma_n. \tag{17}$$

Claim 26. *In a (unique) decomposition of* b_λ, $\lambda \in \Sigma_{n-1}$, *into a linear combination of the basis* $\{e_\sigma\}_{\sigma \in \Sigma_n}$, *the element*

$$e_{\lambda \times 1} = x_{\lambda(1)} \otimes \cdots \otimes x_{\lambda(n-1)} \otimes x_n$$

appears with coefficient 1 *and* b_λ *is the only basis element (16) whose decomposition contains* $e_{\lambda \times 1}$.

Proof. A simple induction on n. □

The following proposition is based on famous Theorem 2.2 of [20] that, however, assumes the existence of a solution $\xi \in R$ of the equation $n\xi = \alpha$ for each natural $n \geq 1$ and each $\alpha \in R$, in the ground ring R. We show that this assumption is not necessary when this theorem is applied to the subspace $\mathbb{T}_{1,\ldots,1}(x_1, \ldots, x_n)$ not to the whole $\mathbb{T}(x_1, \ldots, x_n)$.

Proposition 27. *An element* $F = \sum_{\sigma \in \Sigma_n} a_\sigma \cdot x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \in \mathbb{T}_{1,\ldots,1}(x_1, \ldots, x_n)$, $a_\sigma \in \mathbb{Z}$, *belongs to the subspace* $\mathbb{L}_{1,\ldots,1}(x_1, \ldots, x_n)$ *if and only if*

$$\textstyle\sum_{\tau \in \Sigma_{s,t}} a_{\rho\tau} = 0, \tag{18}$$

for each permutation $\rho \in \Sigma_n$ *and each* $s, t \geq 1$ *such that* $s + t = n$.

Proof. By analyzing the proof of [20, Theorem 2.2], one sees that (18) in fact implies $nF \in \mathbb{L}_{1,\ldots,1}(x_1,\ldots,x_n)$. By Claim 25 this means that $nF = \sum_{\lambda\in\Sigma_{n-1}} \beta_\lambda \cdot b_\lambda$, for some $\beta_\lambda \in \mathbb{Z}$ and b_λ the commutators in (16). On the other hand, it follows from Claim 26 that, in the expression

$$nF = \textstyle\sum_{\sigma\in\Sigma_n} na_\sigma \cdot x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},$$

one has $na_{\lambda\times 1} = \beta_\lambda$, for each $\lambda \in \Sigma_{n-1}$. This means that $F = \sum_{\lambda\in\Sigma_{n-1}} a_{\lambda\times 1} \cdot b_\lambda$, so F is a Lie element. □

Another important piece of the proof of Theorem 24 is

Claim 28. *The restriction* $r : \mathbb{T}_{1,\ldots,1}(x_1,\ldots,x_n)' \to \mathbb{L}_{1,\ldots,1}(x_1,\ldots,x_n)'$ *is an epimorphism.*

Proof. We need to show that an arbitrary linear map $\varphi : \mathbb{L}_{1,\ldots,1}(x_1,\ldots,x_n) \to \mathbb{Z}$ extends into a linear map $\widetilde{\varphi} : \mathbb{T}_{1,\ldots,1}(x_1,\ldots,x_n) \to \mathbb{Z}$. Let $\{b_\lambda\}_{\lambda\in\Sigma_{n-1}}$ be the basis (16) of $\mathbb{L}_{1,\ldots,1}(x_1,\ldots,x_n)$, $\{e_\sigma\}_{\sigma\in\Sigma_n}$ the basis (17) of $\mathbb{T}_{1,\ldots,1}(x_1,\ldots,x_n)$ and put

$$\widetilde{\varphi}(e_\sigma) := \begin{cases} \phi(b_\lambda), & \text{if } \sigma = \lambda\times 1 \text{ for some } \lambda\in\Sigma_{n-1}, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

By Claim 26, $\widetilde{\varphi}$ defined in this way extends φ. □

Let K denote the kernel of the composition

$$\mathbb{T}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n) \xrightarrow{\Phi} \mathbb{T}_{1,\ldots,1}(x_1,\ldots,x_n)' \xrightarrow{r} \mathbb{L}_{1,\ldots,1}(x_1,\ldots,x_n)',$$

where Φ is as in (15) and r the restriction.

Claim 29. *An element* $x \in \mathbb{T}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)$ *belongs to the kernel* K *if and only if there exists a natural* N *such that* $N\cdot x \in \mathrm{Ush}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)$.

Proof. It is an elementary consequence of Proposition 27 that

$$K = (\mathrm{Ush}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)^\perp)^\perp \supset \mathrm{Ush}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n),$$

where $^\perp$ denotes the annihilator in the dual space. It is another elementary fact that, for any subspace A of a finite-dimensional vector space V, one has $(A^\perp)^\perp \cong A$, therefore, after extending the scalars to $\mathbb{Q} = \mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}$, the inclusion in the above display becomes an isomorphism. The claim follows. □

Claim 29 implies that, in the composition

$$\underline{\Phi} : \mathbb{T}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)/\mathrm{Ush}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n) \overset{\pi}{\twoheadrightarrow} \mathbb{T}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)/K \xrightarrow{\cong} \mathbb{L}_{1,\ldots,1}(x_1,\ldots,x_n)',$$

the kernel of the projection π consists of torsion elements. The second map, induced by r, is an isomorphism by Claim 28. Theorem 24 will thus be established if we prove

Claim 30. *The abelian group* $T_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)/\mathrm{Ush}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)$ *is torsion-free.*

To prove the claim we need some properties of configuration spaces. We do not know a purely algebraic proof. As in Section 3, let $\mathrm{Cnf}(\mathbb{R}^h, n)$ be the configuration space of n distinct labeled points in the h-dimensional Euclidean space $\mathbb{R}^h$, $n, h \geq 1$. It is known [24] that (4) induces a cell decomposition

$$\overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n) = \textstyle\bigcup_{\mathbf{T} \in \mathrm{Tree}^h(n)} [\overset{\bullet}{\mathbf{T}}],$$

of the one-point compactification $\overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n)$ of $\mathrm{Cnf}(\mathbb{R}^h, n)$, where $[\overset{\bullet}{\mathbf{T}}]$ is the closure of $[\mathbf{T}]$ in $\overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n)$. Let $F_p \overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n) := \bigcup\{[\overset{\bullet}{\mathbf{T}}];\ \dim([\overset{\bullet}{\mathbf{T}}]) \leq p\}$. We have an increasing bounded filtration

$$\emptyset \subset F_{h+n-1} \overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n) \subset \cdots \subset F_{hn} \overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n) = \overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n)$$

which induces a spectral sequence converging to the reduced homology $\overline{H}_*(\overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n))$. Let us denote by $(G^h_*(n), \partial^h)$ the E^1-term of this spectral sequence desuspended $(h+1)$-times,[6]

$$G^h_*(n) := \bigoplus_{p+q=*+h+1} \overline{H}_{p+q}(F_p \overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n)/F_{p-1} \overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n)),$$

with the induced differential. The quotient $F_p \overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n)/F_{p-1} \overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n)$ is isomorphic to the cluster of p-dimensional spheres indexed by trees in $\mathrm{Tree}^h(n)$ with p edges, therefore $G^h_d(n)$ is spanned by labeled pruned h-trees with $d + h + 1$ edges, $G^h_d(n) = \mathrm{Span}(\mathrm{Tree}^h_d(n))$. So $G^h_*(n)$ agrees with the graded abelian group introduced under the same name on page 5. Our spectral sequence degenerates at the E^2-level, therefore,

$$H_*(G^h_*(n), \partial^h) \cong \overline{H}_{*+h+1}(\overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n)),$$

while, by the Poincaré-Lefschetz duality [23, Section 13.3],

$$\overline{H}_*(\overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^h, n)) \cong H^{hn-*}(\mathrm{Cnf}(\mathbb{R}^h, n)).$$

The cohomology in the right hand side of the above display is known [7, Theorem 1.6]; for the purpose of this paper, it is enough to recall that $H^*(\mathrm{Cnf}(\mathbb{R}^h, n))$ is torsion-free and nontrivial only in degrees $i(h-1)$, $0 \leq i \leq n-1$. The above results combine into

Claim 31. *The homology $H_*(G^h_*(n), \partial^h)$ is torsion-free and concentrated in degrees $(n-2) + i(h-1)$, $0 \leq i \leq n-1$.*

There is a natural degree zero dg-monomorphism $\iota : (G^h_*(n), \partial^h) \hookrightarrow (G^{h+1}_*(n), \partial^{h+1})$ that sends the generator e_T indexed by $T \in \mathrm{Tree}^h(n)$ into the generator $e_{s(T)}$ indexed by the suspension $s(T) \in \mathrm{Tree}^{h+1}(n)$. By simple combinatorics, $G^h_d(n) = G^{h+1}_d(n)$ whenever $d \leq h(n-1) - 1$. Let us denote

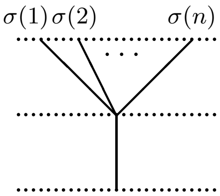
$$(G_*(n), \partial) := \varinjlim\, (G^h_*(n), \partial^h).$$

It is clear that $G_*(n)$ is the span of the graded set $\mathrm{Tree}_*(n)$ so it coincides with the graded abelian group introduced in Definition 3. It is not difficult to see that the differential ∂ is the differential ∂_{lin} of Definition 5 but we will not use this fact. Observe that $G_d(n) = 0$ for $d < n-2$. The above results imply

[6]The number $h+1$ equals the dimension of the affine group of $\mathbb{R}^h$.

Claim 32. *One has* $H_d(G_*(n), \partial) = 0$ *for* $d \neq n-2$ *while* $H_{n-2}(G_*(n), \partial)$ *is torsion-free.*

Let us calculate $H_{n-2}(G_*(n), \partial)$. Since $G^h_d(n) = G^2_d(n)$ for $d \leq n-1$, clearly $H_{n-2}(G_*(n), \partial) \cong H_{n-2}(G^2_*(n), \partial^2)$. The space $G^2_{n-2}(n)$ is spanned by labeled corollas of height two

(19) 

$, \sigma \in \Sigma_n,$

therefore $G_{n-2}(n) \cong \mathbb{T}_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n)$, with the Σ_n-equivariant isomorphism sending the above corolla into the generator $\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(n)}$. The corolla (19) corresponds to the cell of $\overset{\bullet}{\mathrm{Cnf}}(\mathbb{R}^2, n)$ whose generic point is shown in Figure 19 (left).

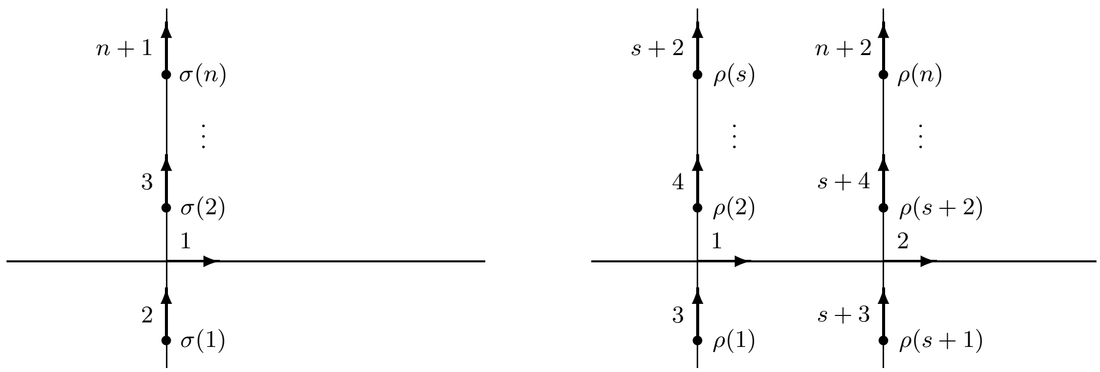
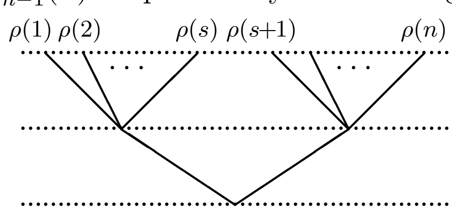


FIGURE 19. Generic points of the Fox-Neuwirth cells.

The little arrows numbered $1, \ldots, n+1$ indicate a frame in the tangent bundle determining the orientation. Likewise, $G^2_{n-1}(n)$ is spanned by trees of height two

(20) 

$, \rho \in \Sigma_n,\ 1 \leq s < n,$

representing the cell whose generic point is shown in Figure 19 (right).

By imagining how a generic point of the cell corresponding to the tree (20) moves to the boundary, one sees that the differential ∂ sends this tree into the element that, under the isomorphism $G_{n-2}(n) \cong \mathbb{T}_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n)$, equals

$$\sum_{\tau \in \Sigma_{s,t}} \mathrm{sgn}(\tau) \cdot \alpha_{\rho\tau(1)} \otimes \cdots \otimes \alpha_{\rho\tau(n)} \tag{21}$$

(observe the sgn(τ)-factor), therefore

$$H_{n-1}(G_*, \partial) \cong \mathbb{T}_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n)/\widetilde{\mathrm{Ush}}_{1,\ldots,1}(\alpha_1, \ldots, \alpha_n), \tag{22}$$

where $\widetilde{\mathrm{Ush}}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)$ denotes the span of elements in (21). We, however, have

Claim 33. *Let* sgn *denote the signum representation. There is a Σ_n-equivariant isomorphism*

$$\mathbb{T}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)/\widetilde{\mathrm{Ush}}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n) \cong \mathrm{sgn}\otimes\mathbb{T}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)/\mathrm{Ush}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n).$$

Proof. The isomorphism $\Psi : \mathbb{T}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n) \stackrel{\cong}{\to} \mathrm{sgn}\otimes\mathbb{T}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)$ given by

$$\Psi(\alpha_{\sigma(1)}\otimes\cdots\otimes\alpha_{\sigma(n)}) := \mathrm{sgn}(\sigma)\otimes\alpha_{\sigma(1)}\otimes\cdots\otimes\alpha_{\sigma(n)},\ \sigma\in\Sigma_n,$$

clearly restricts to an isomorphism $\widetilde{\mathrm{Ush}}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n) \cong \mathrm{Ush}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)$ and induces the isomorphism of the claim. □

By Claim 32 and isomorphism (22), $\mathbb{T}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)/\widetilde{\mathrm{Ush}}_{1,\ldots,1}(\alpha_1,\ldots,\alpha_n)$ is torsion-free, which, by Claim 33, proves Claim 30 and therefore also Theorem 24. Recall that the space $G_*(n)$ inherits a natural free Σ_n-action given by relabeling the spanning trees. As a combination of the above results we get

Theorem 34. *The tree complex $(G_*(n),\partial)$ is a Σ_n-free resolution of the Σ_n-module* $\mathrm{sgn}\otimes\mathcal{L}ie(n)'$.

Tensoring $(G_*(n),\partial)$ with the signum representation therefore leads to a free resolution of $\mathcal{L}ie(n)'$. The following example shows that $(G_*(n),\partial)$ is not the smallest possible.

Example 35. Inspecting Figure 3, one sees that $G_1(3)$ has one Σ_3-generator [1|2|3], $G_2(3)$ two Σ_3-generators [1|2||3] and [1||2|3], and $G_3(3)$ three Σ_3-generators [1||2||3],[1|2|||3] and [1|||2|3]. In general, the number of generators of the free $\mathbb{Z}[\Sigma_3]$-module $G_d(3)$ equals d. One has

$$\partial([1|2||3]) = [1|2|3] - [1|3|2] + [3|1|2] = \partial([3||1|2])$$

and

$$\partial([1|2|||3]) = [1|2||3] - [3||1|2].$$

This shows that one of the generators of $G_2(3)$ is superfluous, so there is a resolution with only one Σ_3-generator in degree 2. We believe in the existence of a free Σ_3-resolution $(\widetilde{G}_*(3),\partial)$ of $\mathcal{L}ie(3)'$ in which $\widetilde{G}_d(3)$ has $[\frac{d-1}{2}]+1$ Σ_3-generators, where $[-]$ denotes the integral part, $d\geq 1$.

Notice that the group ring $R[\Sigma_n]$ does not have good properties even for R a characteristic zero field. For the augmentation ideal $\mathcal{I}$ of $R[\Sigma_n]$ one has $\mathcal{I}/\mathcal{I}^2 = 0$, so there is no good notion of minimality of Σ_n-resolutions of Σ_n-modules.

[October 2, 2009]

Glossary

$\mathbb{F}(-)$,	free operad functor,	page 1
$\underline{\mathbb{F}}(-)$,	free non-Σ operad functor,	page 2
$\underline{\mathrm{Tree}}^h(n)$,	set of pruned h-trees with n tips,	page 4
$\mathrm{Tree}^h(n)$,	set of labeled pruned h-trees with n tips,	page 4
$\underline{\mathrm{Tree}}(n)$,	set of reduced trees of arbitrary height, equals $\varinjlim \underline{\mathrm{Tree}}^h(n)$,	page 4
$\mathrm{Tree}(n)$,	set of reduced labeled trees of arbitrary height, equals $\varinjlim \mathrm{Tree}^h(n)$,	page 4
$G^h(n)$,	right Σ_n-module spanned by $\mathrm{Tree}^h(n)$,	page 5
$G(n)$,	right Σ_n-module spanned by $\mathrm{Tree}(n)$, equals $\varinjlim G^h$,	page 4
$\mathsf{B}(A)$,	bar construction of an associative algebra A,	page 5
$\mathsf{B}^h(A)$,	hth iterate of the bar construction of an ass. comm. algebra,	page 6
$\widehat{\mathsf{B}}^h_*(A)$,	desuspension $\downarrow^{h+1} \mathsf{B}^h_*(A)$,	page 6
$\widehat{\mathsf{B}}^\infty(A)$,	direct limit $\varinjlim \widehat{\mathsf{B}}^h_*(A)$,	page 6
$\mathrm{Cnf}(\mathbb{R}^h, n)$,	configuration space of distinct labeled points in $\mathbb{R}^n$,	page 2
$\mathring{\mathsf{F}}_h(n)$,	moduli space $\mathrm{Cnf}(\mathbb{R}^h, n)/\mathrm{Aff}(\mathbb{R}^h)$,	page 8
$\mathsf{F}_h(n)$,	Fulton-MacPherson compactification of $\mathring{\mathsf{F}}_h(n)$,	page 8
$\mathsf{F}(n)$,	direct limit $\varinjlim \mathsf{F}_h(n)$,	page 8
$[\mathbf{T}]$,	cell of $\mathrm{Cnf}(\mathbb{R}^h, n)$ indexed by a labeled tree $\mathbf{T} = (T, \ell)$,	page 10
$\mu[\mathbf{T}]$,	quotient $[\mathbf{T}]/\mathrm{Aff}(\mathbb{R}^h)$,	page 10
$\mathrm{Tree}^{\mathrm{reg},h}(n)$,	subset of $\mathrm{Tree}^h(n)$ of trees of dimension $< d^h_{\mathrm{crit}}(n)$,	page 11
$\mathrm{Tree}^{\mathrm{reg}}(n)$,	subset of $\mathrm{Tree}(n)$ of trees of dimension $< d_{\mathrm{crit}}(n)$, equals $\varinjlim \mathrm{Tree}^{\mathrm{reg},h}_*$,	page 12
$\mathsf{F}^{\mathrm{reg}}_h$, $\mathsf{F}^{\mathrm{reg}}$,	regular skeleton of the configuration operad F_h resp. F,	page 12
$\mu[\sigma]$,	cell of F corresponding to a face $\sigma : T \to S$ of T,	page 16
$(\sigma_1, \ldots, \sigma_n)$,	notation for a permutation $\sigma \in \Sigma_n$, $\sigma_i := \sigma^{-1}(i)$, $1 \le i \le n$,	page 18
$\mathscr{T}$,	Tamarkin cell, equals $\mu[1\|2\|\|3\|4\|\|5\|6]$,	page 20
$\mathscr{M}$,	a 4-dimensional bad cell, equals $\mu[1\|2\|\|\|3\|4]$,	page 23

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