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LOCALIZATION EFFECTS FOR EIGENFUNCTIONS NEAR TO THE EDGE OF A THIN DOMAIN

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Abstract. It is proved that the first eigenfunction of the mixed boundary-value problem for the Laplacian in a thin domain Ω_h is localized either at the whole lateral surface Γ_h of the domain, or at a point of Γ_h, while the eigenfunction decays exponentially inside Ω_h. Other effects, attributed to the high-frequency range of the spectrum, are discussed for eigenfunctions of the mixed boundary-value and Neumann problems, too.

Keywords: spectral problem, thin domain, boundary layer, trapped mode, localized eigenfunction

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1. Plate-like domains and spectral problems

Let $\omega \subset \mathbb{R}^2$ be a domain bounded by a simple smooth contour $\partial\omega$ and $\Omega_h^0 = \omega \times (-h/2, h/2)$ a cylindrical plate of a small thickness $h \in (0,1]$. Owing to separating variables, the spectral problem in Ω_h^0,

$$\begin{aligned} -\Delta_x u(h,x) &= \Lambda(h) u(h,x),\ x \in \Omega_h, \\ u(h,x) &= 0,\ x \in \Sigma_h^{\pm};\ \partial_\nu u(h,x) = 0,\ x \in \Gamma_h, \end{aligned} \tag{1}$$

only admits the solutions

$$\begin{aligned} \Lambda_{k,j} &= \pi^2 k^2 h^{-2} + \lambda_j, \\ u_{k,j}(h,x) &= w_j(y) \sin\left\{\pi k[h^{-1}z + 1/2]\right\} \end{aligned} \tag{2}$$

where $k, j \in \mathbb{N} = \{1,2,\dots\}$ and the couple $\{\lambda_j, w_j\}$ verifies the spectral problem

$$-\Delta_y w(y) = \lambda w(y),\ y \in \omega;\ \partial_n w(y) = 0,\ y \in \partial\omega. \tag{3}$$

Here ∂_ν and ∂_n stand for derivatives along the outward normals ν and n to the surfaces $\partial\Omega_h$ and $\partial\omega$, respectively, $\Delta_x = \Delta_y + \partial^2/\partial z^2$ and Δ_y are the Laplacians in $\mathbb{R}^3$ and $\mathbb{R}^2$ while, for the cylindrical plate $\Omega_h = \Omega_h^0$, the sets $\Sigma_h^\pm$ and Γ_h denote the bases $\omega \times \{\pm h/2\}$ and the lateral side $\partial\omega \times (-h/2, h/2)$. It is a fair interpretation of the problem (3) as a two-dimensional model for the spatial problem (1). Moreover, formulas (2) express high-frequency asymptotic forms for eigenvalues and eigenfunctions in the thin domain Ω_h. Although the spectral elasticity problem for the isotropic plate Ω_h^0 possesses low-frequency $h^2\mu_{-1,j} + O(h^3)$ and middle-frequency $h^0\mu_{0,j} + O(h^1)$ eigenvalue subsequences (cf. [1]–[3]), problem (1) used to be regarded as a scalar analogue for demonstrating basic properties of high-frequency range of the plate spectrum. In particular, a formal procedure was proposed in [4], [5] to derive the so-called two-dimensional models of high-frequency long-wave oscillations of isotropic elastic plates. As discussed in [6], the boundary layer phenomenon destroys the asymptotic forms employed in [4], [5] and, therefore, the above mentioned model cannot be justified rigorously. Indicating high-frequency eigenfunctions which concentrate near the edge of a thin domain and therefore avoid the asymptotic form $(2)_2$, this paper as well as the preceding papers [7], [8] further contribute to the conclusion on the invalidity of the models introduced in [4], [5].

In the sequel we consider a non-cylindrical plate with the perturbed lateral surface

$$\Gamma_h = \{x\colon\ s \in \partial\omega,\ \eta_1 = -\Upsilon(\eta_2),\ |\eta_2| < 1/2\} \tag{4}$$

where $\eta_1 = h^{-1}n$, $\eta_2 = h^{-1}z$ are rapid variables; $|n| = \operatorname{dist}(y, \partial\omega)$, $n > 0$ outside ω; s stands for both, a point on $\partial\omega$ and the arc length along the contour $\partial\omega$; $\Upsilon \in C^\infty[-1/2, 1/2]$ is a profile function, $\Upsilon > 0$ as $|\eta_2| < 1/2$, $\Upsilon(\pm 1/2) = 0$.

If Υ is even in z, then, by restricting an even eigenfunction u to the upper half-plate $\Omega_h^+ = \{x \in \Omega_h\colon\ z > 0\}$ and using the odd extension over the plane $\{x\colon z = h/2\} \supset \Sigma_h^+$, one gets rid of the Dirichlet conditions and changes them to the Neumann conditions on bases of the new plate. This arguing demonstrates that the Neumann spectral problem in a thin domain inherits localized eigenfunctions from the mixed boundary-value problem (1) which has been singled out here only because the localization effects observed below attribute to the first eigenfunction.

2. Trapped modes

In accordance with [9; Chapter 15], rewriting the problem (1) in the curvilinear coordinates (s, η), omitting lower-order terms, replacing Λ by $h^{-2}\mu$, and putting $h = 0$ leads to the limit problem in the half-strip Π with a curved end

$$\begin{gathered} -\Delta_\eta w(\eta) - \mu w(\eta) = 0,\ \eta \in \Pi = \{\eta\colon\ \eta_1 > -\Upsilon(\eta_2), |\eta_2| < 1/2\}, \\ w(\eta_1, \pm 1/2) = 0,\ \eta_1 > 0;\ \partial_\nu w(\eta) = 0,\ \eta \in \partial\Pi,\ \eta_1 < 0. \end{gathered} \tag{5}$$

In order to initiate further asymptotic analysis we formulate an assertion proved in [8] with the help of an approach developed in [10].

Lemma 1. *Under the condition*

$$\int_{-1/2}^{1/2} \cos(2\pi\eta_2)\Upsilon(\eta_2)\,\mathrm{d}\eta_2 > 0, \tag{6}$$

the problem (5) *has an eigenvalue* $\mu \in (0, \pi^2)$ *to which there corresponds an eigenfunction* $w \in H^1(\Pi)$ *such that* $\|w; L_2(\Pi)\| = 1$ *and* $w(\eta) = O(\exp[-(\pi^2 - \mu)^{1/2}\eta_1])$ *as* $\eta_1 \to \infty$.

The eigenfunction w is called a *trapped mode*. We consider the first eigenvalue μ_0 which is known to be simple (see [8] for the case of a multiple eigenvalue).

By separating variables, any function which satisfies the scalar problem (5) in the half-strip $\Pi^0 = (-1/2, 1/2) \times \mathbb{R}_+$ and decays exponentially as $\eta_1 \to +\infty$, vanishes in Π^0 totally, i.e., the straight end of the half-strip denies a trapped mode in the problem (5). In contrast to scalar problems, the elasticity problem in Π^0 does not admit separation of variables and, moreover, trapped modes have been detected in [11] for an isotropic half-strip whose boundary is free of traction and in [8] for the half-strip which is clamped over its infinite sides[1]. Thus, it is readily predictable that spectral problems related to oscillating cylindrical plates and gaskets enjoy the same localization effects as the scalar problem (1) in the case of non-cylindrical lateral side.

[1] In the first case the corresponding eigenvalue lies in the continuous spectrum.

3. A rough result about the localization of eigenfunctions

For the cylindrical plate Ω_h^0, the first eigenvalue $\Lambda_{1,1}(h)$ is equal to $\pi^2 h^{-2}$ and the eigenfunction $u_{1,1}(h,x) = \sin\{\pi[h^{-1}z + 1/2]\}$ is uniformly distributed along the plate. The next assertion proves that, if an eigenvalue of problem (1) satisfies the inequality

$$\Lambda \leqslant \pi^2 h^{-2}(1-\varepsilon), \quad \varepsilon > 0, \tag{7}$$

the eigenfunction decays exponentially in a distance from the lateral side (4).

Proposition 2. *For an eigenfunction u which corresponds to the eigenvalue* (7) *and is normalized in $L_2(\Omega_h)$, the estimates*

$$\begin{aligned} \|\mathcal{E}_\delta \nabla_x u; L_2(\Omega_h)\| &\leqslant \Lambda^{1/2}(\varepsilon - 2\delta\pi^{-1})^{-1/2}, \\ \|\mathcal{E}_\delta u; L_2(\Omega_h)\| &\leqslant (1 + h^{-1}\delta\Lambda^{-1/2})(\varepsilon - 2\delta\pi^{-1})^{-1/2} \end{aligned} \tag{8}$$

hold, where $\delta \in (0, \pi\varepsilon/2)$ and $\mathcal{E}_\delta$ is a Lipschitz exponential weight function,

$$\begin{aligned} &\mathcal{E}_\delta(x) = 1 \text{ as } x \in \Omega_h \setminus \Omega_h^0, \\ &\mathcal{E}_\delta(x) = \exp\{h^{-1}\delta \operatorname{dist}(y, \partial\omega)\} \text{ as } x \in \Omega_h^0. \end{aligned} \tag{9}$$

P r o o f. Multiplying equation $(1)_1$ by $\mathcal{E}_\delta^2 u$ and integrating by parts in Ω_h yields

$$\|\mathcal{E}_\delta \nabla_x u; L_2(\Omega_h)\|^2 + 2J_\delta(u; \Omega_h^0) = \Lambda \|\mathcal{E}_\delta u; L_2(\Omega_h)\|^2 \tag{10}$$

where, by the relations $|\nabla_x \mathcal{E}_\delta(x)| \leqslant h^{-1}\delta\mathcal{E}_\delta(x)$ and $\nabla_x\mathcal{E}_\delta(x) = 0$ for $x \in \Omega_h \setminus \Omega_h^0$,

$$\begin{aligned} 2|J_\delta(u;\Omega_h^0)| &= 2\left|\int_{\Omega_h^0} \mathcal{E}_\delta u \nabla_x \mathcal{E}_\delta \cdot \nabla_x u \, \mathrm{d}x\right| \\ &\leqslant h^{-2}\pi\delta \|\mathcal{E}_\delta u; L_2(\Omega_h^0)\|^2 + \pi^{-1}\delta \|\mathcal{E}_\delta \nabla_x u; L_2(\Omega_h^0)\|^2. \end{aligned} \tag{11}$$

The Friedrichs inequality $\|\mathcal{E}_\delta u; L_2(\Omega_h)\| \leqslant \pi^{-1}h\|\mathcal{E}_\delta \nabla_x u; L_2(\Omega_h)\|$ now yields

$$\begin{aligned} &\{1 - \pi^{-1}\delta - \pi^{-2}h^2(\Lambda + h^2\pi\delta)\}\|\mathcal{E}_\delta \nabla_x u; L_2(\Omega_h)\|^2 \\ &\qquad \leqslant (1-\pi^{-1}\delta)\|\mathcal{E}_\delta \nabla_x u; L_2(\Omega_h)\|^2 - (\Lambda + h^{-2}\pi\delta)\|\mathcal{E}_\delta u; L_2(\Omega_h^0)\|^2 \\ &\qquad \leqslant \Lambda \|\mathcal{E}_\delta u; L_2(\Omega_h \setminus \Omega_h^0)\|^2 \leqslant \Lambda. \end{aligned}$$

Since the sum in the curly brackets is equal to $1 - h^2\pi^{-2}\Lambda - 2\delta\pi^{-1} \geqslant \varepsilon - 2\delta\pi^{-1}$, the first formula in (8) is verified. The second formula follows from (10) and (11). $\square$

4. Asymptotic reduction to an intermediate problem

In view of Lemma 1, we search for an eigenvalue of problem (1) in the form

$$\Lambda(h) = h^{-2}\mu_0 + \lambda_*(h) \tag{12}$$

where $\lambda_*(h) = o(h^{-2})$ as $h \to 0+$. Since $\mu_0 < \pi^2$, the eigenvalue (12) satisfies (7) and, by virtue of Proposition 2, the corresponding eigenfunction concentrates near the lateral side (4). Recalling the general asymptotic procedure developed in [12] (see also [9; Chapter 16]), we accept the following asymptotic expansion for the eigenfunction:

$$u(h,x) = w(\eta)v_*(h,s) + h^1 W_1(v_*; s,\eta) + h^2 W_2(v_*; s,\eta) + \dots \tag{13}$$

Here w stands for the eigenfunction of problem (5) corresponding to the eigenvalue μ_0, v_* for a smooth function on the contour $\partial\omega$ and W_i for a lower-order asymptotic term. The asymptotic structures of $\lambda_*(h)$ and $v_*(h,s)$ are not fixed yet and in the next three sections we will complete the assumption (12) and the expansion (13) in different ways.

In the coordinates (s,η) introduced in (4), the Laplacian allows the following decomposition with respect to the small parameter h:

$$\begin{gathered} \Delta_x = h^{-2}L_0 + h^{-1}L_1 + h^0 L_2 + \dots, \\ L_0(\nabla_\eta) = \Delta_\eta, \quad L_1(s,\nabla_\eta) = -k(s)\partial_1, \quad L_2(s,\eta,\partial s,\nabla_\eta) = k(s)^2\eta_1\partial_1 + \partial_s^2 \end{gathered} \tag{14}$$

where $k(s)$ is the curvature of the arc $\partial\omega$ at the point s and $\partial_s = \partial/\partial s, \partial_i = \partial/\partial\eta_i$.

We now insert the decompositions (12)–(14) into equalities (1), gather terms of similar order in h, and finally obtain a recurrent sequence of inhomogeneous problems of type (5) depending on the parameter $s \in \partial\omega$. Since μ_0 is a simple eigenvalue, each of the limit problems needs one compatibility condition, more precisely, a family of alike compatibility conditions parameterized by $s \in \partial\omega$. In order to fulfil these conditions we employ a projection trick proposed in [12] (see also [13] and [9; Chapter 16]) and thus we subtract from the right-hand sides the expressions $w(\eta)M_j(s,\partial_s)v_*(h,s)$ with appropriate j-th order differential operators M_j. The following intermediate problem, i.e., an ordinary differential equation on the contour $\partial\omega$, is intended to compensate for additional discrepancies introduced artificially:

$$\sum_{j=0}^{2} h^j M_j(s,\partial_s)v_*(h,s) = 0, \quad s \in \partial\omega. \tag{15}$$

Equation (15) still contains the small parameter h and accurate descriptions of an asymptotic algorithm serving to derive intermediate problems can be found in [12], [13], [9].

The limit problems in question take the form

$$
\begin{aligned}
&-\Delta_\eta W_j - \mu_0 W_j = F_j - wM_j v_* \text{ in } \Pi, \\
&\qquad W_j = 0 \text{ on } \{\eta \in \partial\Pi\colon\ \eta_1 > 0\},\ \partial_\nu W_j = 0 \text{ on } \{\eta \in \partial\Pi\colon\ \eta_1 < 0\}.
\end{aligned}
\tag{16}
$$

Since $F_0 = 0$ and v_* does not depend on η, the product $W_0 = wv_*$ becomes a solution of the problem (16) at $j = 0$; thus, $M_0 = 0$. By virtue of the formula $(14)_2$, we have

$$
F_1 = L_1 wv_* = -v_* k\partial_1 w, \quad F_2 = \lambda_*(h)wv_* + L_1 W_1 + L_2 wv_*. \tag{17}
$$

Hence, the compatibility conditions for the problem (16) at $j = 1$ lead to the formulas

$$
\begin{aligned}
M_1(s,\partial_s)v_*(h,s) &= \int_\Pi w(\eta)F_1(h,s,\eta)\,\mathrm{d}x = \frac{1}{2}k(s)v_*(h,s)\int_\Pi \frac{\partial}{\partial\eta_1}w(\eta)^2\,\mathrm{d}\eta \\
&= b_0 k(s)v_*(h,s), \quad b_0 = \frac{1}{2}\int_{-1/2}^{1/2} w(-\Upsilon(\eta_2),\eta_2)^2\,\mathrm{d}\eta_2 > 0.
\end{aligned}
\tag{18}
$$

We readily conclude that $W_1(v_*; s,\eta) = k(s)w_1(\eta)v_*(h,s)$ where w_1 is a function in $H^1(\Pi)$, chosen to be orthogonal to w in $L_2(\Pi)$. A direct calculation of the integral $\int wF_2\,\mathrm{d}\eta$ furnishes the differential operator M_2,

$$
M_2(s,\partial_s)v_* = \lambda_* v_* + b_1 k^2 v_* + \partial_s^2 v_*, b_1 = \int_\Pi w(\partial_1 w_1 - \eta_1\partial_1 w)\,\mathrm{d}\eta. \tag{19}
$$

Thus, in view of (18) and (19) the intermediate problem (15) reads

$$
-\partial_s^2 v_*(h,s) - (h^{-1}b_0 k(s) + b_1 k(s)^2)v_*(h,s) = \lambda_*(h)v_*(h,s), \quad s \in \partial\omega. \tag{20}
$$

5. A circular plate Ω_h

The simplest case for asymptotic analysis of equation (20) is $k(s) = R^{-1}$, i.e., ω is a circle of radius R. We set

$$
v_*(h,s) = v(s) + \ldots, \quad \lambda^*(h) = -h^{-1}b_0 k_0 + \lambda + \ldots
$$

and reduce promptly the intermediate problem (20) to the limit problem

$$
-\partial_s^2 v(s) - b_1 R^{-2} v(s) = \lambda v(s), \quad s \in \partial\omega = \mathbb{S}_R, \tag{21}
$$

solutions of which are evidently of the form

$$v_p(s) = \exp(\mathrm{i}ps/R), \quad \lambda_p = (p^2 - b_1)R^{-2}$$

where $p \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and $\mathrm{i} = \sqrt{-1}$.

Theorem 3. *Let the hypothesis* (6) *be fulfilled. If $\partial\omega$ is the circle of radius R and μ_0 the first eigenvalue of the problem* (5)*, then the entries of the eigenvalue sequence of the spectral problem* (1),

$$0 < \Lambda_0(h) < \Lambda_1(h) \leqslant \ldots \leqslant \Lambda_p(h) \leqslant \ldots \to +\infty, \tag{22}$$

verify the following estimates with a constant c_p independent of $h \in (0,1]$:

$$\left|\Lambda_{2p+q}(h) - \{h^{-2}\mu_0 - h^{-1}b_0R^{-1} + (p^2 - b_1)R^{-2}\}\right| \leqslant c_p h^{1/2}$$

where $q = 0$ as $p = 0$ and $q = -1, 0$ as $p > 0$.

6. Arbitrary shape of the middle-section ω

Let a point s_0 constitute a local maximum of the curvature function $\partial\omega \ni s \to k(s)$,

$$k(s) = k(s_0) - K_0(s - s_0)^2 + O(|s - s_0|^3). \tag{23}$$

Assuming $K_0 > 0$ and $\alpha = (b_0K_0)^{1/4}$, we introduce the rapid variable $\sigma = h^{-1/4}(s - s_0)$ and assume that the solution of the intermediate problem (20) has the following form:

$$v_*(h, s) = \mathbf{v}(\sigma) + \ldots, \quad \lambda_*(h) = -h^{-1}b_0k_0(s_0) + h^{-1/2}\lambda + \ldots \tag{24}$$

Inserting (23) and (24) into (20), we collect coefficients at $h^{-1/2}$, put $h = 0$, and finally obtain the limit problem, posed on the whole real axis,

$$-\partial_\sigma^2\mathbf{v}(\sigma) + b_0K_0\sigma^2\mathbf{v}(\sigma) = \lambda\mathbf{v}(\sigma), \quad \sigma \in \mathbb{R}. \tag{25}$$

Solutions to the spectral problem (25) are known to be of the form

$$\lambda_m = \alpha^2(2m - 1), \quad \mathbf{v}(\sigma) = \exp\left(\frac{1}{2}\alpha^2\sigma^2\right)\left[\frac{\mathrm{d}^{m-1}}{\mathrm{d}t^{m-1}}\exp(-t^2)\right]\Big|_{t=\alpha\sigma}, \quad m \in \mathbb{N}.$$

Theorem 4. *Let the hypothesis (6) be fulfilled. If $\mu \in (0, \pi^2)$ is a simple eigenvalue of the problem (5) and the formula (23) holds true with $K_0 > 0$, then there exists at least one eigenvalue $\Lambda_p(h)$ of the problem (1) such that*

$$(26) \quad |\Lambda_p(h) - [h^{-2}\mu - h^{-1}b_0k(s_0) + h^{-1/2}(b_0K_0)^{1/2}(2m-1)]| \leqslant c(\mu, s_0, m)h^{-1/4}$$

where $c(\mu, s_0, m)$ is independent of $h \in (0, 1]$. Moreover, in the case that $\mu = \mu_0$ is the first eigenvalue and the curvature k admits its global maximum at the only point s_0 with $K_0 > 0$ in (23), the eigenvalue $\Lambda_m(h)$ from the sequence (22) satisfies the inequality (26) with $\mu = \mu_0$, $p = m$ and there is no other eigenvalue subject to (26).

In view of the exponential behavior of the functions w and $\mathbf{v}_m$, the eigenfunction $u_m(h, x)$ corresponding to $\Lambda_m(h)$ in (22) concentrates in the vicinity of the point $x_0 = (s_0, 0) \in \mathbb{R}^3$ and decays as $o(\exp[-\beta h^{-1}n] \times \exp[-\beta h^{-1/2}|s - s_0|])$ with $\beta > 0$ while x moves from x_0. The decay rates are different in the normal and tangential directions. Theorem 4 can be easily reduced to the convergence statement

$$h^{1/2}\left\{\Lambda_m(h) - \left[h^{-2}\mu - h^{-1}b_0k(s_0)\right]\right\} \to (b_0K_0)^{1/2}(2m-1) \quad \text{as} \quad h \to +0, \quad m \in \mathbb{N}.$$

7. Eigenfunctions oscillating along the edge

Since the equation (20) still involves a (large) parameter, we can employ the WKB assumptions for eigenfunctions with an asymptotic behavior which crucially differs from (24),

$$(27) \qquad v_*(h, s) \sim \exp\left[\mathrm{i}h^{-1/2}A(h, s)\right]\sum_{j=0}^{\infty} h^{j/2}v_*^{(j)}(h, s).$$

Preserving the eigenvalue in the form (12), an application of the WKB method yields

$$(28) \quad A(h, s) = \int_{s_0}^{s}\left(b_0k(t) + h\lambda_*(h)\right)^{1/2}\mathrm{d}t, \quad v_*^0 = (\partial_s A)^{-1/2} = (b_0k + h\lambda_*(h))^{-1/4}.$$

Furthermore, the quantum condition of the Bohr-Sommerfeld type

$$(29) \qquad \int_{s_0}^{s_0+|\partial\omega|}(b_0k(t) + h\lambda_*^{(n)}(h))^{1/2}\,\mathrm{d}t = 2\pi nh^{1/2}$$

where $n \in \mathbb{N}$ and $|\partial\omega|$ denotes the length of the contour, yields the necessary periodicity of functions in (27), (28) but requires $n = O(h^{-1/2})$, $\lambda_*^{(n)}(h) = O(h^{-1})$.

Theorem 5. *If μ is a simple eigenvalue of the problem* (5) *and $\lambda_*^{(n)}(h)$ satisfies* (29) *with $n \in \mathbb{N}$, there exists an eigenvalue of the problem* (1) *such that*

$$|\Lambda_p(h) - [h^{-2}\mu - \lambda_*^{(n)}(h)]| \leqslant c_\mu.$$

The rate $O(h^{-1/2})$ of the oscillations along plate's edge is lower than the rate $O(h^{-1})$ of the exponential decay in the normal direction. This fact becomes the key point in justifying the WKB asymptotic forms.

8. Remarks on the justification

We have avoided to reproduce here any asymptotic formula for eigenfunctions of the problem (1) for the sake of brevity only. Using the assumptions for the asymptotics expansions and appropriate cut-off functions, asymptotic representations can be easily presented.

The exponential decays of the constructed approximations for eigenfunctions make it impossible to apply the standard approach (see, e.g., [1], [3], [6]) to deriving the convergence theorems for the spectrum of the problem (1). In order to prove Theorems 3–5 another method based on *direct and inverse reductions* of singularly perturbed spectral problems has been successfully employed in [8]. As usual, the *inverse* reduction implies a construction of an approximate solution to the problem (1) from a solution to a limit problem and an application of the classical lemma on "almost eigenvalues and eigenvectors". Consequently, this reduction only delivers proofs of Theorem 5 and the first assertion in Theorem 4. In contrast, the *direct* reduction employs a solution to the original problem (1) for constructing an approximate solution to either the intermediate problem (20), or to a limit problem. The repeated application of the above mentioned lemma provides then a comparison of the spectra of these problems and finally furnishes proofs of Theorem 3 and the second assertion of Theorem 4.

For the Neumann-Beltrami Laplacian in a thin curved two-dimensional domain, this way how to justify asymptotic representations of eigenvalues and eigenfunctions was indicated in [14]. Similar approaches have been used in [15] (see also [9]) for spectral problems in domains with singular perturbation of their boundaries and in [16] for problems with a small parameter at the highest derivatives. All estimates obtained in the cited papers as well as in [8] and here in Theorems 3–5 involve constants which depend on the eigenvalue number in the sequence (22). However, the book [17] presents a new approach which, using scrupulous weighted estimates of higher-order derivatives of the eigenfunctions u_m and both reductions, is able to instate estimates of the remainders in asymptotic representations of eigenvalues and eigenfunctions with bounds depending explicitly on attributes of the "limit" spectrum such as the eigenvalue number, the multiplicity, and the inverse distance to the other eigenvalues.

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