CHARACTERIZATIONS OF 0-DISTRIBUTIVE POSETS

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Abstract. The concept of a 0-distributive poset is introduced. It is shown that a section semicomplemented poset is distributive if and only if it is 0-distributive. It is also proved that every pseudocomplemented poset is 0-distributive. Further, 0-distributive posets are characterized in terms of their ideal lattices.

Keywords: 0-distributive, pseudocomplement, sectionally semi-complemented poset, ideal lattice

MSC 2000: 06A06, 06A11, 06C15, 06C20, 06D15

1. Introduction

Grillet and Varlet [1967] introduced the concepts of 0-distributive lattice as a generalization of distributive lattices.

A lattice L with 0 is called 0-*distributive* if, for $a, b, c \in L$, $a \wedge b = a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$. Dually, one can define 1-*distributive* lattice.

In this paper, we define the concept of 0-distributive poset which is distinct from the concept of 0-distributive poset defined by Pawar and Dhamke [1989]. It is proved that a distributive poset is 0-distributive and the converse need not be true. But, if we consider a sectionally semi-complemented poset then the converse is true. Further, we have shown that a poset is 0-distributive if and only if its ideal lattice is pseudocomplemented (equivalently, 0-distributive).

For undefined notations and terminology, the reader is referred to Grätzer [1998].

We begin with necessary definitions and terminologies in a poset P.

Let $A \subseteq P$. The set $A^u = \{x \in P;\ x \geqslant a \text{ for every } a \in A\}$ is called the *upper cone* of A. Dually, we have a concept of the *lower cone* A^l of A. A^{ul} shall mean $\{A^u\}^l$ and A^{lu} shall mean $\{A^l\}^u$. The lower cone $\{a\}^l$ is simply denoted by a^l and $\{a, b\}^l$ is denoted by $(a, b)^l$. Similar notations are used for upper cones. Further, for $A, B \subseteq P$, $\{A \cup B\}^u$ is denoted by $\{A, B\}^u$ and for $x \in P$, the set $\{A \cup \{x\}\}^u$ is denoted by $\{A, x\}^u$. Similar notations are used for lower cones. We note that $A^{lul} = A^l$, $A^{ulu} = A^u$ and $\{a^u\}^l = \{a\}^l = a^l$. Moreover, $A \subseteq A^{ul}$ and $A \subseteq A^{lu}$. If $A \subseteq B$ then $B^l \subseteq A^l$ and $B^u \subseteq A^u$.

2. 0-distributive posets

The concept of 0-distributive lattices is introduced by Grillet and Varlet [1967] which is further extended by Varlet [1972] and also by Pawar and Thakare [1978] to semilattices; see also C. Jayaram [1980], Hoo and Shum [1982]. Pawar and Dhamke [1989] extended the concept of 0-distributive semilattices to 0-distributive posets as follows.

Definition 2.1 (Pawar and Dhamke [1989]). A poset P with 0 is called 0-*distributive* (in the sense of Pawar and Dhamke) if, for $a, x_1, \ldots, x_n \in P$ (n finite), $(a, x_i)^l = \{0\}$ for every i, $1 \leqslant i \leqslant n$ imply $(a, x_1 \vee \ldots \vee x_n)^l = \{0\}$ whenever $x_1 \vee \ldots \vee x_n$ exists in P.

Now, we define the concept of 0-distributive poset as follows, without assuming the existence of join of finitely many elements:

Definition 2.2. A poset P with 0 is called 0-*distributive* if, for $a, b, c \in P$, $(a, b)^l = \{0\} = (a, c)^l$ together imply $\{a, (b, c)^u\}^l = \{0\}$.

Remark 2.3. From the following example it is clear that these two concepts of 0-distributivity are not equivalent.

Consider the poset depicted in Figure 1 which is 0-distributive in the sense of Pawar and Dhamke but it is not 0-distributive in our sense. Indeed, $(a, b)^l = (a, c)^l = \{0\}$ but $\{a, (b, c)^u\}^l \neq \{0\}$.

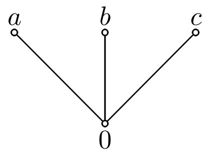

Figure 1

However, if P is an atomic poset then we have:

Proposition 2.4. *Let P be an atomic poset. If P is* 0*-distributive in our sense then it is* 0*-distributive in the sense of Pawar and Dhamke.*

Proof. Let $(a,b)^l = (a,c)^l = (a,d)^l = \{0\}$ and assume that $b \vee c \vee d$ exists. To show that P is 0-distributive in the sense of Pawar and Dhamke, we have to show that $(a, b \vee c \vee d)^l = \{0\}$. Assume to the contrary that $(a, b \vee c \vee d)^l \neq \{0\}$. Since P is atomic, there exists an atom $p \in P$ such that $p \in (a, b \vee c \vee d)^l$. This will imply that $(p,b)^l = (p,c)^l = (p,d)^l = \{0\}$, as $p \leqslant a$. By 0-distributivity in our sense, $\{p, (b,c)^u\}^l = \{p, (c,d)^u\}^l = \{0\}$. Hence, there exist elements d_1 and d_2 in P such that $d_1 \in (b,c)^u$, $d_2 \in (c,d)^u$ and $(p,d_1)^l = (p,d_2)^l = \{0\}$. By 0-distributivity in our sense, $\{p, (d_1,d_2)^u\}^l = \{0\}$. Again there exists $d_3 \in P$ such that $(p,d_3)^l = \{0\}$ and $d_3 \in (d_1,d_2)^u$. But then $d_3 \geqslant b,c,d$ and therefore $d_3 \geqslant b \vee c \vee d$. Hence $(p,d_3)^l = \{0\}$ gives $(p, b \vee c \vee d)^l = \{0\}$, a contradiction to $p \leqslant b \vee c \vee d$. The general case follows by induction. □

Remark 2.5. The converse of Proposition 2.4 is *not* true. The poset depicted in Figure 2 is finite and bounded 0-distributive in the sense of Pawar and Dhamke but not in our sense.

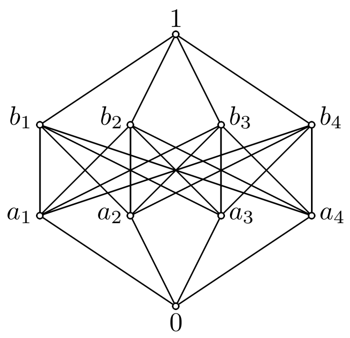

Figure 2

Henceforth, a 0-distributive poset will mean 0-distributive poset in our sense. Throughout this section, P denotes a poset with 0.

The following result gives some more examples of 0-distributive posets. For that we need:

Definition 2.6. A poset P is said to be *distributive* if, for all $a,b,c \in P$, $\{(a,b)^u, c\}^l = \{(a,c)^l, (b,c)^l\}^{ul}$ holds; see Larmerová and Rachůnek [1988].

Let P be a poset with 0. An element $x^* \in P$ is said to be the *pseudocomplement* of $x \in P$, if $(x,x^*)^l = \{0\}$ and for $y \in P$, $(x,y)^l = \{0\}$ implies $y \leqslant x^*$. A poset

P is called *pseudocomplemented* if each element of P has a pseudocomplement; see Venkatanarasimhan [1971] (see also Halaš [1993], Pawar and Waphare [2001]).

A poset P with 0 is called *sectionally semi-complemented* (in brief SSC) if, for $a, b \in P$, $a \nleqslant b$, there exists an element $c \in P$ such that $0 < c \leqslant a$ and $(b, c)^l = \{0\}$.

Lemma 2.7. *A distributive poset is* 0*-distributive.*

P r o o f. Let P be a distributive poset. Let $a, b, c \in P$ be such that $(a, b)^l = (a, c)^l = \{0\}$. By the distributivity of P, we have $\{a, (b, c)^u\}^l = \{(a, b)^l, (a, c)^l\}^{ul}$. But $(a, b)^l = (a, c)^l = \{0\}$ and hence $\{a, (b, c)^u\}^l = \{0\}$. Thus P is a 0-distributive poset. $\square$

R e m a r k 2.8. It is well known that a 0-distributive lattice need not be distributive; see the lattice of Figure 3 which is 0-distributive but not distributive.

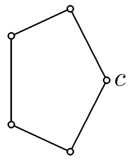

Figure 3

However, the converse of Lemma 2.7 is true in an SSC poset. Explicitly, we have:

Theorem 2.9. *An SSC poset is distributive if and only if it is* 0*-distributive.*

P r o o f. Let P be an SSC poset. Moreover, assume that P is 0-distributive. Let $x \in \{(a, b)^u, c\}^l$ and $y \in \{(a, c)^l, (b, c)^l\}^u$ for $a, b, c \in P$. To show that P is distributive, it is sufficient to show that $x \leqslant y$. Suppose $x \nleqslant y$. As P is SSC, there exists $z \in P$ such that $0 < z \leqslant x$ and $(z, y)^l = \{0\}$. Since $y \in (a, c)^{lu}$ as well as $y \in (b, c)^{lu}$ we have $(a, c)^l \subseteq y^l$ and $(b, c)^l \subseteq y^l$. This yields, after taking intersection with z^l on both sides, $(z, a)^l = \{0\}$ and $(z, b)^l = \{0\}$, as $z \leqslant x \leqslant c$. Now, by 0-distributivity of P, $\{z, (a, b)^u\}^l = \{0\}$. But since $z \leqslant x \in (a, b)^{ul}$, we have $z^l = \{z, (a, b)^u\}^l = \{0\}$, a contradiction to $0 < z$. The converse follows from Lemma 2.7. $\square$

Theorem 2.10. *Every pseudocomplemented poset is* 0*-distributive.*

P r o o f. Let P be a pseudocomplemented poset. Let a^* be the pseudocomplement of a. Moreover, suppose that $(a, b)^l = (a, c)^l = \{0\}$. By the definition of pseudocomplement, $b \leqslant a^*$ and $c \leqslant a^*$, and this yields $(b, c)^{ul} \subseteq \{a^*\}^l$. Taking intersection with a^l on both sides, we get $\{a, (b, c)^u\}^l = (a, a^*)^l = \{0\}$. Thus P is a 0-distributive poset. $\square$

R e m a r k 2.11. It is well known that a 0-distributive lattice need not be pseudocomplemented; see the lattice of Figure 4, which is 0-distributive but not pseudocomplemented.

For $a \in P$, we denote by $\{a\}^{\perp} = \{x \in P;\ (a,x)^l = \{0\}\}$. Now, we characterize 0-distributive posets in terms of ideals. Halaš [1995] defined a concept of an ideal as follows.

Definition 2.12. A non-empty subset I of a poset P is called an *ideal* if $a, b \in I$ implies $(a,b)^{ul} \subseteq I$.

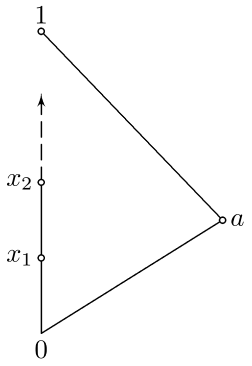

Figure 4

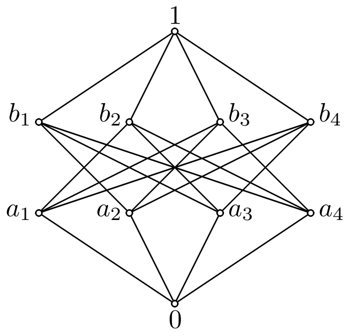

Figure 5

Venkatanarasimhan [1971] also defined the concept of an ideal as follows:

A non-empty subset I of a poset P is called an *ideal* if, $a \in I$, $b \leqslant a \Rightarrow b \in I$ and if the least upper bound of any finite number of elements of I, whenever it exists, belongs to I.

The subset $I = \{0, a, b\}$ of the poset depicted in Figure 1 is an ideal in the sense of Venkatanarasimhan [1971] but not in the sense of Halaš [1995], as $(a,b)^{ul} = P \nsubseteq I$.

But if we consider the subset $I = \{0, a_1, a_2, a_3\}$ of the poset depicted in Figure 5, then it is an ideal in the sense of Halaš [1995] but not in the sense of Venkatanarasimhan [1971], as $a_1 \vee a_2 \vee a_3 \notin I$.

Theorem 2.13. *A poset P is 0-distributive if and only if $\{a\}^{\perp}$ is an ideal (in the sense of Halaš) for every $a \in P$.*

P r o o f. Let $x, y \in \{a\}^{\perp}$. To show that $\{a\}^{\perp}$ is an ideal, we have to show that $(x,y)^{ul} \subseteq \{a\}^{\perp}$. Since $x, y \in \{a\}^{\perp}$, we get $(a,x)^l = (a,y)^l = \{0\}$. By 0-distributivity, $\{a, (x,y)^u\}^l = \{0\}$. Let $z \in (x,y)^{ul}$. Then clearly, $(a,z)^l = \{0\}$. Thus $z \in \{a\}^{\perp}$ which gives $(x,y)^{ul} \subseteq \{a\}^{\perp}$. Therefore $\{a\}^{\perp}$ is an ideal.

Conversely, suppose that $\{a\}^{\perp}$ is an ideal for every $a \in P$. To show P is 0-distributive, let's assume that $(a,x)^l = (a,y)^l = \{0\}$ for $x, y \in P$. Since $(a,x)^l = (a,y)^l = \{0\}$ we have $x, y \in \{a\}^{\perp}$. Since $\{a\}^{\perp}$ is an ideal, we have $(x,y)^{ul} \subseteq \{a\}^{\perp}$. Taking intersection with a^l on both sides, we get $\{a, (x,y)^u\}^l \subseteq \{a\}^{\perp} \cap a^l$. Clearly, $\{a\}^{\perp} \cap a^l = \{0\}$. Therefore $\{a, (x,y)^u\}^l = \{0\}$ and the 0-distributivity of P follows. $\square$

For any subset A of P, we denote by $A^{\perp} = \{x \in P;\ (a,x)^l = \{0\}$ for all $a \in A\}$. It is clear that $A^{\perp} = \bigcap\limits_{a \in A} \{a\}^{\perp}$.

The following corollary is an easy consequence of Theorem 2.13.

Corollary 2.14. *A poset P is 0-distributive if and only if $A^{\perp}$ is an ideal for any subset A of P.*

The results similar to Theorem 2.13 and Corollary 2.14 are also obtained by Pawar and Dhamke [1989] but they have considered the definition of ideal given by Venkatanarasimhan [1971].

Remark 2.15. It is well-known that the ideal lattice of a distributive lattice is pseudocomplemented; see Varlet [1968]. However, the converse is not true; see the lattice depicted in Figure 4 which is not distributive but whose ideal lattice is pseudocomplemented. This example is due to Varlet [1968]. Further, Varlet [1968] proved that a bounded below lattice is 0-distributive if and only if its ideal lattice is pseudocomplemented. It is proved that the set of ideals (in the sense of Halaš) of a poset P, denoted by $\mathrm{Id}(P)$, forms a complete lattice under inclusion; see Halaš [1995].

Now, we characterize 0-distributive posets in terms of their ideal lattice.

Theorem 2.16. *A poset P is 0-distributive if and only if $\mathrm{Id}(P)$ is pseudocomplemented.*

Proof. Let P be a 0-distributive poset and $A \in \mathrm{Id}(P)$. By Corollary 2.14, $A^{\perp}$ is an ideal in P. We claim that $A^{\perp}$ is the pseudocomplement of A in $\mathrm{Id}(P)$. Clearly, $A \wedge A^{\perp} = (0]$. Assume that $A \wedge B = (0]$ for $B \in \mathrm{Id}(P)$. To show that $A^{\perp}$ is the pseudocomplement of A, we have to show that $B \leqslant A^{\perp}$. Let $b \in B$. If $t \in (a,b)^l$ for some $a \in A$, then clearly $t \in A$ as well as $t \in B$; hence $t \in A \wedge B = (0]$. Therefore $(a,b)^l = \{0\}$ for every $a \in A$. Thus $b \in A^{\perp}$ and we get $B \leqslant A^{\perp}$ as required.

Conversely, suppose that $\mathrm{Id}(P)$ is pseudocomplemented. To show P is 0-distributive, assume that $(a,x)^l = (a,y)^l = \{0\}$. Hence $(a] \wedge (x] = (a] \wedge (y] = (0]$. Since $\mathrm{Id}(P)$ is pseudocomplemented, we have $(x] \leqslant (a]^*$ and $(y] \leqslant (a]^*$. Thus we are led to $(x] \vee (y] \leqslant (a]^*$. Taking meet with $(a]$, we get $((x] \vee (y]) \wedge (a] = (a] \wedge (a]^* = (0]$ yielding $\{(x,y)^u, a\}^l = \{0\}$. Thus P is a 0-distributive poset. □

Theorem 2.17. *A poset P is 0-distributive if and only if $\mathrm{Id}(P)$ is a 0-distributive lattice.*

Proof. Suppose P is 0-distributive. By Theorem 2.16 and Theorem 2.10, $\mathrm{Id}(P)$ is 0-distributive.

Conversely, suppose that $\mathrm{Id}(P)$ is a 0-distributive lattice. To show P is 0-distributive, let $(a,x)^l = (a,y)^l = \{0\}$. That means $(a] \wedge (x] = (a] \wedge (y] = (0]$. By 0-distributivity of $\mathrm{Id}(P)$, $(a] \wedge ((x] \vee (y]) = (0]$, i.e., $\{a, (x,y)^u\}^l = \{0\}$. Hence P is a 0-distributive poset. □

Now, we add one more characterization of 0-distributivity which is even new in the lattice context.

Theorem 2.18. *A poset P with 0 is 0-distributive if and only if it satisfies the following condition D_0.*

(D_0) *If $(a,b)^l = (a,c)^l = \{0\}$ and $(a,b)^{ul} \subseteq (b,c)^{ul}$ for $a,b,c \in P$ then $a = 0$.*

Proof. Let P be a 0-distributive poset. To prove the condition (D_0), assume $a,b,c, \in P$ are such that $(a,b)^l = (a,c)^l = \{0\}$ and $(a,b)^{ul} \subseteq (b,c)^{ul}$. By 0-distributivity, we have $\{a,(b,c)^u\}^l = \{0\}$. Since $(a,b)^{ul} \subseteq (b,c)^{ul}$, we get $\{0\} = \{a,(b,c)^u\}^l \supseteq \{a,(a,b)^u\}^l = a^l$. Thus $a = 0$.

Conversely, suppose the condition (D_0) holds. To prove that P is 0-distributive, let $a,b,c \in P$ be such that $(a,b)^l = (a,c)^l = \{0\}$. Let $d \in \{a,(b,c)^u\}^l$. Then clearly $(d,b)^l = (d,c)^l = \{0\}$ and $(d,b)^{ul} \subseteq (b,c)^{ul}$ and $(d,c)^{ul} \subseteq (b,c)^{ul}$. By the condition ($\mathrm{D}_0$), $d = 0$ which yields $\{a,(b,c)^u\}^l = \{0\}$. □

Corollary 2.19. *A lattice L with 0 is 0-distributive if and only if it satisfies the following condition D_0.*

(D_0) *If $a \wedge b = a \wedge c = 0$ and $a \vee b \leqslant b \vee c$ for $a,b,c \in L$ then $a = 0$.*

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