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SOME PROBLEMS ON ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. We survey some results on ordinary differential equations in Banach spaces and pose several related open problems. This survey has been prepared for a special volume celebrating the profound mathematical work of Professor Manuel Valdivia. We wish Professor Valdivia many years of wonderful mathematics in the future.

1. Introduction

The purpose of the present note is to draw attention of Banach space theorists to the topic of ordinary differential equations, particularly in the Banach space setting. Our approach is to consider a few fundamental theorems of the finite-dimensional theory and survey their analogues in the infinite-dimensional setting. By doing so we will present some well-known as well as some new theorems and open problems. The main difference between the finite and the infinite-dimensional situation lies in the absence of compactness in the latter case. So in order to obtain positive results strong additional assumptions are usually needed. On the other hand, the lack of compactness is utilized in constructions of counterexamples. However, and this is perhaps the most interesting part, the infinite dimensional theory in its arguments often rely on nontrivial structural properties and results from Banach space theory. For example reflexivity and James' theorem, Markushevich and Schauder basis, smooth renormings or properties of operators. Theorems on infinite dimensional spaces usually call for restrictive assumptions on the Banach space involved. So it is not completely clear, at the moment, if the infinite dimensional theory is "trivial", in the sense that it depends only on the infinite dimensionality, or if it distinguishes among particular classes of Banach spaces. This seems to be the main broad question of the subject in the infinite dimensional case, and the main motivation for our survey.

Let us start by formulating the classical initial value problem for ordinary differential equations. Let X be a real Banach space and $f : \mathbb{R} \times X \to X$ be a continuous mapping. We have the corresponding ordinary differential equation

$$x' = f(t, x) \tag{1}$$

together with the initial condition

$$x(t_0) = x_0 \tag{2}$$

Given an open interval $J \subset \mathbb{R}$, we say that $x : J \to X$ is a solution to (1) if x is a differentiable function and $x'(t) = f(t, x)$. If $t_0 \in J$ and (2) holds, we say that the solution satisfies the initial condition. If $J = \mathbb{R}$, we say that the solution is global. If there exists no interval $L \supsetneq J$ admitting a solution $y : L \to X$ such that $x = y$

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on J, then we say that J is a maximal interval of existence and x is a maximal solution.
In the finite-dimensional case $X = \mathbb{R}^n$, there are two main approaches to the problem (1). The classical one, systematically employed in the monograph of Hartmann [Ha82], is based on the analytical study of approximate solutions and their convergence properties usually in the form of the Arzela-Ascoli theorem. The modern functional analytic approach is better suited for generalizations into infinite dimensional Banach spaces (see e.g. [P91] [D77] and [DMNZ]). The idea is to study the continuous (but not necessarily linear) operator $T : C(I) \to C(I)$ on the Banach space $(C(I), \|\cdot\|_\infty)$, where I is a suitable interval containing t_0, defined by

$$T(x)(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau \tag{3}$$

A function x on I is a solution of (1), (2) if and only if it is a fixed point of the operator T of (3), i.e. $Tx = x$. This brings into play abstract fixed point theorems like Banach contraction principle or Schauder fixed point theorem. Both the classical (implicitly) and the modern approach rely on the presence of some compactness properties.

Let us begin by stating the results for finite dimensional Banach spaces first, and then pass to the discussion of the infinite dimensional case in subsequent sections.

Theorem 1. *(Peano)*
Let $X = \mathbb{R}^n$, $f : \mathbb{R} \times X \to X$ be a continuous mapping, $t_0 \in \mathbb{R}, x_0 \in X$. Then the ordinary differential equation

$$x' = f(t, x) \tag{4}$$

together with the initial condition

$$x(t_0) = x_0 \tag{5}$$

has a solution on some open interval containing t_0.

Under additional assumptions on f one gets the uniqueness and/or extendability of x to a global solution. The classical condition due to Picard-Lindeloff is when f is Lipschitz in variable x, ie. $\|f(t, x) - f(t, y)\| \le L\|x - y\|$ holds for all $x, y \in X$, where L is a constant. Under this assumption, all solutions to (1) are unique and extendable to global solutions. An optimal result in this direction belongs to Osgood. Let $\omega : [0, \infty) \to [0, \infty)$ be an increasing function, $\omega(0) = 0$, $\omega(t + s) \le \omega(t) + \omega(s)$ whenever $s, t \in [0, \infty)$.

Theorem 2. *(Osgood; Szufla, Shkarin)*
Let X be a Banach space. Suppose that $\int_0^1 \frac{dt}{\omega(t)} = \infty$, and $\|f(t, x) - f(t, y)\| \le \omega(\|x - y\|)$ for all t, x, y. Then (1) *has a unique global solution for every initial condition* (2).

Osgood ([Ha82]) proved his theorem for $X = \mathbb{R}^n$. Szufla [Sz98] and Shkarin [S03] observed that Osgood's theorem remains valid for any Banach space X. Shkarin [S03] proved that the theorem is optimal in the class of spaces admitting a complemented subspace with an unconditional basis, see Theorem 20. If X is finite dimensional, then every solution can be extended into a maximal solution. If we only assume the continuity of f, then maximal solutions are not necessarily global solutions, i.e. their maximal interval of existence is a proper subset of $\mathbb{R}$. Moreover, for a given initial condition, the solutions may not be unique. In fact there are examples when no uniqueness holds for any initial condition (first by Lavrentieff see [Ha82]). On the other hand, given an initial condition $x(t_0) = x_0$ there exists an open interval $(a, b) = I \ni t_0$ such that every solution of (1) with $x(t_0) = x_0$ is extendable to the whole I. It is therefore natural to study the solution sets which share the same initial condition and the same domain.

Definition 3. *Let $(t_0, x_0) \in \mathbb{R} \times X$, $t_0 \in I$ where I is an interval. Suppose that every solution to* (1) *with* (2) *is extendable to I. The overlying solution funnel on I is the set of all solutions on I, i.e.*

$$F(f, (t_0, x_0)) = \{x \in C(I \times X) : x(t_0) = x_0, x'(t) = f(t, x) \text{ on } I\} \tag{6}$$

Definition 4. *Given $t \in I$, we say that $S_t(f, (t_0, x_0)) = \{x(t) : \text{ where } x \in F(f, (t_0, x_0))\}$ is a cross-section of the solution funnel at time t.*

Definition 5. *The solution funnel is the set*

$$S(f, (t_0, x_0)) = \{(t, x(t)) \in I \times X : x(t) \in S_t(f, (t_0, x_0))\} \tag{7}$$

The most complete information is contained in the overlying solution funnels. Solution funnels are their images, and their cross-sections are still more special objects that were studied first by Kneser. In order to facilitate the study of solution funnels we remove the issue of interval I by introducing the following class.

Definition 6. *$\mathcal{F}^m$ is the set of all continuous $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ which have supports in sets of the form $\mathbb{R} \times$ compact.*

If $f \in \mathcal{F}^m$ then every solution can be extended into a global solution. It is known that studying solution funnels for the class $\mathcal{F}^m$ is essentially equivalent to studying the local funnels on intervals [P75]. With this definition we can formulate the main results concerning funnel properties in the finite dimension. Recall that a topological space is called continuum if it is nonempty, compact and connected.

Theorem 7. *(Kneser* [Kn23]*)*
If $f \in \mathcal{F}^m$ then every cross-section of the solution funnel of f is a continuum (in fact, from Theorem 9 we see that it is a continuous image of an $\mathcal{R}_\delta$).

Peano proved the result first for $\mathcal{F}^1$. A more precise description of the more general overlying solution funnels was obtained by Aronszajn [A42]. Recall that an absolute retract is a topological space that is a continuous retract for every overspace.

Definition 8. *(Aronszajn) A topological space belongs to class $\mathcal{R}$ if it is compact and absolute retract. A topological spaces belongs to class $\mathcal{R}_\delta$ if it is an intersection of a decreasing sequence of members of $\mathcal{R}$, i.e. $R = \cap_{n=1}^{\infty} R_n$, where $R_n \in \mathcal{R}$.*

It is well-known that every member of $\mathcal{R}_\delta$ is a continuum.

Theorem 9. *(Aronszajn* [A42]*)*
If $f \in \mathcal{F}^m$ then each overlying solution funnel of f is R_δ.

Hukuhara [Hu28] proved earlier that this set is a continuum. Aronszajn's paper uses modern functional analytic approach. It consists of a study of regular approximations of the operator T from (3) that suits well infinite dimensional generalizations. Pugh [P75] and Horst [H86] gave examples showing that the overlying funnel need not be arcwise connected in Theorem 9.

A detailed study of finer properties of cross-sections of solution funnels was carried out by Pugh [P75], and Rogers [R77]. Pugh constructed continua in $\mathbb{R}^n, n \geq 2$ that are not cross-sections of the solution funnels (a spiral tending to a circle), and cross-sections that are not arcwise connected. Every C^1-smooth polyhedron is a cross-section of the solution funnel. Pugh introduced the funnel cobordism between compact sets $A, B \subset \mathbb{R}^m$ so that A and B are funnel cobordant if there exists an

$f \in \mathcal{F}^m$ such that $\cup_{x\in A} F(f,(0,x)) = \cup_{y\in B} F(f,(1,y))$. Of course, A is funnel cobordant to a point $\{0\}$ iff there exists $f \in \mathcal{F}^m$ such that A is a funnel section at $t = 1$ for the initial condition $(t_0, x_0) = (0,0)$, and $x(1) \notin A$ for all solutions with $x(0) \neq 0$. For this special class of cross-sections of the solution funnels, Pugh obtained a complete topological characterization.

Theorem 10. *(Pugh)*
If A is funnel cobordant to a point $\{0\}$, then there is a C^∞-diffeomorphism from $\mathbb{R}^m \setminus A$ onto $\mathbb{R}^m \setminus \{x_0\}$, which is constant outside of some large ball. If A is compact and there is a C^∞-diffeomorphism (if $n \neq 4$ it suffices a homeomorphism) from $R^m \setminus A$ onto $\mathbb{R}^m \setminus \{x_0\}$ then A is funnel cobordant to a point $\{0\}$, and thus a cross-section of some solution funnel. In particular, A is funnel cobordant to a point iff $\mathbb{R}^m \setminus A$ is diffeomorphic to $S^{m-1} \times \mathbb{R}$.

In spite of the wealth of information contained in [P75], a characterization of cross-sections of solution funnels in finite-dimensional spaces remains open. The following problems come from [P75].

Open problem 11. *Pugh conjectured* [P75] *that Theorem 7 is a characterization, i.e. a continuous image of an $\mathcal{R}_\delta$ set is a cross-section of the solution funnel for some equation from $\mathcal{F}^m$. It seems to be unknown even for $\mathcal{R}$ sets.*

Open problem 12. *If $A \subset \mathbb{R}^k$ is not a cross-section, is it true that it is not a cross-section in $\mathbb{R}^m \hookleftarrow R^k$?*

Open problem 13. *Is being the cross-section of the solution funnel property preserved for homeomorphic images that can be extended to diffeomorphisms on some neighbourhoods?*

An important aspect of the global behavior of the solution is described by the following notion of ω-limit set.

Definition 14. *Let $x : [0,\infty) \to X$ be a solution of an autonomous differential equation $x' = f(x)$. We say that $A \subset X$ is a ω-limit set of the solution, if for every $p \in A, \varepsilon > 0, n \in \mathbb{N}$ there is $t_n > n$ such that $\|x(t_n) - p\| < \varepsilon$.*

The structure of ω-limit sets has been studied extensively ever since the times of Poincaré, and forms an extensive field of research in its own right in the setting of dynamical systems and ergodic theory. In [Ha82] the reader can find the following result.

Theorem 15. *Let $X = \mathbb{R}^m$ and $x : [0,\infty) \to X$ be a bounded solution of an autonomous differential equation $x' = f(x)$ in X. Then an ω-limit set of a x is non-empty, compact and connected.*

A complete description is available in $\mathbb{R}^2$.

Theorem 16. *(Poincaré-Bendixon)*
Let A be a ω-limit set of a bounded global solution x in $X = \mathbb{R}^2$. Assume that A contains no critical points of f. Then A is a Jordan curve, which is a periodic solution of the equation.

The study of ω-limit sets in higher dimensions is an extensive field of study, and it is impossible for us to give an adequate description here. We restrict ourselves to just one important fact. The following result appears to be well-known to specialists in ergodic theory. It follows from the results in [Ka79] and [BMK81] applying the torus technique that we outline at the end of our note in the infinite dimensional setting.

Theorem 17. *Let $X = \mathbb{R}^m, m > 2$. Then there exist ω-limit sets with nonempty interior.*

Let us now pass to the description of infinite-dimensional results.

2. Peano theorem

Using an infinite dimensional Banach space $X = c_0$, Dieudonné [D50] constructed the first counterexample to Theorem 1. Many counterexamples in various infinite dimensional Banach spaces followed, e.g. [LL72], [B61], [Y70], [G72], and importantly [C72] for every nonreflexive Banach space by Cellina. Finally, Godunov in [G75] proved that Theorem 1 is false in every infinite dimensional Banach space.

Theorem 18. *(Godunov)*
Let X be an infinite dimensional Banach space. Then there exists a continuous function $f\colon \mathbb{R}\times X \to X$ and an initial condition $x(t_0) = x_0$, such that the equation (1) *has no solution satisfying* (2).

In trying to negate Peano's theorem in infinite-dimensional spaces it is natural to consider the following problem.

Open problem 19. *Let X be an infinite dimensional Banach space. Is there a continuous function $f\colon \mathbb{R}\times X \to X$ such that* (1) *has no solution on any open interval?*

In [G74], Godunov constructed such f in the Hilbert space. Shkarin [S03] proved the following remarkable result.

Theorem 20. *(Shkarin)*
Let X be an infinite dimensional Banach space admitting a complemented subspace with an unconditional basis. Let ω be a function as in Osgood's theorem, satisfying

$$\int_0^1 \frac{dt}{\omega(t)} < \infty. \tag{8}$$

Then there exists a function f satisfying $\|f(t,x) - f(t,y)\| \le \omega(\|x-y\|)$ for all t,x,y, and such that (1) *has no solution for any initial condition* (2).

As a particular case, we get that for any $\alpha < 1$ there exists a α-Holder function f which satisfies the previous theorem. The class of spaces satisfying Shkarin's assumptions contains most of the classical Banach spaces such as L_p, $1 \le p < \infty$, $C[0,1]$. However, due to Gowers and Maurey [GM93] and Ferenczi's [F97] results, is does not even cover all separable reflexive spaces. It also fails to cover ℓ_∞, because all its complemented subspaces are again isomorphic to ℓ_∞, by a result of Rosenthal ([LT77]).

Open problem 21. *Let X be an infinite dimensional Banach space. Let ω be a function as above, satisfying*

$$\int_0^1 \frac{dt}{\omega(t)} < \infty. \tag{9}$$

Does there exists a function f satisfying $\|f(t,x) - f(t,y)\| \le \omega(\|x-y\|)$ for all t,x,y, and such that (1) *has no solution for any initial condition* (2)*?*

The result covering the widest class of Banach spaces is the following one from [HJ]. It applies in particular to all separable and all reflexive (or WCG) Banach spaces, as well as all $C(K)$ spaces. In fact, it is still an open question whether every Banach space has a separable quotient.

Theorem 22. *([HJ]) Let X be a Banach space with an infinite-dimensional separable quotient. Then there is a continuous mapping $f\colon X \to X$ such that the autonomous equation $x' = f(x)$ has no solutions.*

It seems that most of the positive results proceed along the lines of Kneser's and Aronszajn's theorems and methods. We refer to [DMNZ] for a thorough treatment of this subject, and give just two results. Let M be a subset of a Banach space X. Recall the definition of measure of noncompactness $\alpha(M) = \inf\{\varepsilon : M \subset \cup_{i=1}^{n} B(x_i, \varepsilon)\}$.

Theorem 23. *Let $f : [0,1] \times B_X \to X$ be continuous and bounded in norm by M. Suppose that*

$$\alpha(f(I \times A)) \leq L(\alpha(A)) \text{ for all } A \subset B_X \tag{10}$$

where $L > 0$. The the overlying solution funnel for $x(0) = 0$ on $J \times B_X$, $J = [0, min\{a, \frac{1}{M}\})$ is a continuum.

We say that an operator $T : X \to X$ satisfies the Palais-Smale condition if for each sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n - T(x_n) = 0$ there exists a convergent subsequence.

Theorem 24. *Let I be an interval such that the integral operator (3), $T(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$ satisfies the Palais-Smale conditions. Then the overlying solution funnel on I is a compact R_δ set.*

Approaching the problem from another angle, Lasota and Yorke [LY73] (see also Vidossich [V74]) proved that for every Banach space X, and every initial condition $x(t_0) = x_0$, the set of all continuous mappings $f : \mathbb{R}\times X \to X$, such that $x' = f(t,x)$, $x(t_0) = x_0$ has a solution is a generic set. In other words, functions not admitting a solution to an initial problem are rare. Let us describe the result in more detail.

Definition 25. *We say that x is an unlimited solution if it is maximally defined and moreover $(t, x(t))$ has no limit points at the time boundary.*

Theorem 26. *(Lasota-Yorke)*
Let B be a σ-compact set in X. Then the set of continuous functions $f : \mathbb{R}\times X \to X$ for which there fails to exists an unlimited solution for any initial condition from B is meager in the uniform topology.

Open problem 27. *Decide whether the assumptions in Kneser and Aronszajn-type theorems hold true only for topologically small (in the uniform topology) sets of functions f. Describe some conditions which guarantee the solutions of (1) outside these cases.*

Open problem 28. *Given f find a description of the set of initial conditions that admit a solution.*

3. Peano funnel and its cross sections

Studying the solution funnels in infinite-dimensional spaces involves the question of the existence of solutions in the first place. Some examples do not pay attention to the global behavior of the equation, but it is natural to consider functions that behave well with respect to existence.

Definition 29. *Denote by $\mathcal{F}$ the class of continuous functions f such that (1) has a solution for every initial condition, that extends to a global solution.*

An simple early result on cross-sections is the following.

Theorem 30. *(Binding* [B77]*)*
Let X be an infinite dimensional Banach space. Then there is a function $f \in \mathcal{F}$ such that B_X is a cross-section of the solution funnel.

Using ideas of Godunov, Binding constructs non-connected funnel cross-sections (but $f \notin \mathcal{F}$). Further progress in the study of funnels comes from the connection with negligibility theory [BP75]. Let us formulate some important results on this subject in the infinite dimensional setting.

Theorem 31. *(Klee, Bessaga)*
Let X be an infinite dimensional Banach space, A compact set. Then there exists a homeomorphism $h : X$ onto $X \setminus A$, such that $h(x) = x$ whenever $\|x\| > K$.

Theorem 32. *(Dobrowolski)*
Let X be an infinite dimensional Banach space admitting a nonequivalent C^k-smooth norm, $K \subset X$ be compact. Then there exists a C^k-diffeomorphism $h : X \setminus K \to X$.

Mappings h with properties above are known, for obvious reasons, as deleting homeomorphisms (diffeomorphisms). The relevance of negligibility theory was clear already in the work of Pugh in the finite dimensional case. Garay in [Ga91b] applied the infinite dimensional results to disprove the Peano theorem in every infinite dimensional Banach space. The proof splits into the nonreflexive case, motivated by Cellina's approach [C72], and the reflexive case which uses negligibility of sets in Banach spaces.
Similarly, Garay obtains the following result.

Theorem 33. *(Garay* [Ga90]*)*
Let X be an infinite dimensional Banach space , $A \subset X$ be a two-point subset. Then there is a function $f \in \mathcal{F}(X)$ such that A is the cross-section of the solution funnel from $(0, 0)$ and $(0, 0)$ is the only initial condition of non-uniqueness (condition (*)).

The proof of this result again splits into the reflexive and the nonreflexive part. The reflexive case depends on C^1-smooth renormings of reflexive spaces.

Theorem 34. *(Garay* [Ga90]*)*
Let X be a Banach space, $A \subset X$ be nonempty bounded and closed. Assume that there exists a C^1-smooth diffeomorphism $h : X \setminus A$ onto $X \setminus \{0\}$, such that $h(x) = Id(x)$ for $\|x\| > K$. Then there is a function $f \in \mathcal{F}(X)$ such that A is a cross-section of the solution funnel. Moreover, if A has at least two points, then f can be chosen such that $(0, 0)$ is the only initial condition of nonuniquness.

Using some known results, Garay obtained a number of instances covered by Theorem 34. In particular, it holds for every separable Asplund space with a Schauder basis, and A a compact set. The strongest result in this direction seems to be due to Azagra and Dobrowolski, who investigated smooth negligibility.

Theorem 35. *(Azagra-Dobrowolski* [AD98]*)*
Let X be a infinite dimensional Banach space admitting a C^k smooth (not necessarily equivalent) norm $|||\cdot|||$, A compact. Then there exists a C^k-diffeomorphism $h : X$ onto $X \setminus A$, such that $h(x) = x$ for $|||x||| > K$.

Note that the class of spaces admitting a C^∞-smooth norm includes all spaces with an injection into $c_0(\Gamma)$, in particular all spaces having a Markushevich basis [HMVZ].

Open problem 36. *Let X be an Asplund space. Does X admit a (nonequivalent) C^1-smooth norm?*

Combining this result with the work of Garay [Ga90] they obtain the following theorem. Note that in the next result the C^1-smooth norm must be equivalent to the original norm.

Theorem 37. *(Azagra-Dobrowolski)*
Let X be an infinite dimensional Banach space with a C^1-smooth equivalent norm. Let A be either compact or C^1-smooth equivalent unit ball. Then there is a function $f \in \mathcal{F}(X)$ such that A is a cross-section of the solution funnel.

Comparing Theorem 33 with Theorem 34, Garay in [Ga90] asked the natural question if the deleting diffeomorphism exist in ℓ_1. We have the following negative answer. As every Banach space with a C^1-smooth bump function is an Asplund space ([F$^\sim$]), we see that assumptions used in the method of proof of Theorem 37 are nearly optimal

Theorem 38. *Let X be a Banach space admitting no C^1-smooth bump function. Then there is no C^1-smooth diffeomorphism from $X \setminus \{0\}$ onto $X \setminus \{a, 0\}$ which is identity outside a large enough ball.*

Proof. We proceed by contradiction. Let $T : X \to X$ be a diffeomorphism from $X \setminus \{0\}$ onto $X \setminus \{a, 0\}$ which is identity outside kB_X. Clearly, $T(x) - x$ is zero outside kB_X and it is nonzero at a. Choose a suitable $\phi \in X^*$ such that $b(x) = \phi(T(x) - x)$ is a C^1-bump defined away from the origin. Assume WLOG that $b \le 0$, and set $b(0) = inf_{y \to 0} b(y)$. Clearly, b is a non-constant, lower-semicontinuous function, bounded below and defined on the whole X.
Given any $\varepsilon > 0$ and composing b with a suitable nondecreasing and smooth $\phi : \mathbb{R} \to \mathbb{R}$, we may WLOG assume that $b(z) = -1$ for some $0 \neq z \in kB_X$ and $b \geq -1 - \varepsilon$. Consider a Hahn-Banach functional ϕ for z, $\phi \in B_{X^*}$, $\phi(z) = \|z\|$, and compose ϕ again with a suitable C^∞-smooth function $\rho : \mathbb{R} \to \mathbb{R}$, to obtain a C^∞-smooth function $\psi = \rho \circ \phi$, such that $\psi(z) = \min \psi = 0$, $\psi(0) = 1 + \varepsilon$. Then $q = b + \psi$ is a lower semicontinuous function, C^1-smooth away from the origin, bounded below by $-1 - \varepsilon$, and $q \geq 0$ outside kB_X. Next compose q again with a C^∞-smooth nondecreasing function $\eta : \mathbb{R} \to \mathbb{R}$, $\eta(t) = t$ for $t < -\frac{1}{2}$, $\eta(t) = 0$ for $t > -\varepsilon$. The function $\eta \circ q$ is now easily seen to be C^1-smooth, non-constant, and with support contained in kB_X. This is a contradiction. $\square$

The following problem was posed by several authors in connection with negligibility theory.

Open problem 39. *Suppose that X has a C^k-smooth norm. Does X admit a C^k-smooth nonequivalent norm?*

Open problem 40. *Is every compact set K in an infinite dimensional Banach space X a cross-section of a solution funnel?*

Open problem 41. *Describe the most general conditions for some set to be a cross-section of the solution funnel.*

4. ω-limit sets

It is clear that an ω-limit set of a global solution of an autonomous differential equation is always closed, so it is always a Polish space (i.e. separable and completely metrizable space). The first examples of failure of Theorem 15 in infinite dimensional Hilbert space seem to be due to Horst [H86], where noncompact and disconnected sets are obtained. A wealth of results on ω-limit sets in infinite dimensional spaces is in the paper of Garay [Ga91a]. However, his paper is mainly dealing with dynamical systems, and so its importance for the case of autonomous differential equations lies primarily in the techniques used. In fact, the techniques are very natural, and we have discovered our results below independently of Garay's work, by using the same approaches, paying closer attention to the smoothness and Lipschitness issue, motivated by the work of Herzog [H00].

Theorem 42. *(Herzog)*
Let P be any Polish space, $X = \ell_2 \oplus c_0$. Then there exists a locally Lipschitz function $f : X \to X$ such that P is homeomorphic to a ω-limit set of some solution to the autonomous equation $x' = f(x)$.

This theorem is a characterization, up to homeomorphism, of all possible ω-limit sets in X. Herzog relied on a theorem of Aharoni in his proof and his ω-limit set is contained in a hyperplane of X. Recall that Aharoni's theorem asserts that every Polish space (i.e. separable complete metric space) embeds into c_0 by means of a bi-Lipschitz mapping ([F~]). By a theorem of Kadets ([F~]) all separable infinite dimensional Banach spaces are mutually homeomorphic. Consequently, every Polish space is homeomorphic to a subset of a hyperplane (with empty interior, of course) in every infinite dimensional Banach space. Thus our next theorem is a substantial improvement of Theorem 42. Apart from analogous topological characterization of all ω-limit sets, it also identifies the precise position (for some of them) in the space X.

Theorem 43. *(Garay,* [HV]*) Let X be a Banach space, $S \subset X$ be a separable and closed subset, such that there exists an open set $U \subset X$ with the properties:*
1. U is arcwise connected.
2. $S \subset \partial U$.
Then there exists an autonomous differential equation $x' = f(x)$, where $f : X \to X$ is a Lipschitz mapping, and its solution $x(t)$, with the property $S = \omega(x)$. (S consists of all ω-limit points of $x(t)$, $t \in [0, \infty)$. Moreover, if X admits a C^k-smooth renorming, then f may be chosen C^k-smooth as well.

The idea of the proof is to construct a smooth curve $\gamma : [0, \infty) \to X \setminus S$, together with a fast thinning out tubular neighbourhood not intersecting itself and the set S, that keeps returning progressively closer to all points of S. In order to define the field inside the tube we use the direction of the tangent to the curve and its parallels, with decreasing norm to zero as we approach the topological boundary and as the parameter grows to infinity. In the rest of the space we put the field to be zero.

Theorem 44. *(*[HV]*) Let X be a separable infinite dimensional Asplund space. Then there exists a C^1-smooth and Lipschitz autonomous equation with a solution whose ω-limit set has nonempty interior. Moreover, if the original space has C^k-smooth norm, then the function f can be chosen C^k-smooth as well.*

In the proof we utilize hypercyclic operators on these spaces ([An97], [Sa95]) in order to create an open torus, inside which some trajectory is dense. The geometrical idea is to create a torus-like body which results from rotation of a ball of codimension one around an axis. The hypercyclic operator serves to create a diffeomorphism on the ball, that is transformed into a homotopy parametrized by the rotation angle. The vector field than consistes of tangent vectors to the homotopical mapping.

Using the continuous dependence of a solution on the initial condition, which holds for locally Lipschitz equations, the next result follows by standard argument.

Proposition 45. *(*[HV]*) Let S be a ω-limit set for a locally Lipschitz autonomous equation $x' = f(x)$. If $intS \neq \emptyset$, then there is a solution $x(t)$ such that*

$$S = \overline{\cup_{t\in[0,\infty)} x(t)} \tag{11}$$

In particular, S is connected.

More generally,

Proposition 46. *([HV]) Let S be a ω-limit set for a locally Lipschitz autonomous equation $x' = f(x)$. Suppose that there exists $p \in S$ such that $f(p) \neq 0$. Then again, there is a solution $x(t)$ such that*

$$S = \overline{\cup_{t\in[0,\infty)} x(t)} \tag{12}$$

In particular, S is connected.

This allows to find examples of sets that cannot be ω-limit sets for locally Lipschitz autonomous equations. Take e.g. as set S the union of a unit sphere, a non-empty set inside the interior of unit ball, and a point outside, everything connected with a curve.

Open problem 47. *Let X be a separable Banach space, $U \subset X$ be an open and arcwise connected set. Is there a Lipschitz autonomous equation whose solution has $\overline{U}$ as its ω-limit set?*

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