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Cauchy's stress theorem for stresses represented by measures

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Abstract A version of Cauchy's stress theorem is given in which the stress describing the system of forces in a continuous body is represented by a tensor valued measure with weak divergence a vector valued measure. The system of forces is formalized in the notion of an unbounded Cauchy flux generalizing the bounded Cauchy flux by Gurtin & Martins [12]. The main result of the paper says that unbounded Cauchy fluxes are in one-to-one correspondence with tensor valued measures with weak divergence a vector valued measure. Unavoidably, the force transmitted by a surface generally cannot be defined for all surfaces but only for almost every translation of the surface. Also conditions are given guaranteeing that the transmitted force is represented by a measure. These results are proved by using a new homotopy formula for tensor valued measure with weak divergence a vector valued measure.

Keywords Cauchy stress theorem, divergence measure vectorfield, Newton homotopy

Contents

1 Introduction

Cauchy's stress theorem asserts that the system of contact forces in a continuous body $U \subset \mathbf{R}^n$ is described by a tensor function $\boldsymbol{T}$ on U such that the force $\boldsymbol{F}(A)$ transmitted by an oriented surface A in U with normal $\boldsymbol{n}$ is given by

$$\boldsymbol{F}(A) = \int_A \boldsymbol{T}\boldsymbol{n}\, d\mathscr{H}^{n-1} \tag{1.1}$$

where $\mathscr{H}^{n-1}$ is the area measure. The question of the rigorous derivation of Cauchy's stress theorem from a set of elementary postulates on the mapping $A \mapsto \boldsymbol{F}(A)$ was raised by Noll [22]. The notion of Cauchy flux by Gurtin & Martins [12] (here called bounded Cauchy flux; see Definition 2.1, below) is such an axiomatization of $\boldsymbol{F}(\cdot)$ which permits one to prove that $\boldsymbol{F}(\cdot)$ is represented by an essentially bounded, Lebesgue measurable function $\boldsymbol{T}$ in the sense of (1.1); a result by Ziemer [37] later showed that the weak divergence of $\boldsymbol{T}$ is represented by a bounded Lebesgue measurable function (see Theorem 2.3, below, for a precise formulation).

Subsequent efforts were directed towards generalizations of the notion of Cauchy flux so that unbounded stressfields $\boldsymbol{T}$ can occur with divergence represented by either an unbounded function [29] or even by a measure [8]. A characteristic feature of these generalizations is that the force $\boldsymbol{F}(A)$ cannot be defined for all (no matter how smooth, see Remark 2.17, below) surfaces; suitable notions of almost every surface have been introduced in the cited papers to deal with this situation. The reader is referred to [16–18, 25, 19–20, 31, 27] for further developments related to the Cauchy stress theorem.

In the mentioned references the stress $\boldsymbol{T}$ is represented by a function (possibly with singularities). Here I attempt to proceed a step further and introduce the notion of an (unbounded) Cauchy flux so that the stress itself, and not only its divergence, is represented by a measure.

My motivation comes from the theory of material surfaces [13, 11] with internal structure, from the statics of masonry bodies [14–15], and from the analysis of concentrated contact interactions in [23–24]. In the theory of material surfaces the stress is not described by a tensor valued function $\boldsymbol{T}$ only. The last is true only of the bulk stress $\boldsymbol{T}_b$ but in addition there is also the stress tensor $\boldsymbol{T}_s$ acting in the material surface; these two stresses do not play equivalent roles: the former is distributed (absolutely continuous with respect to the volume measure $\mathscr{L}^n$) while the latter is concentrated on the surface and thus described by a density with respect to the area measure. The most natural unifying concept is the stress represented by a measure with a stress concentration on the material surface. Likewise, to assert the absence of collapse of a masonry body, one seeks admissible stressfields described by a negative semidefinite tensor that are equilibrated with the given loads. It turns out that the admissible equilibrated stressfields are easier to find in the class of measures rather than in the class of ordinary functions; in addition, the former often have a direct statical interpretation [14]. Finally, in [23–24] Podio-Guidugli showed that certain stressfields represented by ordinary functions with singularities give rise to concentrated contact interactions: the surface tractions on certain surfaces exhibit Dirac δ type atoms, i.e., are not absolutely continuous with respect to area.

There is no loss in generality in passing from the vector valued quantity $\boldsymbol{F}(\cdot)$ to a scalar valued quantity $F(\cdot)$; then the analog of the stress tensor $\boldsymbol{T}$ is the vector valued flux vector $\boldsymbol{q}$; thus I seek to axiomatize $F(\cdot)$ in such a way that the flux vector is represented by a vector valued measure $\mathbf{q}$ with the weak divergence represented by a scalar valued measure $\operatorname{div}\mathbf{q}$. I call such measures divergence measure vectorfields; they have been introduced in [6–7]. The postulates on $F(\cdot)$ are formalized in Definition 2.4 of an unbounded Cauchy flux and the resulting notion is shown to be in one to one correspondence with divergence measure vectorfields in Theorem 2.6. Unavoidably, the flux $F(A)$ cannot be postulated to exist for every "good" surface; in Definition 2.4 $F(A)$ is postulated to exist only for $\mathscr{L}^n$ almost every translation of an oriented planar polyhedron $A \subset \mathbf{R}^n$; this relatively small class enables one to prove the existence of the divergence measure vectorfield $\mathbf{q}$ representing $F(\cdot)$; conversely a given $\mathbf{q}$ enables one to define $F(A)$ for almost every translation of an oriented planar polyhedron. The class of oriented planar polyhedra is thus sufficient to establish a one to one correspondence between the class of unbounded Cauchy fluxes and divergence measure vectorfields. Nevertheless, it turns out that every unbounded

Cauchy flux can be extended naturally into a much larger class of objects. Indeed, Theorem 2.11 shows that $F(\cdot)$ can be extended to some normal $n-1$ dimensional currents [10] which are called generalized surfaces here. These are measure–theoretic generalizations of oriented $n-1$ dimensional (curved) surfaces inasmuch as the latter are special cases of the former. (With an extra effort the flux can be extended to an even larger class, viz., to some flat chains [36, 10] thus including fractal surfaces; this will be the subject of a future paper.) The next question addressed in this paper is that of whether the flux through a generalized surface can be represented by a measure; a sufficient condition is given in Theorem 2.14 and it comes as a corollary that for a given generalized surface the flux is well defined and represented by a measure for almost every translation of the surface. Finally, Theorem 2.16 shows that if the surface in question is the boundary of a set of finite perimeter then the measure arising in Theorem 2.14 satisfies an appropriate form of the divergence theorem.

The main novel feature of the analysis is the homotopy formula

$$\mathbf{q} = \mathrm{H_N}\,\mathrm{div}\,\mathbf{q} + \mathrm{div}\,\mathrm{H_N}\,\mathbf{q} \tag{1.2}$$

for divergence measure vectorfields and the associated Riesz kernels $\boldsymbol{x}/|\boldsymbol{x}|^n$, $1/|\boldsymbol{x}|^{n-1}$ used extensively in the multiplied notes [30] in the broader context of normal currents, but so far not published in the periodical literature.

The following section summarizes the main notions and results without proof; the proofs are given in the subsequent sections.

2 The main results

We denote by $\mathscr{L}^n$ the Lebesgue measure in $\mathbf{R}^n$ ([10; Subsection 2.6.5]. If r is an integer, $0 \le r \le n$, we denote by $\mathscr{H}^r$ the r-dimensional Hausdorff measure in $\mathbf{R}^n$ [10; Subsections 2.10.2–2.10.6]. *Throughout we assume that the dimension n of $\mathbf{R}^n$ satisfies $n \ge 2$.*

We initially use (oriented) planar polyhedra as our model of oriented surfaces. A ***planar polyhedron*** is a pair $A = (P, \boldsymbol{n})$ where P is a closed bounded polyhedral set of positive area contained in some $n-1$ dimensional plane H and $\boldsymbol{n}$ is one of the two unit normals to H. We denote by $\mathfrak{S}$ the set of all planar polyhedra. If $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ we define $\mathscr{H}^{n-1}(A) := \mathscr{H}^{n-1}(P)$. We say that two planar polyhedra $A_1 = (P_1, \boldsymbol{n}_1), A_2 = (P_2, \boldsymbol{n}_2)$ are ***compatible*** if P_1, P_2 are contained in the same $n-1$ dimensional plane and $\boldsymbol{n}_1 = \boldsymbol{n}_2$. We then define the ***union*** $A_1 \cup A_2$ to be the pair $(P_1 \cup P_2, \boldsymbol{n}_1)$. We say that A_1 and A_2 are ***essentially disjoint*** if $\mathscr{H}^{n-1}(P_1 \cap P_2) = 0$.

A ***body*** B is a closed bounded polyhedron in $\mathbf{R}^n$ of dimension n of positive volume. We denote by $\mathfrak{B}$ the collection of all bodies. If ∂B is the topological boundary of B and $\boldsymbol{n}$ the map giving the unit outward normal then the ***face*** P of B is the maximal planar polyhedral set contained in ∂B such that $\boldsymbol{n}$ is constant on it. An ***oriented face*** of B is any pair $A = (P, \boldsymbol{m}) \in \mathfrak{S}$ where P is a face and $\boldsymbol{m}$ the constant value of $\boldsymbol{n}$ on P. We denote by A_i, $i = 1, \dots, q$, the collection of all oriented faces of B.

We refer to [12] for the motivation of the following definition.

Definition 2.1. A ***bounded Cauchy flux*** is a function $F : \mathfrak{S} \to \mathbf{R}$ such that

(i) F is additive, i.e., if $A_1, A_2 \in \mathfrak{S}$ are essentially disjoint and compatible then

$$F(A_1 \cup A_2) = F(A_1) + F(A_2);$$

(ii) there exists a constant b such that if $A \in \mathfrak{S}$ then

$$|F(A)| \le b\mathscr{H}^{n-1}(A);$$

(iii) there exists a constant c such that if $B \in \mathfrak{B}$ then

$$|F(\partial B)| \le c\mathscr{L}^n(B)$$

where we have put

$$F(\partial B) := \sum_{i=1}^{q} F(A_i) \tag{2.1}$$

where $A_i, i = 1, \ldots, q$, is the collection of all oriented faces of B.

We denote by $L^\infty(\mathbf{R}^n, \mathbf{R}^n)$ and $L^\infty(\mathbf{R}^n, \mathbf{R})$ the usual Lebesgue spaces of (classes of equivalence) of essentially bounded $\mathscr{L}^n$ measurable functions on $\mathbf{R}^n$ with values in $\mathbf{R}^n$ and $\mathbf{R}$, respectively. Throughout the paper the integrals with unspecified domains of integration denote integrals over $\mathbf{R}^n$, $\int \ldots \equiv \int_{\mathbf{R}^n} \ldots$; furthermore, if Z is a finite dimensional vectorspace then $C_0(\mathbf{R}^n, Z)$ denotes the set of all continuous Z valued compactly supported functions on $\mathbf{R}^n$ and $C_0^\infty(\mathbf{R}^n, Z)$ the set of all infinitely differentiable functions from $C_0(\mathbf{R}^n, Z)$.

Definition 2.2. We denote by $\mathscr{D}L^\infty(\mathbf{R}^n)$ the set of all $\boldsymbol{q} \in L^\infty(\mathbf{R}^n, \mathbf{R}^n)$ for which there exists a function $\operatorname{div} \boldsymbol{q} \in L^\infty(\mathbf{R}^n, \mathbf{R})$ such that

$$\int \boldsymbol{q} \cdot \nabla\varphi \, d\mathscr{L}^n = -\int \varphi \operatorname{div} \boldsymbol{q} \, d\mathscr{L}^n$$

for each $\varphi \in C_0^\infty(\mathbf{R}^n, \mathbf{R})$.

Let $\omega : \mathbf{R}^n \to \mathbf{R}$ be a mollifier, i.e., a nonnegative infinitely differentiable function supported by the unit ball in $\mathbf{R}^n$ such that $\int \omega \, d\mathscr{L}^n = 1$. If $\rho > 0$, we define $\omega_\rho : \mathbf{R}^n \to \mathbf{R}$ by $\omega_\rho(\boldsymbol{x}) = \rho^{-n}\omega(\boldsymbol{x}/\rho), \boldsymbol{x} \in \mathbf{R}^n$; if $\boldsymbol{q}$ is a locally $\mathscr{L}^n$ integrable function with values in a finite dimensional vectorspace we define the ρ mollification $\boldsymbol{q}_\rho$ of $\boldsymbol{q}$ as a function defined on $\mathbf{R}^n$ by

$$\boldsymbol{q}_\rho(\boldsymbol{x}) = \int \boldsymbol{q}(\boldsymbol{x} - \boldsymbol{y})\omega_\rho(\boldsymbol{y}) \, d\mathscr{L}^n(\boldsymbol{y}),$$

$\boldsymbol{x} \in \mathbf{R}^n$.

The main results on bounded Cauchy fluxes, already available in the literature, are summarized in the following

Theorem 2.3.

(i) *If F is a bounded Cauchy flux then there exists a unique $\boldsymbol{q} \in \mathscr{D}L^\infty(\mathbf{R}^n)$ such that*

$$F(A) = \lim_{\rho \to 0} \int_P \boldsymbol{q}_\rho \cdot \boldsymbol{n} \, d\mathscr{H}^{n-1} \tag{2.2}$$

*for every $A = (P, \boldsymbol{n}) \in \mathfrak{S}$. We call $\boldsymbol{q}$ the **flux vector of** F.*

(ii) *For every $\boldsymbol{q} \in \mathscr{D}L^\infty(\mathbf{R}^n)$ there exists a unique bounded Cauchy flux F with the flux vector $\boldsymbol{q}$.*

(iii) *If F is a bounded Cauchy flux and $\boldsymbol{q}$ a representative of its flux vector $\boldsymbol{q}$ then there exists a set $N \subset \mathbf{R}^n$ of null $\mathscr{L}^n$ measure such that*

$$F(A) = \int_P \boldsymbol{q} \cdot \boldsymbol{n}\, d\mathscr{H}^{n-1}, \quad F(\partial B) = \int_B \operatorname{div} \boldsymbol{q}\, d\mathscr{L}^n \tag{2.3}$$

for every $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ with $\mathscr{H}^{n-1}(P \cap N) = 0$ and every $B \in \mathfrak{B}$ with $\mathscr{H}^{n-1}(\partial B \cap N) = 0$.

The exceptional set N can be chosen to be $\emptyset$ if the representative $\boldsymbol{q}$ is continuous; however, that degree of regularity is not guaranteed. The proof of this proposition is obtained by combining results from [12, 37] and [28]; the details are omitted. Alternatively, one can observe [25] that F gives rise to a flat $n-1$ dimensional cochain [36, 10] and use the representation theorem for flat cochains to obtain the same relationship between F and $\boldsymbol{q}$. The latter approach has the advantage that it is immediately clear that any bounded Cauchy flux can be extended from the class of planar polyhedra to generalized $n-1$ dimensional surfaces represented by flat chains. (See also Theorem 2.11, below).

If $M \subset \mathbf{R}^n$ and $\boldsymbol{a} \in \mathbf{R}^n$, we denote by $\mathrm{T}_{\boldsymbol{a}} M$ the translation of M by $\boldsymbol{a}$. If $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ we define $\mathrm{T}_{\boldsymbol{a}} A = (\mathrm{T}_{\boldsymbol{a}} P, \boldsymbol{n})$.

If Z is a finite dimensional inner product space, we denote by $\mathscr{M}(\mathbf{R}^n, Z)$ the set of all Z valued measures on $\mathbf{R}^n$ [1; Chapter 1], i.e., the set of all Z valued countably additive functions defined on the system of all Borel subsets of $\mathbf{R}^n$. If $\mathbf{q} \in \mathscr{M}(\mathbf{R}^n, Z)$ and $\rho > 0$, we define the ρ mollification $\mathbf{q}_\rho$ of $\mathbf{q}$ as a function on $\mathbf{R}^n$ given by

$$\mathbf{q}_\rho(\boldsymbol{x}) = \int \omega_\rho(\boldsymbol{x} - \boldsymbol{y})\, d\mathbf{q}(\boldsymbol{y})$$

$\boldsymbol{x} \in \mathbf{R}^n$. We denote by $|\mathbf{q}|$ the total variation of $\mathbf{q}$, i.e., the smallest nonnegative measure such that $|\mathbf{q}(A)| \le |\mathbf{q}|(A)$ for every Borel subset A of $\mathbf{R}^n$. We further denote by $\mathrm{M}(\mathbf{q})$ the ***mass*** of $\mathbf{q}$, defined by $\mathrm{M}(\mathbf{q}) := |\mathbf{q}|(\mathbf{R}^n)$. If $\mathbf{q} \in \mathscr{M}(\mathbf{R}^n, Z)$ and if $\boldsymbol{\omega} : \mathbf{R}^n \to Z$ is a $\mathbf{q}$ integrable function (i.e., $\boldsymbol{\omega}$ is $|\mathbf{q}|$ measurable and $\int |\boldsymbol{\omega}|\, d|\mathbf{q}| < \infty$) then $\int \boldsymbol{\omega} \cdot d\mathbf{q}$ is a well defined number. We denote by $\mathbf{q} \llcorner B$ the restriction of the measure $\mathbf{q}$ to a Borel set $B \subset \mathbf{R}^n$.

If $M \subset \mathbf{R}^n$ is a Borel set, ϕ is a nonnegative Radon measure and r a nonnegative integer we define the ***lower r dimensional content*** of ϕ in M by

$$\phi^r(M) = \liminf_{\rho \to 0} \int_M \phi_\rho\, d\mathscr{H}^r.$$

We note that in case $\phi = \mathscr{L}^n$ we have

$$\phi^r(M) = \mathscr{H}^r(M)$$

for every Borel set and in Definition 2.4 (below) we use the lower $n-1$ and n dimensional content of two measures θ, η to obtain suitable generalizations of the boundedness assumptions in Definition 2.1(ii) and (iii). If $A = (P, \boldsymbol{n}) \in \mathfrak{S}$, we define the lower r dimensional content of ϕ in A by $\phi^r(A) = \phi^r(P)$.

We now introduce the central notion of this paper.

Definition 2.4. A function $F : \mathfrak{D} \to \mathbf{R}$, where $\mathfrak{D} \subset \mathfrak{S}$, is said to be an (unbounded) ***Cauchy flux*** if the following conditions hold:

(i) if $A \in \mathfrak{S}$ then $\mathrm{T}_{\boldsymbol{a}} A \in \mathfrak{D}$ for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$ and the function $\boldsymbol{a} \mapsto F(\mathrm{T}_{\boldsymbol{a}} A)$ is $\mathscr{L}^n$ measurable;

(ii) F is additive, i.e., if $A_1, A_2 \in \mathfrak{D}$ are essentially disjoint and compatible then $A_1 \cup A_2 \in \mathfrak{D}$ and

$$F(A_1 \cup A_2) = F(A_1) + F(A_2);$$

(iii) there exists a finite nonnegative measure θ such that if $A \in \mathfrak{S}$ then

$$|F(\mathrm{T}_{\boldsymbol{a}} A)| \le \theta^{n-1}(\mathrm{T}_{\boldsymbol{a}} A) \tag{2.4}$$

for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$;

(iv) there exists a finite nonnegative measure η such that if $B \in \mathfrak{B}$ then

$$|F(\partial\, \mathrm{T}_{\boldsymbol{a}} B)| \le \eta^n(\mathrm{T}_{\boldsymbol{a}} B) \tag{2.5}$$

for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$ where we use the notation (2.1) for every $B \in \mathfrak{B}$.

We note that a bounded Cauchy flux is a particular case of an unbounded Cauchy flux: indeed, given a bounded Cauchy flux F, one easily checks that the above definition holds with the domain $\mathfrak{D} = \mathfrak{S}$, with the measures $\theta = \eta = \mathscr{L}^n$, and with (iii), (iv) holding for every translation.

Definition 2.5. We denote by $\mathscr{D}\mathscr{M}(\mathbf{R}^n)$ the set of all $\mathbf{R}^n$ valued measures $\mathbf{q}$ on $\mathbf{R}^n$ for which there exists an $\mathbf{R}$ valued measure $\operatorname{div} \mathbf{q}$ on $\mathbf{R}^n$ such that

$$\int \nabla\varphi \cdot d\mathbf{q} = -\int \varphi \, d \operatorname{div} \mathbf{q} \tag{2.6}$$

for each $\varphi \in C_0^\infty(\mathbf{R}^n, \mathbf{R})$. We call the elements of $\mathscr{D}\mathscr{M}(\mathbf{R}^n)$ the ***divergence measure vectorfields.***

The reader is referred to [6–7, 32, 9, 33] for the divergence theorem for divergence measure vectorfields and to [34] for some description of their structure.

The following theorem, the main result of the paper, is an analog of Theorem 2.3 for unbounded Cauchy fluxes.

Theorem 2.6.

(i) *If $F : \mathfrak{D} \to \mathbf{R}$ is a Cauchy flux then there exists a unique $\mathbf{q} \in \mathscr{D}\mathscr{M}(\mathbf{R}^n)$ such that*

$$F(\mathrm{T}_{\boldsymbol{a}} A) = \lim_{\rho \to 0} \int_{\mathrm{T}_{\boldsymbol{a}} P} \mathbf{q}_\rho \cdot \boldsymbol{n} \, d\mathscr{H}^{n-1} \tag{2.7}$$

*for every $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ and $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$. We call $\mathbf{q}$ the **flux vector of** F.*

(ii) *For every $\mathbf{q} \in \mathscr{D}\mathscr{M}(\mathbf{R}^n)$ there exists an essentially unique Cauchy flux F with the flux vector $\mathbf{q}$; here essentially unique means that any two Cauchy fluxes F, G with the same flux vector satisfy*

$$F(\mathrm{T}_{\boldsymbol{a}} A) = G(\mathrm{T}_{\boldsymbol{a}} A)$$

for every $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ and $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$. One such a flux $F : \mathfrak{D} \to \mathbf{R}$ is defined by

$$F(A) = \lim_{\rho \to 0} \int_P \mathbf{q}_\rho \cdot \boldsymbol{n} \, d\mathscr{H}^{n-1} \tag{2.8}$$

on the set $\mathfrak{D}$ of all $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ for which the limit (2.8) exists and is finite.

(iii) *If F is a Cauchy flux and* $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ *its flux vector then we have*

$$F(\partial \mathrm{T}_{\boldsymbol{a}} B) = \int_{\mathrm{T}_{\boldsymbol{a}} B} d \operatorname{div} \mathbf{q}$$

for every $B \in \mathfrak{B}$ *and* $\mathscr{L}^n$ *a.e.* $\boldsymbol{a} \in \mathbf{R}^n$.

Items (i) and (ii) establish a one-to-one correspondence between unbounded Cauchy fluxes and divergence measure vectorfields. Here (2.7) is the weak analog of (2.2); however, there is no direct analog of (2.3) (as there cannot be). We emphasize that the flux vector $\mathbf{q}$ is unique in the class $\mathscr{DM}(\mathbf{R}^n)$ but not unique in the set $\mathscr{M}(\mathbf{R}^n, \mathbf{R}^n)$, as the following example shows.

Example 2.7. Let $\boldsymbol{a} \neq \mathbf{0}$ and let $\mathbf{q} = \boldsymbol{a}\delta$ where δ is the Dirac measure at the origin $\mathbf{0} \in \mathbf{R}^n$. Then

$$\lim_{\rho \to 0} \int_{\mathrm{T}_{\boldsymbol{a}} P} \mathbf{q}_\rho \cdot \boldsymbol{n} \, d\mathscr{H}^{n-1} = 0 \tag{2.9}$$

for every $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ and $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$. Indeed we have $\mathbf{q}_\rho \to \mathbf{0}$ as $\rho \to 0$ uniformly on all compact sets $K \subset \mathbf{R}^n$ which do not contain the origin. If $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ then since P is compact and $\mathscr{L}^n(P) = 0$ we have $\mathrm{T}_{\boldsymbol{a}} P \cap \{\mathbf{0}\} = \emptyset$ for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$ and for all such $\boldsymbol{a}$ we have (2.9). Thus if $F : \mathfrak{S} \to \mathbf{R}$ is a flux identically equal to zero, the nonzero measure $\mathbf{q}$ satisfies (2.7) for every $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ and $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$. This, however is not in contradiction with the uniqueness statement of Theorem 2.6(i) since $\mathbf{q} \notin \mathscr{DM}(\mathbf{R}^n)$.

Our next goal is the extension of unbounded Cauchy fluxes to a more general class of surfaces and bodies than polyhedra. Our choice is the set of normal $n-1$ currents [10] for the former, called generalized surfaces here, and n dimensional currents for the latter, called generalized bodies here. Let Skw be the set of all antisymmetric linear transformations on $\mathbf{R}^n$, which can be identified with the set of all antisymmetric $n \times n$ matrices. If $\boldsymbol{\omega} \in C_0^\infty(\mathbf{R}^n, \mathrm{Skw})$ we define the divergence of $\boldsymbol{\omega}$ as the unique function $\operatorname{div} \boldsymbol{\omega} : \mathbf{R}^n \to \mathbf{R}^n$ such that

$$\boldsymbol{a} \cdot \operatorname{div} \boldsymbol{\omega} = \operatorname{div}(\boldsymbol{\omega}^{\mathsf{T}} \boldsymbol{a})$$

for every $\boldsymbol{a} \in \mathbf{R}^n$ where the divergence on the right hand side is the usual divergence of a vectorfield.

Definitions 2.8.

(i) An $\mathbf{R}^n$ valued measure $\boldsymbol{v}$ on $\mathbf{R}^n$ is said to be a ***generalized*** ($n-1$ dimensional) ***surface*** if there exists a Skw valued measure $\partial \boldsymbol{v}$ on $\mathbf{R}^n$ such that

$$\int \operatorname{div} \boldsymbol{\omega} \cdot d\boldsymbol{v} = \int \boldsymbol{\omega} \cdot d\,\partial \boldsymbol{v} \tag{2.10}$$

for each $\boldsymbol{\omega} \in C_0^\infty(\mathbf{R}^n, \mathrm{Skw})$. We call $\partial \boldsymbol{v}$ the ($n-2$ dimensional) ***boundary*** of $\boldsymbol{v}$. We denote by $\mathscr{N}_{n-1}$ the set of all generalized surfaces.

(ii) An $\mathbf{R}$ valued measure β on $\mathbf{R}^n$ is said to be a ***generalized*** (n dimensional) ***body*** if there exists an $\mathbf{R}^n$ valued measure $\partial\beta$ on $\mathbf{R}^n$ such that

$$\int \operatorname{div} \boldsymbol{v} \, d\beta = \int \boldsymbol{v} \cdot d\,\partial\beta \tag{2.11}$$

for each $\boldsymbol{v} \in C^\infty(\mathbf{R}^n, \mathbf{R}^n)$. We call $\partial\beta$ the $(n-1$ dimensional) ***boundary*** of β. We denote by $\mathscr{N}_n$ the set of all generalized bodies.

One easily checks that if β is a generalized body then $\partial\beta$ is a generalized surface with $\partial\partial\beta = \mathbf{0}$. The meaning of the above definitions is explained in the following examples and in Proposition 2.10, below.

Examples 2.9.

(i) If $P \subset \mathbf{R}^n$ is a compact oriented $n-1$ dimensional oriented surface of class C^2 with Lipschitz boundary ∂P and $\boldsymbol{n} : P \to \mathbf{R}^n$ is the orienting normal, then the measure

$$\boldsymbol{\nu} := \boldsymbol{n}\,\mathscr{H}^{n-1} \llcorner P \tag{2.12}$$

is a generalized surface; the Stokes theorem shows that the boundary is given by

$$\partial\boldsymbol{\nu} = \boldsymbol{n} \wedge \boldsymbol{m}\,\mathscr{H}^{n-2} \llcorner \partial P$$

where $\boldsymbol{m}$ is the (inplane) normal to ∂P and $\boldsymbol{a} \wedge \boldsymbol{b} = \frac{1}{2}(\boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a})$ for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbf{R}^n$. In this sense classical surfaces and in particular planar polyhedra from $\mathfrak{S}$ are embedded in $\mathscr{N}_{n-1}$. More generally, let P be a countably $\mathscr{H}^{n-1}$ rectifiable set [1, 10] and $\boldsymbol{n} : P \to \mathbf{R}^n$ a $\mathscr{H}^{n-1}$ integrable function with values in the approximate normal space of P with integer multiplicity (a rectifiable $n-1$ dimensional current). Then (2.12) defines a generalized surface provided the distribution on the left hand side of (2.10) is representable by a measure. By [10; Theorem 4.2.16(2)] this occurs if and only if the boundary $\partial\boldsymbol{\nu}$ is an $n-2$ dimensional rectifiable current in the sense that $\partial\boldsymbol{\nu} = \boldsymbol{\xi}\mathscr{H}^{n-2} \llcorner M$ where M is a countably $\mathscr{H}^{n-2}$ rectifiable set and $\boldsymbol{\xi}$ a function with values in the approximate normal space to M of integer multiplicity. The generalized surfaces $\boldsymbol{\nu}$ of this type are called integral currents [10]; a particular case is the measure $\boldsymbol{\nu}$ as in (2.12) representing the measure theoretic boundary $P = \partial B$ of a set B of finite perimeter with $\boldsymbol{n}$ the measure theoretic normal [10, 1]; then $\boldsymbol{\nu} \in \mathscr{N}_{n-1}$ and $\partial\boldsymbol{\nu} = \mathbf{0}$. See below for more details on sets of finite perimeter and their boundaries.

(ii) If $\boldsymbol{\alpha} \in C_0^1(\mathbf{R}^n, \mathbf{R}^n)$ then

$$\boldsymbol{\nu} := \boldsymbol{\alpha}\,\mathscr{L}^n$$

is a generalized surface with

$$\partial\boldsymbol{\nu} = -\operatorname{curl}\boldsymbol{\alpha}\,\mathscr{L}^n;$$

this is an example of a “thick” (distributed) $n-1$ dimensional surface occupying an n dimensional set.

(iii) If $B \subset \mathbf{R}^n$ is a bounded open set with Lipschitz boundary ∂B of normal $\boldsymbol{n}$ then the measure

$$\beta := \mathscr{L}^n \llcorner B \tag{2.13}$$

is a generalized body; the divergence theorem shows that the boundary is given by

$$\partial\beta = \boldsymbol{n}\mathscr{H}^{n-1} \llcorner \partial B. \tag{2.14}$$

More generally if B is a bounded set of finite perimeter then (2.13) provides a generalized body with the boundary (2.14) where $\boldsymbol{n}$ and ∂B are the measure theoretic normal and boundary of B.

(iv) If $\gamma \in C_0^1(\mathbf{R}^n, \mathbf{R})$ then

$$\beta = \gamma \mathscr{L}^n \tag{2.15}$$

is a generalized body with

$$\partial\beta = -\nabla\gamma \mathscr{L}^n .$$

This is an example of a "fuzzy" body [5, 9]. More generally, if $\gamma \in BV(\mathbf{R}^n, \mathbf{R})$ is a function of bounded variation [1] then (2.15) provides a generalized body with the boundary given by

$$\partial\beta = -\nabla\gamma \tag{2.16}$$

where now $\nabla\gamma$ is the measure representing the weak derivative of γ. A particular case is the characteristic function $\gamma = 1_B$ of a bounded set of finite perimeter in which case there is no "fuzziness" at all.

The following proposition delineates the classes of generalized surfaces and bodies. In particular it shows that the dimensions $n-1$ and n indicated parenthetically in Definitions 2.8 emerge as a consequence of (2.10) and (2.11) despite the fact that these relations contain no explicit information about n.

Proposition 2.10.

(i) ([34; Section 7]) *Each $\boldsymbol{v} \in \mathscr{N}_{n-1}$ is of the form*

$$\boldsymbol{v} = \boldsymbol{n}\, \mathscr{H}^{n-1} \llcorner P + \boldsymbol{v}_c + \boldsymbol{m}\mathscr{L}^n \tag{2.17}$$

where P is a countably $\mathscr{H}^{n-1}$ rectifiable subset of $\mathbf{R}^n$, $\boldsymbol{n} : P \to \mathbf{R}^n$ is an $\mathscr{H}^{n-1}$ integrable function with values in the approximate normal space to P, $\boldsymbol{v}_c$ is an $\mathscr{L}^n$ singular measure which vanishes on sets of finite $\mathscr{H}^{n-1}$ measure, and $\boldsymbol{m}$ is an $\mathscr{L}^n$ integrable vector valued function.

(ii) *We have $\beta \in \mathscr{N}_n$ if and only if $\beta = \gamma\mathscr{L}^n$ where $\gamma \in BV(\mathbf{R}^n, \mathbf{R})$ and if it is the case then $\partial\beta$ is given by* (2.16) *where $\nabla\gamma$ is the measure representing the weak derivative of γ.*

Here the first term on the right hand side of (2.17) corresponds essentially to the standard notion of a surface (in which case the remaining two terms vanish), the second term corresponds to the Cantor part, as its dimension is between $n-1$ and n, and the third term corresponds to distributed surfaces. In particular, a nonvanishing $\boldsymbol{v} \in \mathscr{N}_{n-1}$ cannot be supported by a set of Hausdorff's dimension less than $n-1$ and likewise a nonvanishing $\beta \in \mathscr{N}_n$ cannot be supported by a set of Hausdorff's dimension less than n.

We note in passing that the space $\mathscr{N}_{n-1}$ is isomorphic, but not identical to the $n-1$ dimensional currents as defined in [10]. Namely, we have identified the space of $n-1$ vectors with $\mathbf{R}^n$ in case of $\boldsymbol{v}$ and the space of $n-2$ vectors with Skw in case of $\partial\boldsymbol{v}$ via the Hodge $*$ map. Under the Hodge $*$ map the exterior and interior derivatives exchange their roles and thus the boundary $\partial\boldsymbol{v}$ defined above is mapped to the boundary defined in [10]. The same applies to $\mathscr{N}_n$.

Let $k : \mathbf{R}^n \to (0, \infty]$ be the (Riesz) kernel given by

$$k(\boldsymbol{x}) = n^{-1}\kappa_n^{-1}|\boldsymbol{x}|^{-n+1},$$

$\boldsymbol{x} \in \mathbf{R}^n$, where we put $|\boldsymbol{x}|^{-n+1} = \infty$ if $\boldsymbol{x} = \mathbf{0}$ and $n > 1$ and where κ_n is the volume of the unit ball in $\mathbf{R}^n$.

The following result gives a natural extension of the Cauchy flux to some generalized surfaces.

Theorem 2.11. *If* $\mathsf{q} \in \mathscr{DM}(\mathbf{R}^n)$ *and* $\boldsymbol{\nu} \in \mathscr{N}_{n-1}$ *satisfy*

$$\left.\begin{aligned} &\int\int k(\boldsymbol{x}-\boldsymbol{y})\, d\,|\boldsymbol{\nu}|(\boldsymbol{x})\, d\,|\operatorname{div}\mathsf{q}|(\boldsymbol{y}) < \infty,\\ &\int\int k(\boldsymbol{x}-\boldsymbol{y})\, d\,|\partial\boldsymbol{\nu}|(\boldsymbol{x})\, d\,|\mathsf{q}|(\boldsymbol{y}) < \infty \end{aligned}\right\} \tag{2.18}$$

then the limit

$$\tilde{F}(\boldsymbol{\nu}) := \lim_{\rho\to 0} \int \mathsf{q}_\rho \cdot d\boldsymbol{\nu} \tag{2.19}$$

exists and is finite. In particular, if $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ *and*

$$\left.\begin{aligned} &\int_P\int k(\boldsymbol{x}-\boldsymbol{y})\, d\mathscr{H}^{n-1}(\boldsymbol{x})\, d\,|\operatorname{div}\mathsf{q}|(\boldsymbol{y}) < \infty,\\ &\int\int_{\partial P} k(\boldsymbol{x}-\boldsymbol{y})\, d\mathscr{H}^{n-2}(\boldsymbol{x})\, d\,|\mathsf{q}|(\boldsymbol{y}) < \infty \end{aligned}\right\} \tag{2.20}$$

then the limit (2.8) *exists and is finite.*

The sought extension is (2.19). We may call $\tilde{F}(\boldsymbol{\nu})$ the flux through the generalized surface $\boldsymbol{\nu}$.

Remarks 2.12.

(i) In Section 4 we shall give an explicit formula for the limits (2.19) and (2.8); that formula does not involve the mollifier ω and thus the limits (2.19) and (2.8) are independent of it.

(ii) In view of the singularity of the kernel $k(\boldsymbol{x}-\boldsymbol{y})$ near $\boldsymbol{x} = \boldsymbol{y}$ the conditions (2.18) say that q and $\operatorname{div}\mathsf{q}$ are not too singular on the surface $\boldsymbol{\nu}$; in the particular case of the boundary $\boldsymbol{\nu} = \boldsymbol{n}\mathscr{H}^{n-1} \llcorner \partial B$ of a set B of finite perimeter we shall see [in (2.30)] that $(2.18)_1$ implies that no mass of the measure $\operatorname{div}\mathsf{q}$ is contained in ∂B (but $(2.18)_1$ requires more). Although Conditions (2.18) are new, the need of *some* conditions emerged previously in particular cases (see [29] for $\boldsymbol{q} \in L^1(\mathbf{R}^n, \mathbf{R}^n)$, $\operatorname{div}\boldsymbol{q} \in L^1(\mathbf{R}^n, \mathbf{R})$ and [8] for $\boldsymbol{q} \in L^1(\mathbf{R}^n, \mathbf{R}^n)$, $\operatorname{div}\boldsymbol{q} \in \mathscr{M}(\mathbf{R}^n, \mathbf{R})$. See also [31–33, 9]).

(iii) If $\mathsf{q} = \boldsymbol{q}\mathscr{L}^n$ where $\boldsymbol{q} \in \mathscr{D}L^\infty(\mathbf{R}^n)$ and if additionally $\boldsymbol{q}$ vanishes outside a bounded set then (4.4) (below) shows that the integrals

$$\int k(\boldsymbol{x}-\boldsymbol{y})\, d\,|\operatorname{div}\mathsf{q}|(\boldsymbol{y}) \equiv \int k(\boldsymbol{x}-\boldsymbol{y})|\operatorname{div}\boldsymbol{q}(\boldsymbol{y})|\, d\mathscr{L}^n(\boldsymbol{y}),$$
$$\int\int k(\boldsymbol{x}-\boldsymbol{y})\, d\,|\mathsf{q}|(\boldsymbol{y}) \equiv \int k(\boldsymbol{x}-\boldsymbol{y})|\boldsymbol{q}(\boldsymbol{y})|\, d\mathscr{L}^n(\boldsymbol{y})$$

are bounded functions of $\boldsymbol{x}$ on $\mathbf{R}^n$ and hence (2.18) is satisfied by all $\boldsymbol{\nu} \in \mathscr{N}_{n-1}$ and thus the limit in (2.19) always exists; hence $\tilde{F}(\boldsymbol{\nu})$ exists for every generalized surface.

(iv) In the general case of $\mathsf{q} \in \mathscr{DM}(\mathbf{R}^n)$, it is a matter of the relative position of the measures $\boldsymbol{\nu}$ and q whether the conditions (2.18) are satisfied or not. However, the next proposition shows that (2.18) is surely satisfied by $\mathscr{L}^n$ almost every translation of $\boldsymbol{\nu}$.

If ϕ is a measure on $\mathbf{R}^n$ with values in a finite dimensional vectorspace and $\boldsymbol{a} \in \mathbf{R}^n$, we define the translation $\mathrm{T}_{\boldsymbol{a}}\,\phi$ of ϕ by $\boldsymbol{a}$ by $\mathrm{T}_{\boldsymbol{a}}\,\phi(M) = \phi(\mathrm{T}_{-\boldsymbol{a}} M)$ for every Borel set $M \subset \mathbf{R}^n$.

Remark 2.13. *If ϕ, ψ are nonnegative finite measures on $\mathbf{R}^n$ then*

$$\int\int k(\boldsymbol{x}-\boldsymbol{y})\, d\,\mathrm{T}_{\boldsymbol{a}}\,\phi(\boldsymbol{x})\, d\psi(\boldsymbol{y}) < \infty \tag{2.21}$$

and

$$\liminf_{\rho\to 0} \int \phi_\rho\, d\mathrm{T}_{\boldsymbol{a}}\psi < \infty \tag{2.22}$$

for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$.

As a corollary we obtain that if $\boldsymbol{\nu} \in \mathscr{N}_{n-1}$ and $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ then the flux $\tilde{F}(\mathrm{T}_{\boldsymbol{a}}\,\boldsymbol{\nu})$ is well defined for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$.

Next, we consider the situation when the flux through the generalized surface $\boldsymbol{\nu}$ is represented by a measure:

Theorem 2.14. *If $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ and $\boldsymbol{\nu} \in \mathscr{N}_{n-1}$ satisfies* (2.18) *and*

$$\int\int k(\boldsymbol{x}-\boldsymbol{y})\, d\,|\boldsymbol{\nu}|(\boldsymbol{x})\, d\,|\mathbf{q}|(\boldsymbol{y}) < \infty, \quad \liminf_{\rho\to 0}\int |\mathbf{q}|_\rho\, d\,|\boldsymbol{\nu}| < \infty \tag{2.23}$$

then the limit

$$\tilde{F}(f\boldsymbol{\nu}) := \lim_{\rho\to 0}\int f\mathbf{q}_\rho \cdot d\boldsymbol{\nu} \tag{2.24}$$

exists and is finite for every $f \in C_0^\infty(\mathbf{R}^n, \mathbf{R})$ and there exists a signed measure $\mu_{\boldsymbol{\nu}}$ supported by spt $\boldsymbol{\nu}$ *such that*

$$\tilde{F}(f\boldsymbol{\nu}) = \int f\, d\mu_{\boldsymbol{\nu}} \tag{2.25}$$

for every $f \in C_0^\infty(\mathbf{R}^n, \mathbf{R})$. In particular, if $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ satisfies (2.20) *and*

$$\int_P\int k(\boldsymbol{x}-\boldsymbol{y})\, d\mathscr{H}^{n-1}(\boldsymbol{x})\, d\,|\mathbf{q}|(\boldsymbol{y}) < \infty, \quad |\mathbf{q}|^{n-1}(P) < \infty$$

then for $\boldsymbol{\nu} = \boldsymbol{n}\,\mathscr{H}^{n-1} \llcorner P$ the limit (2.24) *exists and is finite for every $f \in C_0^\infty(\mathbf{R}^n, \mathbf{R})$ and there exists a measure $\mu_{\boldsymbol{\nu}}$ supported by P such that* (2.25) *holds.*

In this situation we have a well defined flux not only for the whole surface $\boldsymbol{\nu}$, but also for any Borel set $M \subset \mathbf{R}^n$ by setting

$$\tilde{F}(\boldsymbol{\nu}, M) := \mu_{\boldsymbol{\nu}}(M). \tag{2.26}$$

The interpretation of the value $\tilde{F}(\boldsymbol{\nu}, M)$ is that it is the flux through the set M where the latter is interpreted as a "subset" of $\boldsymbol{\nu}$. In particular, if $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ satisfies the hypothesis of Theorem 2.14 and $\boldsymbol{\nu} := \boldsymbol{n}\,\mathscr{H}^{n-1}$ then for any Borel set M the value $\tilde{F}(\boldsymbol{\nu}, M)$ is the flux through the subset $P \cap M$ of P.

Remarks 2.15.

(i) Conditions (2.18) and $(2.23)_1$ have been discussed above; Condition $(2.23)_2$ is the main boundedness condition guaranteeing the existence of the measure $\mu_{\boldsymbol{\nu}}$; similar conditions are found in [31–33] in more restricted or more general contexts (see also [9]).

(ii) In the particular case $\mathbf{q} = \boldsymbol{q}\mathscr{L}^n$ with $\boldsymbol{q} \in \mathscr{D}L^\infty(\mathbf{R}^n)$ vanishing outside a bounded set, (2.18) and (2.23) are satisfied by every $\boldsymbol{\nu} \in \mathscr{N}_{n-1}$ and the flux $\tilde{F}$ through $\boldsymbol{\nu}$ is represented by a measure for every generalized surface.

(iii) In the general case, Remark 2.13 implies that given $\boldsymbol{\nu} \in \mathscr{N}_{n-1}$ and $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$, the flux $\tilde{F}(\mathrm{T}_{\boldsymbol{a}}\,\boldsymbol{\nu})$ is represented by a measure for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$.

(iv) We note that the localization procedure (2.26) employs the measure $\mu_{\boldsymbol{\nu}}$; no such a localization is possible in the more general context of Theorem 2.11 (see Remark 2.17, below).

If $B \subset \mathbf{R}^n$ is an $\mathscr{L}^n$ measurable set and $\boldsymbol{x} \in \mathbf{R}^n$, we define the n dimensional density $\Theta(\boldsymbol{x}, B)$ of B at $\boldsymbol{x}$ by

$$\Theta(\boldsymbol{x}, B) = \lim_{\rho \to 0} \kappa_n^{-1} \rho^{-n} \mathscr{L}^n(B \cap \mathbf{B}(\boldsymbol{x}, \rho))$$

whenever the limit exists, where $\mathbf{B}(\boldsymbol{x}, \rho)$ is the open ball of center $\boldsymbol{x}$ and radius ρ. We define the measure theoretic interior B_* of B by

$$B_* = \{ \boldsymbol{x} \in \mathbf{R}^n : \Theta(\boldsymbol{x}, B) = 1 \}$$

and recall that B_* is a Borel set that differs from B by a set of $\mathscr{L}^n$ measure 0. We say that B is a normalized set if $B = B_*$, and recall that the measure theoretic boundary ∂B of B is defined by $\partial B = \mathbf{R}^n \sim (B_* \cup (\mathbf{R}^n \sim B)_*$.

We finally show that if the generalized surface $\boldsymbol{\nu}$ represents the measure theoretic boundary of a set of finite perimeter then the measure associated with $\boldsymbol{\nu}$ satisfies the divergence theorem.

Theorem 2.16. *If a normalized set of finite perimeter $B \subset \mathbf{R}^n$ with the measure theoretic normal $\boldsymbol{n}$ and a divergence measure vectorfield $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ satisfy*

$$\left.\begin{aligned} \int \int_{\partial B} k(\boldsymbol{x} - \boldsymbol{y})\, d\mathscr{H}^{n-1}(\boldsymbol{x})\, d\,|\mathbf{q}|(\boldsymbol{y}) < \infty, \\ \int \int_{\partial B} k(\boldsymbol{x} - \boldsymbol{y})\, d\mathscr{H}^{n-1}(\boldsymbol{x})\, d\,|\operatorname{div} \mathbf{q}|(\boldsymbol{y}) < \infty, \\ \liminf_{\rho \to 0} \int_{\partial B} |\mathbf{q}|_\rho \, d\mathscr{H}^{n-1} < \infty \end{aligned}\right\} \qquad (2.27)$$

then the generalized surface $\boldsymbol{\nu} = \boldsymbol{n}\mathscr{H}^{n-1} \llcorner \partial B$ satisfies the hypotheses of Theorem 2.14 *and hence the flux through ∂B is represented by a measure $\mu_{\partial B}$ (supported by* $\operatorname{cl} \partial B$*) in the sense that*

$$\tilde{F}(f\boldsymbol{\nu}) := \lim_{\rho \to 0} \int_{\partial B} f \mathbf{q}_\rho \cdot \boldsymbol{n}\, d\mathscr{H}^{n-1} = \int f\, d\mu_{\partial B} \qquad (2.28)$$

for every $f \in C_0^\infty(\mathbf{R}^n, \mathbf{R})$. The measure $\mu_{\partial B}$ satisfies the divergence theorem

$$\int f\, d\mu_{\partial B} = \int_B f\, d\, \operatorname{div} \mathbf{q} + \int_B \nabla f \cdot d\mathbf{q} \qquad (2.29)$$

for any $f \in C_0^\infty(\mathbf{R}^n, \mathbf{R})$.

Lemma 5.1 (below) shows that the hypotheses $(2.27)_{1,\,2}$ imply that the measures $\mathbf{q}$ and $\operatorname{div} \mathbf{q}$ vanish on ∂B, i.e.,

$$|\mathbf{q}|(\partial B) = 0, \quad |\operatorname{div} \mathbf{q}|(\partial B) = 0. \qquad (2.30)$$

We have chosen to formulate the divergence theorem only for the special case of generalized bodies represented by *sets* (of finite perimeter), as in the classical case. There is also a divergence theorem for generalized bodies of Definition 2.8(ii); this will be the subject of a future paper.

Remark 2.17. There seems to be no natural definition of the flux represented by a divergence measure vectorfield $\mathbf{q}$ for all surfaces from some large class independent of the particular $\mathbf{q}$; by shifting a surface a little as explained above to avoid singularities of one particular $\mathbf{q}$ one encounters singularities of another $\mathbf{q}$. This is in the nature of things and not in the employed mathematical tools, as will be now explained. In defining the flux F through an oriented surface A with normal $\boldsymbol{n}$ (the last two objects interpreted intuitively for the moment) corresponding to $\mathbf{q} \in \mathscr{D}\mathscr{M}(\mathbf{R}^n)$, one seeks to give a meaning to the right hand side of the equation

$$\tilde{F}(A) = \int_A \mathbf{q} \cdot \boldsymbol{n} \, d\mathscr{H}^{n-1}. \tag{2.31}$$

If $\mathbf{q} = \boldsymbol{q}\mathscr{L}^n$ where $\boldsymbol{q} \in \mathscr{D}L^\infty(\mathbf{R}^n)$ is a continuous function, then the right hand side of (2.31) is identified with

$$\int_A \boldsymbol{q} \cdot \boldsymbol{n} \, d\mathscr{H}^{n-1}; \tag{2.32}$$

however, starting from the case $\boldsymbol{q} \in \mathscr{D}L^\infty(\mathbf{R}^n)$, the expression in (2.32) is devoid of an immediate meaning and additional interpretations are needed, such as those in (2.2) and (2.19). The existence of the limits in these formulas is far from obvious and both these rest on the fact that the weak divergence of $\boldsymbol{q}$ or of $\mathbf{q}$ is well behaved in the sense that $\operatorname{div} \boldsymbol{q}$ is a bounded function and $\operatorname{div} \mathbf{q}$ a measure. Moreover, the case $\boldsymbol{q} \in \mathscr{D}L^\infty(\mathbf{R}^n)$ is essentially the most general case in which the limit in (2.2) is guaranteed universally for a class of surfaces that is independent of the particular flux vector. One may take for that class either $\mathfrak{S}$ as the minimal choice or $\mathscr{N}_{n-1}$ as the maximal choice. This universality property comes from the fact that in addition to the boundedness of $\operatorname{div} \boldsymbol{q}$, the function $\boldsymbol{q}$ itself is bounded and thus, roughly speaking, the integral in (2.32) converges. In the case of $\mathbf{q} \in \mathscr{D}\mathscr{M}(\mathbf{R}^n)$ the surface must be good for the particular $\mathbf{q}$ in the sense of the hypotheses Theorems 2.11, 2.14, and 2.16.

Let us now see that *some* additional conditions are needed in case of a general $\mathbf{q} \in \mathscr{D}\mathscr{M}(\mathbf{R}^n)$. Let us restrict ourselves to surfaces modeled as subsets of the boundary ∂B of a subset B of $\mathbf{R}^n$. The flux through the whole of ∂B can be defined using the divergence theorem by putting

$$\tilde{F}(\partial B) = \int_B d \operatorname{div} \mathbf{q} \equiv \mathbf{q}(\partial B).$$

However, the interpretation requires that the flux be defined also for subsets of ∂B, i.e., for parts of the boundary from some reasonably rich class of parts. We have seen in (2.26) that such a localization is possible if the flux through ∂B is represented by a measure, and Theorem 2.16 gives a sufficient condition for the latter to occur. We note that the right hand side of (2.29) is meaningful generally and thus the existence of a measure $\mu_{\partial B}$ as in (2.29) is a well posed question. The general answer is *negative,* though, and has nothing to do with the smoothness of ∂B. The problem of the representation of the right hand side of (2.29) in terms of an object associated with the boundary of B has been considered in the literature. The most general result is that of [33, 32] which shows that if B is an open set (with a boundary that need not be smooth) the value of the right hand side of (2.29) depends only on the values of f on ∂B and in fact *there exists a continuous linear functional $g \mapsto T(g)$, defined on the space of Lipschitz functions g on ∂B, such that*

$$T(f|\partial B) = \int_B f\, d\, \operatorname{div} \mathbf{q} + \int_B \nabla f \cdot d\mathbf{q} \tag{2.33}$$

for any $f \in C_0^\infty(\mathbf{R}^n, \mathbf{R})$ *where* $f|\partial B$ *is the restriction of* f *to* ∂B (and more generally for any Lipschitz continuous function f provided the second term on the right hand side of (2.33) is given an appropriate interpretation). Let us call the functional T the normal trace of $\mathbf{q}$ on ∂B. Example 2.5 in [33] exhibits a vectorfield $\boldsymbol{q}$ in $L^p(\mathbf{R}^2, \mathbf{R}^2)$ for all $p \in [1, 2)$ and a $B \subset \mathbf{R}^2$ with C^∞ boundary such that T is not represented by a measure, in fact T is isomorphic to the one dimensional *distribution* giving the principal value of $1/x$ on $\mathbf{R}$ (which is not a distribution represented by a measure). The vectorfield is a C^∞ function outside the origin and its weak divergence vanishes; the last shows that also the smoothness of $\operatorname{div} \boldsymbol{q}$ is irrelevant. The mentioned example also exhibits a vectorfield in $L^p(\mathbf{R}^2, \mathbf{R}^2)$ such that the normal trace is a distribution of fractional order $q < 1$ arbitrarily close to 1 provided the vectorfield is in $L^p(\mathbf{R}^2, \mathbf{R}^2)$ with $p > 1$ sufficiently close to 1. Thus, roughly, $T(g)$ depends on "fractional derivatives" of g in the tangential directions of ∂B of order q close to 1; the occurrence of derivatives prevents a localization. In the context of normal traces, Theorem 2.16 provides a sufficient condition for the normal trace to be represented by a measure. Related to the non–representability of T by a measure is the result [35; Theorem 1.2, Chapter I] which shows that if B is a region with C^2 boundary, $\mathbf{q} = \boldsymbol{q}\mathscr{L}^n$ where $\boldsymbol{q} \in L^2(B, \mathbf{R}^n)$ and $\operatorname{div} \boldsymbol{q} \in L^2(B, \mathbf{R})$ then T is merely a continuous linear functional on $W^{1/2,\,2}(\partial B, \mathbf{R})$ and hence again, fractional derivatives of order up to $1/2$ intervene and a localization is impossible. Only in case of a bounded $\boldsymbol{q}$ Anzellotti [2; Section 1] shows that the trace of $\mathbf{q} = \boldsymbol{q}\mathscr{L}^n$ on ∂B is given by a $\mathscr{H}^{n-1}$ integration of a bounded function, in accordance with Theorems 2.3 and Theorem 2.11 in this particular case. The reader is also referred to [31–32, 6–7, 9] for additional information on the problem of traces.

We now turn to proofs. In Section 3 we prove that each Cauchy flux is represented by $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ in the sense of (2.7). The rest of Theorems 2.6, 2.14, and 2.16 conversely assumes a given $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ and proves various assertions about objects stemming from it. To prove these, we first establish a homotopy formula (1.2) in Section 4. With (1.2) at our disposal we complete the proofs in Section 5. Specifically, we prove the existence of the limit (2.19) at the asserted generality, the uniqueness of the flux vector of the given Cauchy flux, and after the proof of Remark 2.13, we prove that each $\mathbf{q}$ gives rise to a unique Cauchy flux that satisfies (2.7). The proofs are then completed by proving the representation of the flux by a measure under the hypotheses of Theorem 2.14 which satisfies the divergence theorem 2.16.

3 The existence of the flux vector

We prove the existence part of Assertion (i) of Theorem 2.6. Some preliminary results are needed.

Remark 3.1. *If* ϕ *is a finite nonnegative measure,* r *is an integer,* $0 \le r \le n$, *and* $M \subset \mathbf{R}^n$ *a Borel set then*

$$\int \phi^r(\mathrm{T}_{\boldsymbol{a}} M)\omega_\rho(\boldsymbol{a})\, d\mathscr{L}^n(\boldsymbol{a}) \le \int_M \phi_\rho\, d\mathscr{H}^r$$

for any $\rho > 0$.

Proof We have

$$\phi^r(\mathrm{T}_{\boldsymbol{a}} M) = \liminf_{\tau\to 0} \int_{\mathrm{T}_{\boldsymbol{a}} M} \phi_\tau \, d\mathscr{H}^r = \liminf_{\tau\to 0} \int_M \mathrm{T}_{-\boldsymbol{a}} \phi_\tau \, d\mathscr{H}^r$$

for any $\boldsymbol{a} \in \mathbf{R}^n$ where $\mathrm{T}_{\boldsymbol{a}} \phi_\tau$ denotes the translation of the function ϕ_τ by $\boldsymbol{a}$. Hence by Fatou's lemma and by the properties of mollification

$$\begin{aligned}\int \phi^r(\mathrm{T}_{\boldsymbol{a}} M)\omega_\rho(\boldsymbol{a})\, d\mathscr{L}^n(\boldsymbol{a}) &= \int \omega_\rho(\boldsymbol{a}) \liminf_{\tau\to 0} \int_M \mathrm{T}_{-\boldsymbol{a}} \phi_\tau \, d\mathscr{H}^r d\mathscr{L}^n(\boldsymbol{a}) \\ &\le \liminf_{\tau\to 0} \int \omega_\rho(\boldsymbol{a}) \int_M \mathrm{T}_{-\boldsymbol{a}} \phi_\tau \, d\mathscr{H}^r d\mathscr{L}^n(\boldsymbol{a}) \\ &= \liminf_{\tau\to 0} \int_M (\phi_\tau)_\rho \, d\mathscr{H}^r \\ &= \liminf_{\tau\to 0} \int_M (\phi_\rho)_\tau \, d\mathscr{H}^r \\ &= \int_M \phi_\rho \, d\mathscr{H}^r. \quad \square\end{aligned}$$

To prove the existence of the flux vector $\mathbf{q}$ corresponding to a given (unbounded) Cauchy flux F, we first mollify F to gain more regularity and to profit from Theorem 2.3.

Lemma 3.2. *Let F be a Cauchy flux and for each $\tau > 0$ define $E^\tau : \mathfrak{S} \to \mathbf{R}$ by*

$$E^\tau(A) = \int F(\mathrm{T}_{\boldsymbol{a}} A)\omega_\tau(\boldsymbol{a})\, d\mathscr{L}^n(\boldsymbol{a}), \tag{3.1}$$

$A \in \mathfrak{S}$. Then E^τ is a bounded Cauchy flux and hence there exists a unique $\boldsymbol{p}^\tau \in \mathscr{D}L^\infty(\mathbf{R}^n)$ and $N_\tau \subset \mathbf{R}^n$ with $\mathscr{L}^n(N_\tau) = 0$ such that

$$E^\tau(A) = \int \boldsymbol{p}^\tau \cdot \boldsymbol{n}\, d\mathscr{H}^{n-1} \tag{3.2}$$

for each $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ with $\mathscr{H}^{n-1}(P \cap N_\tau) = 0$.

It is a part of the assertion of the lemma that the function $\boldsymbol{a} \mapsto F(\mathrm{T}_{\boldsymbol{a}} A)$ is locally $\mathscr{L}^n$ integrable for each $A \in \mathfrak{S}$ and so the definition (3.1) is meaningful. We shall see later that the flux vector $\boldsymbol{p}^\tau$ has an infinitely differentiable representative and thus the set N_τ can be chosen an empty set. However, this is not obvious at the present stage.

Proof Letting $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ and combining Definition 2.4(iii) with Remark 3.1 we obtain

$$\int |F(\mathrm{T}_{\boldsymbol{a}} A)|\omega_\tau(\boldsymbol{a})\, d\mathscr{L}^n(\boldsymbol{a}) \le \int \theta^{n-1}(\mathrm{T}_{\boldsymbol{a}} A)\omega_\tau(\boldsymbol{a})\, d\mathscr{L}^n(\boldsymbol{a}) \le \int_P \theta_\tau \, d\mathscr{H}^{n-1}.$$

This shows that the function $\boldsymbol{a} \mapsto F(\mathrm{T}_{\boldsymbol{a}} A)$ is locally $\mathscr{L}^n$ integrable and

$$|E^\tau(A)| \le \int_P \theta_\tau \, d\mathscr{H}^{n-1} \tag{3.3}$$

for each $A = (P, \boldsymbol{n}) \in \mathfrak{S}$. We prove that E^τ is a bounded Cauchy flux by verifying Definition 2.1. Definition 2.1(i) follows immediately from Definition 2.4(ii). Further, using $\theta_\tau(\boldsymbol{x}) \le \tau^{-n}|\omega|_{L^\infty}\mathrm{M}(\theta)$, we have Definition 2.1(ii) with $b = \tau^{-n}|\omega|_{L^\infty}\mathrm{M}(\theta)$ by (3.3). Finally, if $B \in \mathfrak{B}$ then

$$|E^\tau(\partial B)| \le \int_B \eta_\tau \, d\mathscr{L}^n \tag{3.4}$$

by Definition 2.4(iv) and Remark 3.1 so that we have Definition 2.1(iii) with $c = \tau^{-n}|\omega|_{L^\infty}\mathrm{M}(\eta)$. Thus E^τ is a bounded Cauchy flux; the existence of $\boldsymbol{p}^\tau$ follows from Theorem 2.3. $\square$

Lemma 3.3. *If F is a Cauchy flux then there exists a $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ such that for any $\tau > 0$ the vector $\boldsymbol{p}^\tau$ from Lemma 3.2 is the τ mollification of $\mathbf{q}$ and hence*

$$E^\tau(A) = \int_P \mathbf{q}_\tau \cdot \boldsymbol{n}\, d\mathscr{H}^r \tag{3.5}$$

for each $A = (P, \boldsymbol{n}) \in \mathfrak{S}$.

Proof Let θ, η be the measures as in Definition 2.4. From (3.3) and (3.2) we have

$$E^\tau(A) = \int_P \boldsymbol{p}^\tau \cdot \boldsymbol{n}\, d\mathscr{H}^{n-1} \le \int_P \theta_\tau\, d\mathscr{H}^{n-1}$$

for any $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ with $\mathscr{H}^{n-1}(P \cap N_\tau) = 0$. It follows that

$$|\boldsymbol{p}^\tau(\boldsymbol{x})| \le \theta_\tau(\boldsymbol{x}), \tag{3.6}$$

for $\mathscr{L}^n$ a.e. $\boldsymbol{x} \in \mathbf{R}^n$. Similarly, by (3.4) and $(2.3)_2$ we have

$$E^\tau(\partial B) = \int_B \operatorname{div} \boldsymbol{p}^\tau\, d\mathscr{L}^n \le \int_B \eta_\tau\, d\mathscr{L}^n$$

for any $B \in \mathfrak{B}$ such that $\mathscr{H}^{n-1}(\partial B \cap N_\tau) = 0$ and hence

$$|\operatorname{div} \boldsymbol{p}^\tau(\boldsymbol{x})| \le \eta_\tau(\boldsymbol{x}) \tag{3.7}$$

for $\mathscr{L}^n$ a.e. $\boldsymbol{x} \in \mathbf{R}^n$. Integrating (3.6) and (3.7) with respect to $\mathscr{L}^n$ over $\mathbf{R}^n$ we obtain

$$\mathrm{M}(\boldsymbol{p}^\tau \mathscr{L}^n) \le \mathrm{M}(\theta), \quad \mathrm{M}(\operatorname{div} \boldsymbol{p}^\tau \mathscr{L}^n) \le \mathrm{M}(\eta)$$

for all $\tau > 0$. Hence there exists a sequence $\tau_k \to 0$, denoted by τ for brevity, and $\mathbf{q} \in \mathscr{M}(\mathbf{R}^n, \mathbf{R}^n), \phi \in \mathscr{M}(\mathbf{R}^n, \mathbf{R})$ such that

$$\boldsymbol{p}^\tau \mathscr{L}^n \to \mathbf{q}, \quad \operatorname{div} \boldsymbol{p}^\tau \mathscr{L}^n \to \phi$$

weak* in $\mathscr{M}(\mathbf{R}^n, \mathbf{R}^n)$ and $\mathscr{M}(\mathbf{R}^n, \mathbf{R})$, respectively. Since $\operatorname{div} \boldsymbol{p}^\tau \mathscr{L}^n$ is the weak divergence of $\boldsymbol{p}^\tau \mathscr{L}^n$, one easily finds that $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ and $\phi = \operatorname{div} \mathbf{q}$; hence

$$\boldsymbol{p}^\tau \mathscr{L}^n \to \mathbf{q}, \quad \operatorname{div} \boldsymbol{p}^\tau \mathscr{L}^n \to \operatorname{div} \mathbf{q}.$$

Let $A = (P, \boldsymbol{n}) \in \mathfrak{S}$. By (3.2) we have

$$E^\tau(\mathrm{T}_{\boldsymbol{a}} A) = \int_P \mathrm{T}_{\boldsymbol{a}}\, \boldsymbol{p}^\tau \cdot \boldsymbol{n}\, d\mathscr{H}^{n-1}$$

for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$. Letting $\rho > 0$, multiplying by $\omega_\rho(\boldsymbol{a})$ and integrating we obtain

$$\int E^\tau(\mathrm{T}_{\boldsymbol{a}} A)\omega_\rho(\boldsymbol{a})\, d\mathscr{L}^n(\boldsymbol{a}) = \int_P (\boldsymbol{p}^\tau)_\rho \cdot \boldsymbol{n}\, d\mathscr{H}^{n-1}. \tag{3.8}$$

Observing that for a fixed k the function $\boldsymbol{a} \mapsto E^\tau(\mathrm{T}_{\boldsymbol{a}} A)$ is the τ mollification of the locally $\mathscr{L}^n$ integrable function $\boldsymbol{a} \mapsto F(\mathrm{T}_{\boldsymbol{a}} A)$ we deduce that $E^\tau(\mathrm{T}_{\boldsymbol{a}} A) \to F(\mathrm{T}_{\boldsymbol{a}} A)$ as $k \to \infty$ for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$ and locally in L^1. Further, since $\boldsymbol{p}^\tau \mathscr{L}^n \to \mathbf{q}$, weak* in $\mathscr{M}(\mathbf{R}^n, \mathbf{R}^n)$, we have $(\boldsymbol{p}^\tau)_\rho(\boldsymbol{x}) \to \mathbf{q}_\rho(\boldsymbol{x})$ for every $\boldsymbol{x} \in \mathbf{R}^n$ and locally in L^1. Thus the limit in (3.8) gives

$$\int F(\mathrm{T}_{\boldsymbol{a}} A)\omega_\rho(\boldsymbol{a})\, d\mathscr{L}^n(\boldsymbol{a}) = \int_P \mathbf{q}_\rho \cdot \boldsymbol{n}\, d\mathscr{H}^{n-1}.$$

Observing that the left hand side is $E^\rho(A)$ and comparing with (3.2) we thus conclude that $\boldsymbol{p}^\rho = \mathbf{q}_\rho$. Hence $\boldsymbol{p}^\tau$ is representable by a continuous function; consequently the exceptional set N_τ vanishes and (3.5) holds. □

Proof of Theorem* 2.6, *Part* (i), *existence By Lemma 3.3 there exists an $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ such that (3.5) holds for each $\tau > 0$ and $A \in \mathfrak{S}$. Recalling from the proof of Lemma 3.3 that the function $\boldsymbol{a} \mapsto E^\tau(\mathrm{T}_{\boldsymbol{a}} A)$ is the τ mollification of the locally $\mathscr{L}^n$ integrable function $\boldsymbol{a} \mapsto F(\mathrm{T}_{\boldsymbol{a}} A)$ we deduce that $E^\tau(\mathrm{T}_{\boldsymbol{a}} A) \to F(\mathrm{T}_{\boldsymbol{a}} A)$ as $\tau \to 0$ for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$. Then the limit in (3.5) gives (2.7). Thus $\mathbf{q}$ is the flux vector of F. □

4 Homotopy formula

We now introduce the main tool for our treatment of the divergence measure vector-fields.

Let $\boldsymbol{k} : \mathbf{R}^n \sim \{\mathbf{0}\} \to \mathbf{R}^n$ given by

$$\boldsymbol{k}(\boldsymbol{x}) = n^{-1}\kappa_n^{-1}\boldsymbol{x}/|\boldsymbol{x}|^n,$$

$\boldsymbol{x} \in \mathbf{R}^n \sim \{\mathbf{0}\}$. For any $\boldsymbol{x} \in \mathbf{R}^n$ we denote by $k_{\boldsymbol{x}}, \boldsymbol{k}_{\boldsymbol{x}}$ the maps defined by $k_{\boldsymbol{x}}(\boldsymbol{y}) = k(\boldsymbol{x}-\boldsymbol{y}), \boldsymbol{k}_{\boldsymbol{x}}(\boldsymbol{y}) = \boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})$ for every $\boldsymbol{y}$ for which the right hand sides are defined.

If μ is a nonnegative finite measure in $\mathbf{R}^n$, we define $\mathrm{G}(\mu) : \mathbf{R}^n \to [0,\infty]$ by

$$\mathrm{G}(\mu)(\boldsymbol{x}) = \int k_{\boldsymbol{x}}\, d\mu,$$

$\boldsymbol{x} \in \mathbf{R}^n$. An application of Fatou's lemma shows that $\mathrm{G}(\mu)$ is a lower semicontinuous function. If $\phi, \mathbf{q}$ are measures with values in $\mathbf{R}$ and $\mathbf{R}^n$, respectively, we define the ***Newton homotopies*** $\mathrm{H_N}\,\phi$ of ϕ and $\mathrm{H_N}\,\mathbf{q}$ of $\mathbf{q}$ as functions with values in $\mathbf{R}^n$ and Skw by

$$\mathrm{H_N}\,\phi(\boldsymbol{x}) = \int \boldsymbol{k}_{\boldsymbol{x}}\, d\phi, \tag{4.1}$$

$$\mathrm{H_N}\,\mathbf{q}(\boldsymbol{x}) = -2\int \boldsymbol{k}_{\boldsymbol{x}} \wedge d\mathbf{q}, \tag{4.2}$$

for every $\boldsymbol{x} \in \mathbf{R}^n$ for which the integrals in (4.1) and (4.2) are well defined. The integrands in (4.1) and (4.2) are bounded pointwise by $k_{\boldsymbol{x}}$ and hence $\mathrm{H_N}\,\phi$ and $\mathrm{H_N}\,\mathbf{q}$ are bounded pointwise by $\mathrm{G}(|\phi|)$ and $2\mathrm{G}(|\mathbf{q}|)$; we shall later prove the following assertion to see that $\mathrm{G}(|\phi|)$ and $\mathrm{G}(|\mathbf{q}|)$ and hence $\mathrm{H_N}\,\phi$ and $\mathrm{H_N}\,\mathbf{q}$ are $\mathscr{L}^n$ locally integrable functions.

Remark 4.1. *If μ is a nonnegative finite measure then $\mathrm{G}(\mu)$ is a $\mathscr{L}^n$ locally integrable function on $\mathbf{R}^n$; in fact if $A \subset \mathbf{R}^n$ is $\mathscr{L}^n$ measurable then*

$$\int_A \mathrm{G}(\mu)\, d\mathscr{L}^n \le \kappa_n^{-1/n}(\mathscr{L}^n(A))^{1/n}\mathrm{M}(\mu); \tag{4.3}$$

moreover, if $M \subset \mathbf{R}^n$ is a Lebesgue measurable set with $\mathscr{L}^n(M) < \infty$ then

$$\mathrm{G}(\mathscr{L}^n \llcorner M)(\boldsymbol{x}) \le \kappa_n^{-1/n}\,(\mathscr{L}^n(A))^{1/n} \tag{4.4}$$

for every $\boldsymbol{x} \in \mathbf{R}^n$.

The main result of this section is the following

Proposition 4.2. *If* $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ *then*

$$\mathbf{q} = \mathrm{H_N}\,\mathrm{div}\,\mathbf{q} + \mathrm{div}\,\mathrm{H_N}\,\mathbf{q} \tag{4.5}$$

in the sense of distributions, i.e., if $\boldsymbol{m} \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$ *then*

$$\int \boldsymbol{m} \cdot d\mathbf{q} = \int \boldsymbol{m} \cdot \mathrm{H_N}\,\mathrm{div}\,\mathbf{q}\,d\mathscr{L}^n - \int \mathrm{curl}\,\boldsymbol{m} \cdot \mathrm{H_N}\,\mathbf{q}\,d\mathscr{L}^n. \tag{4.6}$$

The two Newton homotopies (4.1) and (4.2) and the homotopy formula (4.5) are special cases of Newton homotopies of measures with values in the space of *r* vectors [30]. The formula (4.5) allows us to reconstruct the measure $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ from the locally integrable functions $\mathrm{H_N}\,\mathbf{q}$ and $\mathrm{H_N}\,\mathrm{div}\,\mathbf{q}$.

***Proof of Remark* 4.1** We have $\int_A \mathrm{G}(\mu)\,d\mathscr{L}^n = n^{-1}\kappa_n^{-1} \int \int_A k_{\boldsymbol{y}}\,d\mathscr{L}^n d\mu(\boldsymbol{y})$ and $\int_A k_{\boldsymbol{y}}\,d\mathscr{L}^n \le n\kappa_n^{(n-1)/n}(\mathscr{L}^n(A))^{1/n}$ for any $\boldsymbol{y} \in \mathbf{R}^n$ by [21; Lemma 3.4.3]. This proves (4.3). To prove (4.4), we denote by $\mathbf{B}(\boldsymbol{x}, r)$ the open ball of center $\boldsymbol{x}$ and radius *r*, put $D(r) = \mathscr{L}^n(M \cap \mathbf{B}(\boldsymbol{x}, r))$, note that $D(r) \le \kappa_n r^n$, and let $a > 0$. We have

$$\begin{aligned} n\kappa_n \mathrm{G}(\mathscr{L}^n \llcorner M)(\boldsymbol{x}) &= \int_0^\infty r^{-n+1}\,dD(r) \\ &= (n-1)\int_0^\infty r^{-n} D(r)\,dr \\ &= (n-1)\int_0^a r^{-n} D(r)\,dr + (n-1)\int_a^\infty r^{-n} D(r)\,dr \\ &\le \kappa_n (n-1)a + a^{-n+1}\mathscr{L}^n(M). \end{aligned}$$

The right hand side of the last inequality has a minimum with respect to a at $a = (\mathscr{L}^n(M)/\kappa_n)^{1/n}$ and the corresponding minimum value gives the right hand side of (4.4). □

***Proof of Proposition* 4.2** If we interpret $\boldsymbol{k_x}$ as a vector valued distribution, i.e., as a linear functional on $C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$ given by $\langle \boldsymbol{k_x}, \boldsymbol{m}\rangle = \int \boldsymbol{m} \cdot \boldsymbol{k_x}\,d\mathscr{L}^n$ for any $\boldsymbol{m} \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$, then a standard result says that the divergence in the sense of distributions of $\boldsymbol{k_x}$ on $\mathbf{R}^n$ satisfies

$$\mathrm{div}\,\boldsymbol{k_x} = -\delta_{\boldsymbol{x}} \tag{4.7}$$

where $\delta_{\boldsymbol{x}}$ is the Dirac measure concentrated at $\boldsymbol{x}$, i.e., $\langle \boldsymbol{k_x}, \nabla\varphi\rangle = \varphi(\boldsymbol{x})$ for any $\varphi \in C_0^\infty(\mathbf{R}^n, \mathbf{R})$. If $\boldsymbol{m} \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$ and $\boldsymbol{S} \in C_0^\infty(\mathbf{R}^n, \mathrm{Skw})$, we define the ***Newton cohomotopies*** $\mathrm{H_N^*}\,\boldsymbol{m}$ of $\boldsymbol{m}$ and $\mathrm{H_N^*}\,\boldsymbol{S}$ of $\boldsymbol{S}$ as functions with values in $\mathbf{R}$ and $\mathbf{R}^n$, respectively, by

$$\mathrm{H_N^*}\,\boldsymbol{m}(\boldsymbol{x}) = \int \boldsymbol{m} \cdot \boldsymbol{k_x}\,d\mathscr{L}^n, \quad \mathrm{H_N^*}\,\boldsymbol{S}(\boldsymbol{x}) = 2\int \boldsymbol{S}\boldsymbol{k_x}\,d\mathscr{L}^n$$

for every $\boldsymbol{x} \in \mathbf{R}^n$. Standard results on the differentiation of convolutions of distributions with C_0^∞ functions [26; Theorem 6.30] imply that $\mathrm{H_N^*}\,\boldsymbol{m}$ and $\mathrm{H_N^*}\,\boldsymbol{S}$ are infinitely differentiable functions, moreover, the compactness of the supports of $\boldsymbol{m}$ and $\boldsymbol{S}$ and immediate estimates give that $\mathrm{H_N^*}\,\boldsymbol{m}$ and $\mathrm{H_N^*}\,\boldsymbol{S}$ decay to $\mathbf{0}$ at ∞ at the rate $|\boldsymbol{x}|^{-n+1}$. We note that

$$\mathrm{H_N^*}\,\boldsymbol{m}(\boldsymbol{x}) = \langle \boldsymbol{k_x}, \boldsymbol{m}\rangle, \quad \boldsymbol{a} \cdot \mathrm{H_N^*}\,\boldsymbol{S}(\boldsymbol{x}) = -2\langle \boldsymbol{k_x}, \boldsymbol{S}\boldsymbol{a}\rangle \tag{4.8}$$

for any $\boldsymbol{a}, \boldsymbol{x} \in \mathbf{R}^n$. We next show that

$$\boldsymbol{m} = \nabla \mathrm{H}_{\mathrm{N}}^* \boldsymbol{m} + \mathrm{H}_{\mathrm{N}}^* \operatorname{curl} \boldsymbol{m} \tag{4.9}$$

for any $\boldsymbol{m} \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$. Indeed, if $\boldsymbol{a}, \boldsymbol{x} \in \mathbf{R}^n$ then $(4.8)_2$ gives $\boldsymbol{a} \cdot \mathrm{H}_{\mathrm{N}}^* \operatorname{curl} \boldsymbol{m}(\boldsymbol{x}) = -2\langle \boldsymbol{k}_{\boldsymbol{x}}, (\operatorname{curl} \boldsymbol{m})\boldsymbol{a}\rangle$. If we denote by $\nabla_{\boldsymbol{a}}$ the derivative in the direction $\boldsymbol{a}$, then the formula $(\operatorname{curl} \boldsymbol{m})\boldsymbol{a} = \frac{1}{2}(\nabla_{\boldsymbol{a}} \boldsymbol{m} - \nabla \boldsymbol{m}^{\mathsf{T}} \boldsymbol{a})$, the integration by parts involved in the definition of the derivative of a distribution and (4.7) give

$$\boldsymbol{a} \cdot \mathrm{H}_{\mathrm{N}}^* \operatorname{curl} \boldsymbol{m}(\boldsymbol{x}) = \langle \nabla_{\boldsymbol{a}} \boldsymbol{k}_{\boldsymbol{x}}, \boldsymbol{m}\rangle + \boldsymbol{a} \cdot \boldsymbol{m}(\boldsymbol{x})$$

which in combination with

$$\nabla_{\boldsymbol{a}} \mathrm{H}_{\mathrm{N}}^* \boldsymbol{m}(\boldsymbol{x}) = -\langle \nabla_{\boldsymbol{a}} \boldsymbol{k}_{\boldsymbol{x}}, \boldsymbol{m}\rangle$$

provides $\boldsymbol{a} \cdot (\nabla \mathrm{H}_{\mathrm{N}}^* \boldsymbol{m}(\boldsymbol{x}) + \mathrm{H}_{\mathrm{N}}^* \operatorname{curl} \boldsymbol{m}(\boldsymbol{x})) = \boldsymbol{a} \cdot \boldsymbol{m}(\boldsymbol{x})$, which is (4.9).

We further note that the cohomotopies and homotopies are dual to each other: we have

$$\int \boldsymbol{m} \cdot \mathrm{H}_{\mathrm{N}}\, \phi \, d\mathscr{L}^n + \int \mathrm{H}_{\mathrm{N}}^* \boldsymbol{m} \cdot d\phi = 0, \tag{4.10}$$

$$\int \boldsymbol{S} \cdot \mathrm{H}_{\mathrm{N}}\, \mathbf{q} \, d\mathscr{L}^n + \int \mathrm{H}_{\mathrm{N}}^* \boldsymbol{S} \cdot d\mathbf{q} = 0. \tag{4.11}$$

This is verified by writing down the definitions of H_{N} and $\mathrm{H}_{\mathrm{N}}^*$, noting that the resulting double integrals are absolutely convergent, and exchanging the orders of integration by Fubini's theorem.

We prove (4.5) by dualizing (4.9). We first note that a simple approximation argument shows that (2.6) holds for every bounded $\varphi \in C^\infty(\mathbf{R}^n, \mathbf{R})$ with a bounded $\nabla\varphi$. Letting $\boldsymbol{m} \in C_0^\infty(\mathbf{R}^n, \mathbf{R}^n)$, we observe that $\varphi := \mathrm{H}_{\mathrm{N}}^* \boldsymbol{m}$ is a bounded function in $C^\infty(\mathbf{R}^n, \mathbf{R})$ with a bounded $\nabla\varphi$, integrate (4.9) with respect to $\mathbf{q}$ and employ (4.10), (4.11), (2.6) to obtain (4.6). □

5 Fluxes defined by the flux vector

We now use (4.5) to complete the proofs.

***Proof of Theorem* 2.11** By (4.5) we have $\mathbf{q}_\rho = (\mathrm{H}_{\mathrm{N}} \operatorname{div} \mathbf{q})_\rho + \operatorname{div}(\mathrm{H}_{\mathrm{N}}\, \mathbf{q})_\rho$ and hence

$$\begin{aligned} \int \mathbf{q}_\rho \cdot d\boldsymbol{v} &= \int (\mathrm{H}_{\mathrm{N}} \operatorname{div} \mathbf{q})_\rho \cdot d\boldsymbol{v} + \int \operatorname{div}(\mathrm{H}_{\mathrm{N}}\, \mathbf{q})_\rho \cdot d\boldsymbol{v} \\ &= \int (\mathrm{H}_{\mathrm{N}} \operatorname{div} \mathbf{q})_\rho \cdot d\boldsymbol{v} + \int (\mathrm{H}_{\mathrm{N}}\, \mathbf{q})_\rho \cdot d\partial\boldsymbol{v} \end{aligned} \tag{5.1}$$

where the second equality follows by applying (2.10) to the function $\boldsymbol{\omega} = (\mathrm{H}_{\mathrm{N}}\, \mathbf{q})_\rho$. We have

$$(\mathrm{H}_{\mathrm{N}} \operatorname{div} \mathbf{q})_\rho(\boldsymbol{x}) = \int (\boldsymbol{k}_{\boldsymbol{x}})_\rho \, d \operatorname{div} \mathbf{q},$$

$$(\mathrm{H}_{\mathrm{N}}\, \mathbf{q})_\rho(\boldsymbol{x}) = -2 \int (\boldsymbol{k}_{\boldsymbol{x}})_\rho \wedge d\mathbf{q}$$

and $(\boldsymbol{k}_{\boldsymbol{x}})_\rho \to \boldsymbol{k}_{\boldsymbol{x}}$ pointwise on $\mathbf{R}^n \sim \{\boldsymbol{x}\}$ by the continuity of $\boldsymbol{k}_{\boldsymbol{x}}$; moreover, elementary scaling arguments show that there exists a constant $c = c(n)$ depending only on the dimension n such that $|(\boldsymbol{k}_{\boldsymbol{x}})_\rho| \le c\boldsymbol{k}_{\boldsymbol{x}}$ on $\mathbf{R}^n$ for all $\boldsymbol{x} \in \mathbf{R}^n$ and $\rho > 0$. Hence, if $\boldsymbol{x}$ is such that $\mathrm{G}(|\operatorname{div} \mathbf{q}|)(\boldsymbol{x}) < \infty$ (which implies $|\operatorname{div} \mathbf{q}|(\{\boldsymbol{x}\}) = 0$), we have $(\boldsymbol{k}_{\boldsymbol{x}})_\rho \to \boldsymbol{k}_{\boldsymbol{x}}$ pointwise for $|\operatorname{div} \mathbf{q}|$ a.e. $\boldsymbol{x} \in \mathbf{R}^n$ and the convergence is majorized by the $|\operatorname{div} \mathbf{q}|$ integrable function $c\boldsymbol{k}_{\boldsymbol{x}}$; thus the Lebesgue theorem yields that

$$(\mathrm{H_N}\operatorname{div}\mathbf{q})_\rho(\boldsymbol{x}) \to \mathrm{H_N}\operatorname{div}\mathbf{q}(\boldsymbol{x}). \tag{5.2}$$

Similarly, if $\boldsymbol{x}$ is such that $\mathrm{G}(|\mathbf{q}|)(\boldsymbol{x}) < \infty$ then

$$(\mathrm{H_N}\,\mathbf{q})_\rho(\boldsymbol{x}) \to \mathrm{H_N}\,\mathbf{q}(\boldsymbol{x}). \tag{5.3}$$

Inequalities (2.18) imply that $\mathrm{G}(|\operatorname{div}\mathbf{q}|)(\boldsymbol{x}) < \infty$ for $|\boldsymbol{v}|$ a.e. $\boldsymbol{x} \in \mathbf{R}^n$ and $\mathrm{G}(|\mathbf{q}|)(\boldsymbol{x}) < \infty$ for $|\partial\boldsymbol{v}|$ a.e. $\boldsymbol{x} \in \mathbf{R}^n$; moreover,

$$|(\mathrm{H_N}\operatorname{div}\mathbf{q})_\rho(\boldsymbol{x})| \le \int |(\boldsymbol{k_x})_\rho|\, d|\operatorname{div}\mathbf{q}| \le c\int k_{\boldsymbol{x}}\, d|\operatorname{div}\mathbf{q}|,$$

$$|(\mathrm{H_N}\,\mathbf{q})_\rho(\boldsymbol{x})| \le 2\int |(\boldsymbol{k_x})_\rho|\, d|\mathbf{q}| \le 2c\int k_{\boldsymbol{x}}\, d|\mathbf{q}|$$

and the right hand sides of the last two inequalities are integrable functions of $\boldsymbol{x}$ with respect to $|\boldsymbol{v}|$ and $|\partial\boldsymbol{v}|$, respectively, by (2.18). Thus we have (5.2) for $|\boldsymbol{v}|$ a.e. $\boldsymbol{x} \in \mathbf{R}^n$ and (5.3) for $|\partial\boldsymbol{v}|$ a.e. $\boldsymbol{x} \in \mathbf{R}^n$ with the existence of integrable majorants; hence the Lebesgue theorem yields

$$\lim_{\rho\to 0}\int \mathbf{q}_\rho \cdot d\boldsymbol{v} = \int \mathrm{H_N}\operatorname{div}\mathbf{q}\cdot d\boldsymbol{v} + \int \mathrm{H_N}\,\mathbf{q}\cdot d\partial\boldsymbol{v}$$

by (5.1). This completes the proof of the existence of the limit (2.19). The particular case (2.20) is obtained by applying the above main case to the normal $n-1$ dimensional current $\boldsymbol{v} = \boldsymbol{n}\,\mathscr{H}^{n-1} \llcorner P$. □

Proof of Theorem* 2.6*, Part* (i)*, uniqueness It suffices to show that if $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$ is a flux vector representing the null flux, then $\mathbf{q} = \mathbf{0}$. Thus by (2.7) we have

$$\lim_{\rho\to 0}\int_{\mathrm{T}_{\boldsymbol{a}} P} \mathbf{q}_\rho \cdot \boldsymbol{n}\, d\mathscr{H}^{n-1} = 0 \tag{5.4}$$

for every $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ and $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$. Let A be fixed and put $\boldsymbol{v} = \boldsymbol{n}\,\mathscr{H}^{n-1} \llcorner P$. By the proof of Theorem 2.11 equation (5.4) reads

$$\int \mathrm{H_N}\operatorname{div}\mathbf{q}\cdot d\,\mathrm{T}_{\boldsymbol{a}}\,\boldsymbol{v} + \int \mathrm{H_N}\,\mathbf{q}\cdot d\,\mathrm{T}_{\boldsymbol{a}}\,\partial\boldsymbol{v} = 0$$

and hence

$$\int \mathrm{T}_{-\boldsymbol{a}}\,\mathrm{H_N}\operatorname{div}\mathbf{q}\cdot d\boldsymbol{v} + \int \mathrm{T}_{-\boldsymbol{a}}\,\mathrm{H_N}\,\mathbf{q}\cdot d\partial\boldsymbol{v} = 0$$

for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$. We now let $\rho > 0$, multiply the last equation by $\omega_\rho(\boldsymbol{a})$, integrate with respect to $\mathscr{L}^n$, exchange the orders of integration, and use the definition of mollification to obtain

$$\int (\mathrm{H_N}\operatorname{div}\mathbf{q})_\rho \cdot d\boldsymbol{v} + \int (\mathrm{H_N}\,\mathbf{q})_\rho \cdot d\partial\boldsymbol{v} = 0.$$

Since $(\mathrm{H_N}\,\mathbf{q})_\rho$ is an infinitely differentiable function decaying at infinity, we can use (2.10) to obtain

$$\int \big((\mathrm{H_N}\operatorname{div}\mathbf{q})_\rho + \operatorname{div}(\mathrm{H_N}\,\mathbf{q})_\rho\big)\cdot d\boldsymbol{v} = 0;$$

this holds for every $\boldsymbol{v} = \boldsymbol{n}\,\mathscr{H}^{n-1} \llcorner P$ where $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ is arbitrary. Since the integrand is a continuous function, we conclude that $(\mathrm{H_N}\operatorname{div}\mathbf{q})_\rho + \operatorname{div}(\mathrm{H_N}\,\mathbf{q})_\rho = 0$ which by the homotopy formula means that $\mathbf{q}_\rho = \mathbf{0}$. Thus all mollifications of $\mathbf{q}$ vanish and since $\mathbf{q}_\rho \to \mathbf{q}$ as $\rho \to 0$ weak* in $\mathscr{M}(\mathbf{R}^n, \mathbf{R}^n)$, we obtain $\mathbf{q} = \mathbf{0}$. □

***Proof of Remark* 2.13** Inequality (2.21): We note that

$$\int\int k(\boldsymbol{x}-\boldsymbol{y})\, d\,\mathrm{T}_{\boldsymbol{a}}\,\phi(\boldsymbol{x})\, d\psi(\boldsymbol{y}) = \int\int k(\boldsymbol{x}-\boldsymbol{y}+\boldsymbol{a})\, d\phi(\boldsymbol{x})\, d\psi(\boldsymbol{y}) \tag{5.5}$$

for any $\boldsymbol{a} \in \mathbf{R}^n$. Let $M \subset \mathbf{R}^n$ be a $\mathscr{L}^n$ measurable set with $\mathscr{L}^n(M) < \infty$. If we show that

$$\int_M \int\int k(\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{a})\, d\phi(\boldsymbol{x})\, d\psi(\boldsymbol{y})\, d\mathscr{L}^n(\boldsymbol{a}) < \infty \tag{5.6}$$

then we have the finiteness of the integrals in (5.5) for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in M$ and the arbitrariness of M gives (2.21). To prove (5.6), we use Fubini's theorem to exchange the orders of integration to obtain the integral

$$\int\int\int_M k(\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{a})\, d\mathscr{L}^n(\boldsymbol{a})\, d\phi(\boldsymbol{x})\, d\psi(\boldsymbol{y})$$

By the second part of Remark 4.1 we have

$$\int_M k(\boldsymbol{x} - \boldsymbol{y} + \boldsymbol{a})\, d\mathscr{L}^n(\boldsymbol{a}) < c$$

for some $c \in \mathbf{R}$ and all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$. Since ϕ, ψ are finite measures, the finiteness of the integral in (5.6) follows. Inequality (2.22): If $\rho > 0$ then

$$\int \phi_\rho\, d\mathrm{T}_{\boldsymbol{a}}\psi = \int \phi_\rho(\boldsymbol{x} + \boldsymbol{a})\, d\psi(\boldsymbol{x})$$

and

$$\int\int \phi_\rho\, d\mathrm{T}_{\boldsymbol{a}}\psi\, d\mathscr{L}^n(\boldsymbol{a}) = \int\int \phi_\rho(\boldsymbol{x} + \boldsymbol{a})\, d\mathscr{L}^n(\boldsymbol{a})\, d\psi(\boldsymbol{x}) = \mathrm{M}(\phi)\mathrm{M}(\psi)$$

Hence

$$\infty > \mathrm{M}(\phi)\mathrm{M}(\psi) = \liminf_{\rho\to 0} \int\int \phi_\rho\, d\mathrm{T}_{\boldsymbol{a}}\psi\, d\mathscr{L}^n(\boldsymbol{a}) \ge \int \liminf_{\rho\to 0} \int \phi_\rho\, d\mathrm{T}_{\boldsymbol{a}}\psi\, d\mathscr{L}^n(\boldsymbol{a})$$

by Fatou's lemma and the assertion follows. □

***Proof of Theorem* 2.6, *Part* (ii)** Given $\mathbf{q} \in \mathscr{DM}(\mathbf{R}^n)$, we define $F : \mathfrak{D} \to \mathbf{R}$ by (2.8) on the set $\mathfrak{D}$ of all $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ for which the limit (2.8) exists and is finite. Prove that F is a Cauchy flux. To verify Definition 2.4(i), we note that if $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ then

$$\left.\begin{aligned} &\int\int_{\mathrm{T}_{\boldsymbol{a}} P} k(\boldsymbol{x} - \boldsymbol{y})\, d\mathscr{H}^{n-1}(\boldsymbol{x})\, d|\operatorname{div}\mathbf{q}|(\boldsymbol{y}) < \infty, \\ &\int\int_{\partial \mathrm{T}_{\boldsymbol{a}} P} k(\boldsymbol{x} - \boldsymbol{y})\, d\mathscr{H}^{n-2}(\boldsymbol{x})\, d|\mathbf{q}|(\boldsymbol{y}) < \infty \end{aligned}\right\} \tag{5.7}$$

for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$ by Remark 2.13. The particular case of Theorem 2.11 says that the limit (2.7) exists for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$ and thus $\mathrm{T}_{\boldsymbol{a}}A \in \mathfrak{D}$ for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$. The function $\boldsymbol{a} \mapsto F(\mathrm{T}_{\boldsymbol{a}}A)$ is $\mathscr{L}^n$ measurable because it is a pointwise limit, $\mathscr{L}^n$ a.e., of the family of continuous functions $\boldsymbol{a} \mapsto f_\rho(\boldsymbol{a}) := \int_{\mathrm{T}_{\boldsymbol{a}} P} \mathbf{q}_\rho \cdot \boldsymbol{n}\, d\mathscr{H}^{n-1}, \rho > 0$. Definition 2.4(ii) is immediate. To verify Definition 2.4(iii), we put $\theta := |\mathbf{q}|$, let $A = (P, \boldsymbol{n}) \in \mathfrak{S}$ and note that

$$\Big|\int_{\mathrm{T}_{\boldsymbol{a}} P} \mathbf{q}_\rho \cdot \boldsymbol{n}\, d\mathscr{H}^{n-1}\Big| \le \int_{\mathrm{T}_{\boldsymbol{a}} P} |\mathbf{q}_\rho|\, d\mathscr{H}^{n-1} \le \int_{\mathrm{T}_{\boldsymbol{a}} P} |\mathbf{q}|_\rho\, d\mathscr{H}^{n-1}.$$

The $\liminf_{\rho\to 0}$ using the existence of the limit (2.7) yields (2.4) for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$. To verify Definition 2.4(iv), we put $\eta = |\operatorname{div}\mathbf{q}|$ and let $B \in \mathfrak{B}$. If $A_i = (P_i, \boldsymbol{n}_i), i = 1, \dots, q$, is the collection of all oriented faces of B, we infer from Definition 2.4(i) that $\mathrm{T}_{\boldsymbol{a}}A_i \in \mathfrak{D}$ for $\mathscr{L}^n$ a.e. $\boldsymbol{a} \in \mathbf{R}^n$ and all $i = 1, \dots, q$, and

$$F(\partial\,\mathrm{T}_{\boldsymbol{a}}B) = \lim_{\rho\to0}\sum_{i=1}^{q}\int_{\mathrm{T}_{\boldsymbol{a}}P_i}\mathbf{q}_\rho\cdot\boldsymbol{n}_i\,d\mathscr{H}^{n-1} = \lim_{\rho\to0}\int_{\mathrm{T}_{\boldsymbol{a}}B}\operatorname{div}\mathbf{q}_\rho\,d\mathscr{L}^n \tag{5.8}$$

by the divergence theorem for the smooth vectorfield $\mathbf{q}_\rho$. One has

$$\Big|\int_{\mathrm{T}_{\boldsymbol{a}}B}\operatorname{div}\mathbf{q}_\rho\,d\mathscr{L}^n\Big| \le \int_{\mathrm{T}_{\boldsymbol{a}}B}|\operatorname{div}\mathbf{q}_\rho|\,d\mathscr{L}^n \le \int_{\mathrm{T}_{\boldsymbol{a}}B}|\operatorname{div}\mathbf{q}|_\rho\,d\mathscr{L}^n$$

and hence (5.8) gives (2.5). The rest of theorem 2.6, Part (ii) is obvious. □

***Proof of Theorem* 2.6, *Part* (iii)** The proof of Theorem 2.6, Part (ii) shows that

$$F(\partial\,\mathrm{T}_{\boldsymbol{a}}B) = \lim_{\rho\to0}\int_{\mathrm{T}_{\boldsymbol{a}}B}\operatorname{div}\mathbf{q}_\rho\,d\mathscr{L}^n$$

for every $B\in\mathfrak{B}$ and $\mathscr{L}^n$ a.e. $\boldsymbol{a}\in\mathbf{R}^n$. Furthermore, we have $\operatorname{div}\mathbf{q}_\rho\,\mathscr{L}^n\to\operatorname{div}\mathbf{q}$ weak* in $\mathscr{M}(\mathbf{R}^n,\mathbf{R})$ and the well known properties of the weak convergence (Portmanteau theorem on weak convergence of measures (e.g., [3; p. 196], [4; p. 7]) imply that

$$\lim_{\rho\to0}\int_{\mathrm{T}_{\boldsymbol{a}}B}\operatorname{div}\mathbf{q}_\rho\,d\mathscr{L}^n \to \int_{\mathrm{T}_{\boldsymbol{a}}B}d\operatorname{div}\mathbf{q}$$

provided $|\operatorname{div}\mathbf{q}|(\partial\,\mathrm{T}_{\boldsymbol{a}}B) = 0$. The proof is now completed by noting that the last relation holds for $\mathscr{L}^n$ a.e. $\boldsymbol{a}\in\mathbf{R}^n$ by Fubini's theorem since $\mathscr{L}^n(\partial\,\mathrm{T}_{\boldsymbol{a}}B) = 0$. □

***Proof of Theorem* 2.14** If $\boldsymbol{\nu}\in\mathscr{N}_{n-1}$ and $f\in C_0^\infty(\mathbf{R}^n,\mathbf{R})$ then $f\boldsymbol{\nu}\in\mathscr{N}_{n-1}$ and

$$\partial(f\boldsymbol{\nu}) = f\partial\boldsymbol{\nu} + \nabla f\wedge\boldsymbol{\nu}.$$

Since ∇f is bounded, we see that $(2.23)_1$ and (2.18) imply that $\tilde{\boldsymbol{\nu}} := f\boldsymbol{\nu}$ satisfies the hypothesis of Theorem 2.11 and thus the limit (2.24) exists and is finite. Further, we have

$$\tilde F(f\boldsymbol{\nu}) = \lim_{\rho\to0}\int f\mathbf{q}_\rho\cdot d\boldsymbol{\nu} \le |f|_{L^\infty}\liminf_{\rho\to0}\int|\mathbf{q}_\rho|\cdot d|\boldsymbol{\nu}| \le |f|_{L^\infty}\liminf_{\rho\to0}\int|\mathbf{q}|_\rho\cdot d|\boldsymbol{\nu}| = c|f|_{L^\infty}$$

for every $f\in C_0^\infty(\mathbf{R}^n,\mathbf{R})$ where c is a finite constant equal to the left hand side of $(2.23)_2$. Thus the functional $f\mapsto\tilde F(f\boldsymbol{\nu})$ is represented by a measure as in (2.25). □

Lemma 5.1. *If $B\subset\mathbf{R}^n$ is a normalized set of finite perimeter then*

$$\int_{\partial B}k(\boldsymbol{x}-\boldsymbol{y})\,d\mathscr{H}^{n-1}(\boldsymbol{x}) = \infty \tag{5.9}$$

for every $\boldsymbol{y}\in\partial B$. *If B and $\mathbf{q}$ satisfy the hypotheses of Theorem* 2.16 *then we have* (2.30) *as a consequence.*

Proof Let $\boldsymbol{y}\in\partial B$. Shifting B we assume that $\boldsymbol{y} = \mathbf{0}\in\partial B$ to simplify the notation. We have

$$k(\boldsymbol{x}) = n^{-1}\kappa_n^{-1}(n-1)\int_0^\infty 1_{\mathbf{U}(\rho)}(\boldsymbol{x})\rho^{-n}\,d\rho$$

for every $\boldsymbol{x}\in\mathbf{R}^n$ where $1_{\mathbf{U}(\rho)}$ is the characteristic function of $\mathbf{U}(\rho) := \mathbf{B}(\mathbf{0},\rho)$. Hence the proof of (5.9) amounts to proving that

$$\int_0^\infty\mathscr{H}^{n-1}(\mathbf{U}(\rho)\cap\partial B)\rho^{-n}\,d\rho = \infty.$$

By the relative isoperimetric inequality (e.g., [10; Corollary 4.5.3]) there exists a constant $c > 0$ such that

$$\mathscr{H}^{n-1}(\mathbf{U}(\rho) \cap \partial B) \geq c \min\left\{[\mathscr{L}^n(\mathbf{U}(\rho) \cap B)]^p, [\mathscr{L}^n(\mathbf{U}(\rho) \sim B)]^p\right\}$$

for all $\rho > 0$ where $p = (n-1)/n$, and thus to within an inessential positive multiplicative constant the integral in (5.9) is bounded from below by

$$I := \int_0^\infty \varphi^p(\rho)\rho^{-1}\, d\rho$$

where

$$\varphi(\rho) = \min\left\{D(\rho), 1 - D(\rho)\right\}, \quad D(\rho) = \kappa_n^{-1}\rho^{-n}\mathscr{L}^n(\mathbf{U}(\rho) \cap B).$$

We note that $0 \leq D(\rho) \leq 1$ and

$$\varphi(\rho) = \begin{cases} D(\rho) & \text{if } D(\rho) \leq 1/2, \\ 1 - D(\rho) & \text{if } D(\rho) \geq 1/2. \end{cases}$$

If $0 < \sigma < \rho$ then $\mathscr{L}^n(\mathbf{U}(\rho) \cap B) - \mathscr{L}^n(\mathbf{U}(\sigma) \cap B) \leq \mathscr{L}^n(\mathbf{U}(\rho) \sim \mathbf{U}(\sigma)) = \kappa_n(\rho^n - \sigma^n)$ from which we obtain

$$D(\sigma) \geq 1 + (\rho/\sigma)^n(D(\rho) - 1)$$

and hence

$$D(\sigma) \geq D(\rho)/2 \quad \text{whenever} \quad q(D(\rho)) \leq \sigma/\rho \leq 1 \tag{5.10}$$

where

$$q(a) = \left((1-a)/(1-a/2)\right)^{1/n} \leq 1$$

for any $a \in [0,1]$. Replacing B by its complement we obtain analogously that

$$1 - D(\rho) \geq (1 - D(\rho))/2 \quad \text{whenever} \quad q(1 - D(\rho)) \leq \sigma/\rho \leq 1. \tag{5.11}$$

We now consider the following three exhaustive possibilities to prove $I = \infty$:

(i) there exists a sequence $\rho_k \to 0$ such that $D(\rho_k) = 1/2$;

(ii) we have $D(\rho) < 1/2$ for all sufficiently small ρ;

(iii) we have $D(\rho) > 1/2$ for all sufficiently small ρ.

Let $q_0 := q(1/2) = (3/4)^{1/n}$ and note that q is a decreasing function of $a \in [0,1]$. Assume that (i) occurs. Passing to a subsequence of the sequence ρ_k we assume that

$$\rho_{k+1} \leq q_0\rho_k, \quad k = 1, \ldots \tag{5.12}$$

Noting that $\varphi(\rho_k) = 1/2$ we use (5.10) and (5.11) to deduce that $\varphi(\sigma) \geq 1/4$ for all $\sigma \in J_k$ where

$$J_k := (q_0\rho_k, \rho_k), \quad k = 1, \ldots \tag{5.13}$$

Noting that the system of intervals J_k is disjoint we deduce that

$$I \geq \sum_{k=1}^\infty \int_{J_k} \varphi^p(\rho)\rho^{-1}\, d\rho \geq (1/4)^p \sum_{k=1}^\infty \int_{J_k} \rho^{-1}\, d\rho = -(1/4)^p \sum_{k=1}^\infty \ln q_0 = \infty. \tag{5.14}$$

Assume that (ii) occurs. Since $\boldsymbol{y} \equiv \mathbf{0} \in \partial B$, there exists an $a > 0$ and a sequence $\rho_k \to 0$ such that $D(\rho_k) \geq a$; indeed the opposite would mean that $\Theta(\mathbf{0}, B) = 0$ and consequently $\boldsymbol{y} = \mathbf{0} \in \mathbf{R}^n \sim (B \cup \partial B)$. We can assume (5.12). Noting that under (ii) we have $\varphi(\rho) = D(\rho)$ for all sufficiently small ρ, we deduce from (5.10) and from $q(D(\rho_k)) \ge q_0$ that $\varphi(\sigma) \ge a/2$ for all $\sigma \in J_k$ where J_k is given by (5.13). An argument similar to that in (5.14) then gives $I = \infty$. Assume that (iii) occurs. Since $\boldsymbol{y} \equiv \mathbf{0} \in \partial B$, there exists a $b < 1$ and a sequence $\rho_k \to 0$ such that $D(\rho_k) \le b$ since otherwise we would have $\Theta(\mathbf{0}, B) = 1$ and hence $\boldsymbol{y} = \mathbf{0} \in B$. We can assume (5.12). Noting that under (iii) we have $\varphi(\rho) = 1 - D(\rho)$ for all sufficiently small ρ we deduce from (5.11) and from $q(1 - D(\rho_k)) \ge q_0$ that $\varphi(\sigma) \ge (1-b)/2$ for all $\sigma \in J_k$ where J_k is given by (5.13). An argument similar to that in (5.14) then gives $I = \infty$, which completes the proof of (5.9).

Equations (2.30) are direct consequences of (5.9) and $(2.27)_{1,2}$: Indeed, the positivity of $|\mathbf{q}|(\partial B)$ and (5.9) would cause the integral in $(2.30)_1$ diverge and similarly for $(2.30)_2$. □

***Proof of Theorem* 2.16** Letting $\mathbf{q}_\rho \in C^\infty(\mathbf{R}^n, \mathbf{R}^n)$ be the ρ mollification of $\mathbf{q}$ and $f \in C_0^\infty(\mathbf{R}^n, \mathbf{R})$ we use the divergence theorem for sets of finite perimeter [10; Theorem 4.5.6] and the commutativity of mollifications to assert that

$$\int_{\partial B} f \mathbf{q}_\rho \cdot \boldsymbol{n}\, d\mathscr{H}^{n-1} = \int_B f \operatorname{div} \mathbf{q}_\rho + \nabla f \cdot \mathbf{q}_\rho \, d\mathscr{L}^n = \int_{\mathbf{R}^n} (f 1_B)_\rho \, d \operatorname{div} \mathbf{q} + \int_{\mathbf{R}^n} (\nabla f 1_B)_\rho \, d\mathbf{q} \tag{5.15}$$

where 1_B is the characteristic function of B and $(f1_B)_\rho$, $(\nabla f 1_B)_\rho$ are the mollifications of the indicated functions. The continuity of f and ∇f imply that for $\rho \to 0$ we have

$$(f1_B)_\rho(\boldsymbol{x}) \to f(\boldsymbol{x}) \lim_{\rho \to 0} (1_B)_\rho(\boldsymbol{x}), \quad (\nabla f 1_B)_\rho \to \nabla f(\boldsymbol{x}) \lim_{\rho \to 0} (1_B)_\rho(\boldsymbol{x})$$

for every $\boldsymbol{x} \in \mathbf{R}^n$ for which the limit $\lim_{\rho \to 0} (1_B)_\rho(\boldsymbol{x})$ exists. Since B is a normalized set, we have

$$\lim_{\rho \to 0} (1_B)_\rho(\boldsymbol{x}) = 1_B(\boldsymbol{x}) \tag{5.16}$$

for every $\boldsymbol{x} \in \mathbf{R}^n \sim \partial B$. This by (2.30) means that we have (5.16) for $|\mathbf{q}|$ a.e. $\boldsymbol{x} \in \mathbf{R}^n$ and for $|\operatorname{div} \mathbf{q}|$ a.e. $\boldsymbol{x} \in \mathbf{R}^n$. The left hand side of (5.15) converges to $\int f \, d\mu_{\partial B}$ by (2.28) and the right hand side of (5.15) to

$$\int_B f \, d \operatorname{div} \mathbf{q} + \int_B \nabla f \cdot d\mathbf{q}$$

by Lebesgue's theorem. □

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