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Finite element approximation of flow of fluids with shear rate and pressure dependent viscosity

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In this paper, we consider a class of incompressible viscous fluids whose viscosity depends on the shear rate and pressure. We deal with the isothermal steady flow and analyze the Galerkin discretization of the corresponding equations. We discuss the existence and uniqueness of discrete solutions, and their convergence to the solution to the original problem. In particular, we derive a priori error estimates which provide optimal rates of convergence with respect to the expected regularity of the solution. Finally, we demonstrate the achieved results by numerical experiments.
The fluid models under consideration appear in many practical problems, for instance in elasto-hydrodynamic lubrication, where very high pressures occur. Here, we consider shear-thinning fluid models, similar to the power-law/Carreau model. A restricted sub-linear dependence of the viscosity on the pressure is allowed. The mathematical theory concerned with the self-consistency of the governing equations has emerged only recently. We adopt the established theory in the context of discrete approximations. To our knowledge, this is the first analysis of the finite element method for fluids with pressure-dependent viscosity. The derived estimates coincide with the optimal error estimates established recently for Carreau-type models, which are covered as a special case.

Keywords: Non-Newtonian fluid, shear rate and pressure dependent viscosity, finite element method, error analysis.

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1. Introduction

The article is devoted to the finite element discretization of equations governing the steady flow of a class of incompressible fluids whose viscosity depends non-linearly on shear rate and pressure. We discuss the well-posedness of the discretized problem and derive a priori estimates of the discretization error.

The isothermal flow of an incompressible viscous fluid is typically described by the Navier–Stokes equations, which embody Newton's hypothesis that the viscosity—the ratio between the shear stress and the shear rate—is constant. Since the early formation of fluid mechanics it has been known that this assumption may not be applicable to all viscous flows. In the last decades many non-Newtonian phenomena have become subject of scientific interest. We will consider models with shear-dependent and pressure-dependent viscosity, which play an important role in many areas such as elasto-hydrodynamic lubrication, the modeling of Earth's mantle, glaciers or avalanches. The viscosity of fluids in such applications varies considerably, even by several orders, with the pressure.

We study the steady isothermal flow of an homogeneous incompressible viscous fluid in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, governed by the following system of PDEs:

$$\left.\begin{aligned} -\operatorname{div} \boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}) + \nabla \pi &= \boldsymbol{f} \\ \operatorname{div} \boldsymbol{v} &= 0 \end{aligned}\right\} \quad \text{in } \Omega, \tag{1.1}$$

where $\boldsymbol{v}$ is the velocity, π denotes the pressure (more specifically, the ratio of the mean normal stress and the density), and $\boldsymbol{f}$ represents the density of an applied body force. Here, $\boldsymbol{D}\boldsymbol{v}$ is the symmetric part of the velocity gradient. Note that we avoid mathematical difficulties related to the convective term by neglecting inertial forces in the first equation. We consider extra stress tensors $\boldsymbol{S}$ of the form

$$\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}) = 2\eta(\pi, |\boldsymbol{D}\boldsymbol{v}|^2)\boldsymbol{D}\boldsymbol{v}, \tag{1.2}$$

where η is the generalized kinematic viscosity. Many details, examples, and an extensive discussion concerning the class of models (1.2) can be found in (Málek & Rajagopal, 2006, 2007).

We assume that the domain boundary $\partial\Omega$ is Lipschitz and consists of two parts, $\partial\Omega = \Gamma_D \cup \Gamma_P$, $|\Gamma_D| > 0$. Then, we complement the system (1.1) by the boundary conditions

$$\boldsymbol{v} = \boldsymbol{v}_D \qquad \text{on } \Gamma_D, \tag{1.3}$$

$$-\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v})\boldsymbol{n} + \pi\boldsymbol{n} = \boldsymbol{b} \qquad \text{on } \Gamma_P, \tag{1.4}$$

where $\boldsymbol{n}$ denotes the unit outer normal vector to $\partial\Omega$. We distinguish two cases:

a) If $|\Gamma_P| = 0$ (i.e., the Dirichlet boundary conditions are prescribed on the whole boundary, $\Gamma_D = \partial\Omega$) then we additionally fix the level of pressure by requiring

$$\fint_\Omega \pi \,\mathrm{d}\boldsymbol{x} = \pi_0 \in \mathbb{R}. \tag{1.5}$$

For simplicity of notations* we assume $\pi_0 = 0$.

b) If $|\Gamma_P| > 0$ then (1.4) suffices to fix the level of pressure. This was shown in (Lanzendörfer & Stebel, 2008, 2009), see also Lemma 2.6 and Theorem 3.2 below.

*The theoretical methods and results of this paper are not restricted to the choice $\pi_0 = 0$.

It is a special feature of piezoviscous fluids in the case a) that the number π_0 affects through $\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v})$ the whole solution, including the velocity field. Hence, the non-physical constraint (1.5) comprises an important input parameter undeterminable by practical applications. By contrast, $\boldsymbol{b}$ in (1.4) represents the force acting on the domain boundary and reflects physically reasonable input data.

While the mathematical self-consistency of the shear-thinning or shear-thickening fluid models has been studied intensively since the 1960's, the rigorous analysis of those with pressure-dependent viscosity has emerged only recently, see (Málek & Rajagopal, 2006) for references. The well-posedness of problems where the viscosity depends solely on the pressure, or grows with the pressure super-linearly, has not been resolved, except under severe restrictions on data size or time interval. When the viscosity changes with the pressure too rapidly, the equations corresponding to the steady flow lose their ellipticity. A breakthrough result appeared in a paper by Málek et al. (Málek *et al.*, 2002), where viscosities depending both on the pressure and the shear rate have been considered. The structure of the viscosity proposed therein has allowed for global and large data existence results for both steady and unsteady motions under various boundary conditions, see e.g. (Franta *et al.*, 2005; Bulíček *et al.*, 2007; Lanzendörfer, 2009; Lanzendörfer & Stebel, 2009).

Our aim is to adopt the established mathematical theory in the framework of Galerkin discretization. The finite element method has been studied extensively in the context of power-law/Carreau models (where the viscosity only depends on the shear rate), see (Baranger & Najib, 1990; Barrett & Liu, 1993, 1994) and the references therein. In particular, Hirn (Hirn, 2010) and Belenki et al. (Belenki *et al.*, 2010) have recently derived optimal a priori error estimates in the case of shear thinning. However, no such analysis is available when the fluid's viscosity depends also on the pressure. To our best knowledge, the present paper provides the first analytical study of the finite element method in the context of fluids with shear rate and pressure dependent viscosity.

This paper is devoted to the finite element discretization of the problem (1.1)–(1.5). We will show that the finite element solutions $(\boldsymbol{v}_h, \pi_h)$ exist, are determined uniquely, and that they converge to the weak solution $(\boldsymbol{v}, \pi)$ strongly in $\mathbf{W}^{1,p}(\Omega) \times \mathrm{L}^{p'}(\Omega)$, $p \in (1,2)$, for diminishing mesh size h. Moreover, if the solution $(\boldsymbol{v}, \pi)$ satisfies the regularity condition

$$\int_\Omega (1 + |\boldsymbol{D}\boldsymbol{v}|)^{p-2} |\nabla \boldsymbol{D}\boldsymbol{v}|^2 \,\mathrm{d}\boldsymbol{x} < \infty \qquad \text{and} \qquad \pi \in \mathrm{W}^{1,p'}(\Omega), \tag{1.6}$$

then an $\mathscr{O}(h)$ error bound for the velocity in $\mathbf{W}^{1,p}(\Omega)$, and an $\mathscr{O}(h^{\frac{2}{p'}})$ error bound for the pressure in $\mathrm{L}^{p'}(\Omega)$ will be established:

$$\|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,p} \leqslant ch, \qquad \|\pi - \pi_h\|_{p'} \leqslant ch^{\frac{2}{p'}}.$$

These estimates will be derived by means of the well-known quasi-norm technique which has originally been developed for the error analysis of the p-Laplace equation, see (Barrett & Liu, 1994). Numerical experiments indicate that these estimates are optimal with respect to the supposed regularity. Moreover, the present paper also covers the case of Carreau-type models, for which the a priori error estimates derived here coincide with those established in (Hirn, 2010) and (Belenki *et al.*, 2010).

The paper is organized as follows: In Section 2 we formulate basic assumptions, introduce tools and define the problem and its discretization. Section 3 deals with the existence and uniqueness of the discrete solutions and their convergence to the weak solution of the problem. A priori error estimates are derived in Section 4 and are applied to the finite element discretization in Section 5. Finally, in Section 6 we demonstrate the theoretical results by numerical experiments.

2. Preliminaries

In this section we introduce the notation, we state our assumptions on the extra stress tensor, indicate how the stress tensor is related to N-functions and we show its resulting properties. Then, we introduce the weak formulation of the system (1.1)–(1.5) and its Galerkin discretization.

NOTATION AND FUNCTION SPACES. The set of all positive real numbers is denoted by $\mathbb{R}^+$. Let $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. The Euclidean scalar product of two vectors $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^d$ is denoted by $\boldsymbol{p} \cdot \boldsymbol{q}$, the scalar product of $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{d\times d}$ is defined by $\boldsymbol{P} : \boldsymbol{Q} := \sum_{i,j=1}^d P_{ij} Q_{ij}$. We set $|\boldsymbol{Q}| := (\boldsymbol{Q} : \boldsymbol{Q})^{1/2}$. Often we use c as a generic constant, whose value may change from line to line but does not depend on important variables. We write $a \sim b$ if there exist positive constants c and C independent of all relevant quantities such that $cb \leqslant a \leqslant Cb$. Similarly, the notation $a \lesssim b$ is used for $a \leqslant Cb$.

For measurable set $\omega \subset \Omega$, $|\omega|$ denotes its d-dimensional Lebesgue measure. For $\nu \in [1,\infty]$, $\mathrm{L}^\nu(\Omega)$ stands for the Lebesgue space and $\mathrm{W}^{m,\nu}(\Omega)$ for the Sobolev space of order m. The space $\mathrm{L}_0^\nu(\Omega)$ contains all $q \in \mathrm{L}^\nu(\Omega)$ with $⨍_\Omega q\,dx := \frac{1}{|\Omega|}\int_\Omega q\,dx = 0$. For $\nu > 1$ we use the notation $\mathrm{W}_0^{1,\nu}(\Omega)$ for the Sobolev space with vanishing traces on $\partial\Omega$. The $\mathrm{L}^\nu(\omega)$-norm is denoted by $\|\cdot\|_{\nu;\omega}$ and the $\mathrm{W}^{m,\nu}(\omega)$-norm is denoted by $\|\cdot\|_{m,\nu;\omega}$. The notation $(u,v)_\omega$ is used for the integral $\int_\omega uv\,\mathrm{d}\boldsymbol{x}$. In case of $\omega = \Omega$, we usually omit the index Ω. Spaces of $\mathbb{R}^d$-valued functions are denoted with boldface type, though no distinction is made in the notation of norms and inner products; the norm in $\mathbf{W}^{m,\nu}(\Omega) \equiv [\mathrm{W}^{m,\nu}(\Omega)]^d$ is given by $\|\boldsymbol{w}\|_{m,\nu} = \big(\sum_{1\leqslant i\leqslant d}\sum_{0\leqslant|\alpha|\leqslant m}\|\partial^\alpha w_i\|_\nu^\nu\big)^{1/\nu}$, etc.

STRUCTURAL ASSUMPTIONS ON THE STRESS TENSOR. Let $p > 1$, $\varepsilon > 0$, and $\gamma_0 \geqslant 0$ be given. We suppose that the extra stress tensor $\boldsymbol{S}$ belongs to the class (1.2) and satisfies the structural assumptions:

(A1) *There exist positive constants σ_0, σ_1 such that for all $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{d\times d}_{sym}$, $q \in \mathbb{R}$ there holds*

$$\sigma_0(\varepsilon^2 + |\boldsymbol{P}|^2)^{\frac{p-2}{2}}|\boldsymbol{Q}|^2 \leqslant \frac{\partial \boldsymbol{S}(q,\boldsymbol{P})}{\partial \boldsymbol{P}} : (\boldsymbol{Q}\otimes\boldsymbol{Q}) \leqslant \sigma_1(\varepsilon^2 + |\boldsymbol{P}|^2)^{\frac{p-2}{2}}|\boldsymbol{Q}|^2,$$

where $\mathbb{R}^{d\times d}_{sym} := \{\boldsymbol{P} \in \mathbb{R}^{d\times d}; \boldsymbol{P} = \boldsymbol{P}^\top\}$ and $(\boldsymbol{Q}\otimes\boldsymbol{Q})_{ijkl} = Q_{ij}Q_{kl}$.

(A2) *For all $\boldsymbol{P} \in \mathbb{R}^{d\times d}_{sym}$ and $q \in \mathbb{R}$ there holds*

$$\Big|\frac{\partial \boldsymbol{S}(q,\boldsymbol{P})}{\partial q}\Big| \leqslant \gamma_0(\varepsilon^2 + |\boldsymbol{P}|^2)^{\frac{p-2}{4}}.$$

REMARK 2.1 Many examples of viscosities fulfilling these assumptions can be found, e.g., in (Málek *et al.*, 2002; Málek & Rajagopal, 2006, 2007). See also Remark 6.1.

We depict how the stress tensor relates to N-functions. A continuous, convex function $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is called N-function if $\psi(0) = 0$, $\psi(t) > 0$ for $t > 0$, $\lim_{t\to 0+}\psi(t)/t = 0$ and $\lim_{t\to\infty}\psi(t)/t = \infty$. Consequently, there exists the right derivative ψ' of ψ, which is non-decreasing and satisfies $\psi'(0) = 0$, $\psi'(t) > 0$ for $t > 0$, and $\lim_{t\to\infty}\psi'(t) = \infty$. We define the complementary N-function ψ^* by $\psi^*(t) := \sup_{s\geqslant 0}(st - \psi(s))$ for all $t \geqslant 0$. If ψ' is strictly increasing, then $(\psi^*)' = (\psi')^{-1}$. An important subclass of N-functions are those that satisfy the Δ_2-condition: ψ satisfies the Δ_2-condition, if there exists $C > 0$ such that $\psi(2t) \leqslant C\psi(t)$ for all $t \geqslant 0$. Here, $\Delta_2(\psi)$ denotes the smallest such constant.

Lemma 32 in (Diening & Ettwein, 2008) provides the following Young-type inequality: For all $\delta > 0$ there exists $c_\delta > 0$, which only depends on $\Delta_2(\psi)$, $\Delta_2(\psi^*) < \infty$, such that for all $s, t \geqslant 0$ there holds

$$s\psi'(t) + \psi'(s)t \leqslant \delta\,\psi(s) + c_\delta\,\psi(t). \tag{2.1}$$

Let us consider the following simple examples: For $p > 1$ we introduce the convex function

$$\varphi \in C(\mathbb{R}_0^+, \mathbb{R}_0^+), \qquad \varphi(t) := \frac{1}{p} t^p. \tag{2.2}$$

Clearly, φ and φ^*, where $\varphi^*(t) = \frac{1}{p'} t^{p'}$, are N-functions satisfying the Δ_2-condition. For given N-function ψ with $\Delta_2(\psi)$, $\Delta_2(\psi^*) < \infty$, we define the family of *shifted* functions $\{\psi_a\}_{a \geqslant 0}$ by

$$\psi_a(t) := \int_0^t \psi_a'(s)\,\mathrm{d}s \qquad \text{with} \qquad \psi_a'(t) := \psi'(a+t)\frac{t}{a+t}. \tag{2.3}$$

Then, Lemma 23 in (Diening & Ettwein, 2008) ensures that $\{\psi_a\}_{a \geqslant 0}$ are again N-functions and satisfy the Δ_2-condition uniformly in $a \geqslant 0$ with Δ_2-constants only depending on $\Delta_2(\psi)$, $\Delta_2(\psi^*)$. Let us return to the case (2.2): The family of shifted N-functions $\{\varphi_a\}_{a \geqslant 0}$ belongs to $C^1(\mathbb{R}_0^+) \cap C^2(\mathbb{R}^+)$ and satisfies the Δ_2-condition uniformly in $a \geqslant 0$ with Δ_2-constants only depending on p. Using the definition of φ_a, we easily conclude

$$\min\{1, p-1\}(a+t)^{p-2} \leqslant \varphi_a''(t) \leqslant \max\{1, p-1\}(a+t)^{p-2} \tag{2.4}$$

and $\varphi_a'(t) \sim \varphi''(a+t)t \sim \varphi_a''(t)t$. Moreover, $\varphi_a(t) \sim \varphi_a'(t)t$ uniformly in $t, a \geqslant 0$. Due to (2.4) the inequalities of Assumption **(A1)** defining the (p,ε)-structure of $\boldsymbol{S}$ can be expressed equivalently in terms of the N-functions φ_ε.

BASIC PROPERTIES OF THE EXTRA STRESS TENSOR. We express several consequences of Assumptions **(A1)** and **(A2)**. Below we formulate the results as general as possible, although in the forthcoming sections we only treat the shear thinning case $p < 2$. We introduce the function $\boldsymbol{F} : \mathbb{R}^{d \times d}_{\mathrm{sym}} \to \mathbb{R}^{d \times d}_{\mathrm{sym}}$ by

$$\boldsymbol{F}(\boldsymbol{P}) := \big(\varepsilon + |\boldsymbol{P}|\big)^{\frac{p-2}{2}} \boldsymbol{P}, \tag{2.5}$$

where p and ε are the same as in Assumptions **(A1)–(A2)**. The quantity $\boldsymbol{F}$ is closely related to the extra stress tensor $\boldsymbol{S}$ as depicted by the following lemma:

LEMMA 2.1 For given $p \in (1, \infty)$ and $\varepsilon \in [0, \infty)$ let $\boldsymbol{S}$ satisfy **(A1)**, let $\boldsymbol{F}$ be defined by (2.5), and let φ be defined by (2.2). Then, uniformly for all $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{R}^{d \times d}_{\mathrm{sym}}$, $q \in \mathbb{R}$, it holds:

$$\begin{aligned}
\Big(\boldsymbol{S}(q,\boldsymbol{P}) - \boldsymbol{S}(q,\boldsymbol{Q})\Big) : (\boldsymbol{P} - \boldsymbol{Q}) &\sim (\varepsilon + |\boldsymbol{P}| + |\boldsymbol{Q}|)^{p-2} |\boldsymbol{P} - \boldsymbol{Q}|^2 \\
&\sim \varphi_{\varepsilon + |\boldsymbol{P}|}(|\boldsymbol{P} - \boldsymbol{Q}|) \sim |\boldsymbol{F}(\boldsymbol{P}) - \boldsymbol{F}(\boldsymbol{Q})|^2, \\
|\boldsymbol{S}(q,\boldsymbol{P}) - \boldsymbol{S}(q,\boldsymbol{Q})| &\sim \varphi'_{\varepsilon + |\boldsymbol{P}|}(|\boldsymbol{P} - \boldsymbol{Q}|),
\end{aligned}$$

where the constants only depend on σ_0, σ_1 and p. In particular, they are independent of $\varepsilon \geqslant 0$. Moreover, the following estimates hold:

$$\boldsymbol{S}(q,\boldsymbol{Q}) : \boldsymbol{Q} \geqslant \frac{\sigma_0}{2p}(|\boldsymbol{Q}|^p - \varepsilon^p) \qquad \text{and} \qquad |\boldsymbol{S}(q,\boldsymbol{Q})| \leqslant \frac{\sigma_1}{p-1} |\boldsymbol{Q}|^{p-1}. \tag{2.6}$$

Proof. For (2.6), see (Málek *et al.*, 1996, Lemma 1.19). All remaining estimates are proven in (Diening & Ettwein, 2008). □

As a straightforward consequence of Assumptions **(A1)** and **(A2)** we also obtain

LEMMA 2.2 For given $p \in (1,\infty)$, $\varepsilon \in (0,\infty)$ and $\gamma_0 \in [0,\infty)$ let $\boldsymbol{S}$ satisfy **(A1), (A2)**. Then, for all $\boldsymbol{P},\boldsymbol{Q} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $\pi, q \in \mathbb{R}$, denoting $\boldsymbol{P}_s := \boldsymbol{Q} + s(\boldsymbol{P}-\boldsymbol{Q})$, it holds:

$$\frac{\sigma_0}{2}\int_0^1 (\varepsilon^2+|\boldsymbol{P}_s|^2)^{\frac{p-2}{2}}|\boldsymbol{P}-\boldsymbol{Q}|^2\mathrm{d}s \leqslant (\boldsymbol{S}(\pi,\boldsymbol{P})-\boldsymbol{S}(q,\boldsymbol{Q})):(\boldsymbol{P}-\boldsymbol{Q}) + \frac{\gamma_0^2}{2\sigma_0}|\pi-q|^2,$$

$$|\boldsymbol{S}(\pi,\boldsymbol{P})-\boldsymbol{S}(q,\boldsymbol{Q})| \leqslant \sigma_1\int_0^1 (\varepsilon^2+|\boldsymbol{P}_s|^2)^{\frac{p-2}{2}}|\boldsymbol{P}-\boldsymbol{Q}|\,\mathrm{d}s + \gamma_0\int_0^1(\varepsilon^2+|\boldsymbol{P}_s|^2)^{\frac{p-2}{4}}|\pi-q|\,\mathrm{d}s.$$

Proof. See, e.g., (Bulíček *et al.*, 2007, Lemma 1.4). □

In view of Lemma 2.2 we define the distance

$$d(\boldsymbol{v},\boldsymbol{u})^2 := \int_\Omega\int_0^1 (\varepsilon^2+|\boldsymbol{D}\boldsymbol{u}+s(\boldsymbol{D}\boldsymbol{v}-\boldsymbol{D}\boldsymbol{u})|^2)^{\frac{p-2}{2}}|\boldsymbol{D}\boldsymbol{v}-\boldsymbol{D}\boldsymbol{u}|^2\,\mathrm{d}s\,\mathrm{d}\boldsymbol{x} \tag{2.7}$$

for all $\boldsymbol{v},\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$. We get the following

COROLLARY 2.1 For given $p \in (1,\infty)$, $\varepsilon \in (0,\infty)$ and $\gamma_0 \in [0,\infty)$ let $\boldsymbol{S}$ satisfy **(A1), (A2)**. Let $d(\cdot,\cdot)$ be defined by (2.7). Then, for all $\boldsymbol{v},\boldsymbol{w} \in \mathbf{W}^{1,p}(\Omega)$ and $\pi,q \in \mathrm{L}^2(\Omega)$ there holds:

$$\frac{\sigma_0}{2}d(\boldsymbol{v},\boldsymbol{w})^2 \leqslant (\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v})-\boldsymbol{S}(q,\boldsymbol{D}\boldsymbol{w}),\boldsymbol{D}\boldsymbol{v}-\boldsymbol{D}\boldsymbol{w})_\Omega + \frac{\gamma_0^2}{2\sigma_0}\|\pi-q\|_2^2. \tag{2.8}$$

Moreover, for each $\delta > 0$ there exists a positive constant $c_\delta = c_\delta(\sigma_1,\gamma_0)$ such that

$$(\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v})-\boldsymbol{S}(q,\boldsymbol{D}\boldsymbol{w}),\boldsymbol{D}\boldsymbol{v}-\boldsymbol{D}\boldsymbol{w})_\Omega \leqslant c_\delta d(\boldsymbol{v},\boldsymbol{w})^2 + \delta\|\pi-q\|_2^2. \tag{2.9}$$

In particular, if $p<2$, then for all $\boldsymbol{v},\boldsymbol{w} \in \mathbf{W}^{1,p}(\Omega)$ and all sufficiently smooth functions π,q there exists a positive constant $c = c(p,\sigma_1)$ such that

$$\|\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v})-\boldsymbol{S}(q,\boldsymbol{D}\boldsymbol{w})\|_2 \leqslant \sigma_1\varepsilon^{\frac{p-2}{2}}d(\boldsymbol{v},\boldsymbol{w}) + \gamma_0\varepsilon^{\frac{p-2}{2}}\|\pi-q\|_2, \tag{2.10}$$

$$\|\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v})-\boldsymbol{S}(q,\boldsymbol{D}\boldsymbol{w})\|_{p'} \leqslant c\,d(\boldsymbol{v},\boldsymbol{w})^{\frac{2}{p'}} + \gamma_0\varepsilon^{\frac{p-2}{2}}\|\pi-q\|_{p'}. \tag{2.11}$$

Proof. The proof is based on Lemma 2.2. In order to derive (2.11), we additionally need Lemma 2.1 in (Acerbi & Fusco, 1989). □

We remark that the distance $d(\cdot,\cdot)$ is equivalent to the so-called *quasi-norm* which was introduced by Barrett/Liu in (Barrett & Liu, 1993). Hence, all results below can also be expressed in terms of quasi-norms. The following lemma indicates that $d(\cdot,\cdot)$ is also equivalent to the $\boldsymbol{F}$–distance:

LEMMA 2.3 For $p \in (1,\infty)$, $\varepsilon \in (0,\infty)$ let $\boldsymbol{S}$ satisfy **(A1)**. Let $d(\cdot,\cdot)$ be defined by (2.7), and let $\boldsymbol{F}$ be defined by (2.5). For all $\boldsymbol{v},\boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega)$ and $\pi \in \mathrm{L}^2(\Omega)$ there holds:

$$d(\boldsymbol{v},\boldsymbol{u})^2 \sim \|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{F}(\boldsymbol{D}\boldsymbol{u})\|_2^2 \sim (\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v})-\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{u}),\boldsymbol{D}\boldsymbol{v}-\boldsymbol{D}\boldsymbol{u})_\Omega. \tag{2.12}$$

All constants only depend on p,σ_0,σ_1.

Proof. The assertion follows from Lemma 2.1 and Lemma 2.1 in (Acerbi & Fusco, 1989). □

The following lemma, whose proof can be found in (Berselli *et al.*, 2010), shows the connection between the quasi-norms and Sobolev norms:

LEMMA 2.4 For $p \in (1,2]$ and $\varepsilon \in (0,\infty)$ let $\boldsymbol{S}$ satisfy **(A1)**, and let $\boldsymbol{F}$ be defined by (2.5). Then, for all sufficiently smooth functions $\boldsymbol{v}$, $\boldsymbol{u}$ and for $\nu \in [1,2]$ there holds:

$$\|\boldsymbol{D}(\boldsymbol{v}-\boldsymbol{u})\|_\nu^2 \lesssim \|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u})\|_2^2 \|(\varepsilon + |\boldsymbol{D}\boldsymbol{v}| + |\boldsymbol{D}\boldsymbol{u}|)^{2-p}\|_{\frac{\nu}{2-\nu}}, \tag{2.13}$$

where the constant only depends on p, σ_0, and σ_1. If $\nu = 2$, then $\frac{\nu}{2-\nu} = \infty$.

WEAK FORMULATION. The natural spaces for the velocity and pressure are given by

$$\begin{aligned} \boldsymbol{X}^p &:= \{\boldsymbol{w} \in \mathbf{W}^{1,p}(\Omega); \operatorname{tr} \boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_D\}, \\ Q^p &:= \{q \in \mathrm{L}^{p'}(\Omega);\ \text{if } |\Gamma_P| = 0 \text{ then } \textstyle\int_\Omega q \,\mathrm{d}\boldsymbol{x} = 0\}, \end{aligned}$$

where $p' := p/(p-1)$. The following Korn inequality holds in $\boldsymbol{X}^p$ as long as $|\Gamma_D| > 0$:

LEMMA 2.5 (Korn's inequality) Let $\nu \in (1,\infty)$, $\Omega \subset \mathbb{R}^d$ be a bounded domain and $\partial\Omega, \Gamma_D \in C^{0,1}$, where $\Gamma_D \subset \partial\Omega$ has nonzero $(d-1)$-dimensional measure. Then there exists a constant $c_K := c_K(\Omega, \Gamma_D, \nu) > 0$ such that

$$c_K \|\boldsymbol{w}\|_{1,\nu} \leqslant \|\boldsymbol{D}\boldsymbol{w}\|_\nu \qquad \forall \boldsymbol{w} \in \boldsymbol{X}^\nu.$$

Proof. The result can be found e.g. in (Málek *et al.*, 1996, Theorem 1.10 on p. 196); although it is formulated for $\Gamma_D = \partial\Omega$ there, its proof covers the case $|\Gamma_D| > 0$. □

Let us summarize the general assumptions that will be used in the following sections.

Assumption 2.1. *We suppose that*

- $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$ *is a bounded domain,* $\partial\Omega = \Gamma_D \cup \Gamma_P$ *and* $\partial\Omega, \Gamma_D, \Gamma_P \in C^{0,1}$, $|\Gamma_D| > 0$*;*
- *For given* $p \in (1,2)$, $\varepsilon \in (0, \varepsilon_0]$ *with* $\varepsilon_0 > 0$ *arbitrary, and* $\gamma_0 \in (0,\infty)$*, Assumptions* **(A1)** – **(A2)** *hold true;*
- *The following data are given:*

$$\begin{aligned} &\boldsymbol{v}_0 \in \mathbf{W}^{1,p}(\Omega), \quad \operatorname{div} \boldsymbol{v}_0 = 0 \textit{ a.e. in } \Omega, \quad \boldsymbol{v}_0 = \boldsymbol{v}_D \textit{ on } \Gamma_D, \\ &\boldsymbol{f} \in \mathbf{L}^{p'}(\Omega) \quad \textit{and} \quad \boldsymbol{b} \in \mathbf{L}^{(p^\#)'}(\Gamma_P), \quad \textit{with } (p^\#)' := \tfrac{(d-1)p}{d(p-1)}. \end{aligned}$$

Here, $p^\# := \frac{(d-1)p}{d-p}$ *is such that* $\operatorname{tr}(\mathrm{W}^{1,p}(\Omega)) \hookrightarrow \mathrm{L}^{p^\#}(\partial\Omega)$.

The weak formulation of the system (1.1)–(1.5) reads:

(pS) *Find* $(\boldsymbol{v}, \pi) \in (\boldsymbol{v}_0 + \boldsymbol{X}^p) \times Q^p$ *(the weak solution) such that*

$$(\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}), \boldsymbol{D}\boldsymbol{w})_\Omega - (\pi, \operatorname{div} \boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega - (\boldsymbol{b}, \boldsymbol{w})_{\Gamma_P} \qquad \forall \boldsymbol{w} \in \boldsymbol{X}^p, \tag{2.14}$$

$$(\operatorname{div} \boldsymbol{v}, q)_\Omega = 0 \qquad \forall q \in Q^p. \tag{2.15}$$

GALERKIN APPROXIMATION. For given $h>0$, let $\boldsymbol{X}_h$, Y_h be finite-dimensional spaces and

$$\boldsymbol{X}_h^p := \boldsymbol{X}_h \cap \boldsymbol{X}^p, \qquad Q_h^p := Y_h \cap Q^p,$$
$$\boldsymbol{V}_h^p := \left\{\boldsymbol{w}_h \in \boldsymbol{X}_h^p;\, (\operatorname{div}\boldsymbol{w}_h, q_h)_\Omega = 0 \text{ for all } q_h \in Q_h^p\right\}.$$

We will specify the spaces in the context of finite elements in Section 5, h will then stand for the mesh parameter. At this stage, we only require that $\boldsymbol{X}_h^p$ and Q_h^p approximate $\boldsymbol{X}^p$ and Q^p in the following sense

$$\lim_{h\searrow 0} \inf_{\boldsymbol{w}_h \in \boldsymbol{X}_h^p} \|\boldsymbol{w}-\boldsymbol{w}_h\|_{1,p} = \lim_{h\searrow 0} \inf_{q_h \in Q_h^p} \|q-q_h\|_{p'} = 0 \qquad \forall \boldsymbol{w} \in \boldsymbol{X}^p, \forall q \in Q^p. \tag{2.16}$$

The pure Galerkin approximation of **(pS)** consists in replacing the Banach spaces $\boldsymbol{X}^p$ and Q^p by their finite dimensional subspaces $\boldsymbol{X}_h^p$ and Q_h^p:

(pS$_h$) *Find* $(\boldsymbol{v}_h, \pi_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{X}_h^p) \times Q_h^p$ *(the discrete solution) such that*

$$(\boldsymbol{S}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\pi_h, \operatorname{div}\boldsymbol{w}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega - (\boldsymbol{b}, \boldsymbol{w}_h)_{\Gamma_P} \qquad \forall \boldsymbol{w}_h \in \boldsymbol{X}_h^p, \tag{2.17}$$
$$(\operatorname{div}\boldsymbol{v}_h, q_h)_\Omega = 0 \qquad \forall q_h \in Q_h^p. \tag{2.18}$$

Here, $\boldsymbol{v}_{0,h}$ *is any*[†] *appropriate approximation of the Dirichlet data which satisfies*

$$(\operatorname{div}\boldsymbol{v}_{0,h}, q_h)_\Omega = 0 \quad \forall q_h \in Q_h^p \qquad \textit{and} \qquad \lim_{h\searrow 0} \|\boldsymbol{v}_0 - \boldsymbol{v}_{0,h}\|_{1,p} = 0. \tag{2.19}$$

INF-SUP CONDITIONS. The following observation plays an essential role in the further analysis.

LEMMA 2.6 Let Assumption 2.1 be satisfied. For any $\nu \in (1,\infty)$ there exists a constant $\beta(\nu)$ (depending on ν, Ω and Γ_P) such that

$$0 < \beta(\nu) \leqslant \inf_{q\in Q^\nu} \sup_{\boldsymbol{w}\in \boldsymbol{X}^\nu} \frac{(q, \operatorname{div}\boldsymbol{w})_\Omega}{\|q\|_{\nu'} \|\boldsymbol{w}\|_{1,\nu}}. \tag{2.20}$$

In particular, there exists a constant $\beta_0(\nu)$ depending on ν and Ω such that

$$0 < \beta_0(\nu) \leqslant \inf_{q\in \mathrm{L}_0^{\nu'}(\Omega)} \sup_{\boldsymbol{w}\in \mathbf{W}_0^{1,\nu}(\Omega)} \frac{(q, \operatorname{div}\boldsymbol{w})_\Omega}{\|q\|_{\nu'} \|\boldsymbol{w}\|_{1,\nu}}. \tag{2.21}$$

If $|\Gamma_P| > 0$ then one possible choice of $\beta(\nu)$ is related to $\beta_0(\nu)$ through (2.22).

Proof. If $|\Gamma_P| = 0$ then $\boldsymbol{X}^\nu = \mathbf{W}_0^{1,\nu}(\Omega)$ and $Q^\nu = \mathrm{L}_0^{\nu'}(\Omega)$. Then, (2.20) and (2.21) are identical, well-known and follow from the properties of the Bogovskii operator, see Remark 2.2.

Let $|\Gamma_P| > 0$. Then, (2.20) can be derived from (2.21), see, e.g., (Haslinger & Stebel, 2010). For $q \in \mathrm{L}^{\nu'}(\Omega)$ arbitrary, we write $q = q_0 + (⨍_\Omega q\,\mathrm{d}\boldsymbol{x})$. Since $q_0 \in \mathrm{L}_0^{\nu'}(\Omega)$, there exists $\boldsymbol{w}_0 \in \mathbf{W}_0^{1,\nu}(\Omega)$, $\|\boldsymbol{w}_0\|_{1,\nu} = 1$ such that $\beta_0(\nu)\|q_0\|_{\nu'} \leqslant (q_0, \operatorname{div}\boldsymbol{w}_0)_\Omega = (q, \operatorname{div}\boldsymbol{w}_0)_\Omega$. Since $\Gamma_P \in C^{0,1}$, $|\Gamma_P| > 0$, there exists some $\boldsymbol{\xi} \in \boldsymbol{X}^\nu$ such that $\int_\Omega \operatorname{div}\boldsymbol{\xi}\,\mathrm{d}\boldsymbol{x} = \int_{\Gamma_P} \boldsymbol{\xi}\cdot\boldsymbol{n}\,\mathrm{d}\boldsymbol{x} = 1$. Taking

$$\boldsymbol{w} := \boldsymbol{w}_0 + \delta\,\mathrm{sign}(⨍_\Omega q\,\mathrm{d}\boldsymbol{x})\,\boldsymbol{\xi}, \qquad \text{with } \delta := \frac{\beta_0(\nu)|\Omega|^{1/\nu'}}{1 + |\Omega|^{1/\nu'}\|\operatorname{div}\boldsymbol{\xi}\|_\nu},$$

[†]For example, $\boldsymbol{v}_{0,h} \in \boldsymbol{X}_h$ is typical in the context of finite elements; but one can also take $\boldsymbol{v}_{0,h} = \boldsymbol{v}_0$.

and using $\|q\|_{\nu'} \leqslant \|q_0\|_{\nu'} + |\Omega|^{1/\nu'} |\fint_\Omega q \,\mathrm{d}\boldsymbol{x}|$, we obtain:

$$\begin{aligned}(q, \operatorname{div} \boldsymbol{w})_\Omega &= (q, \operatorname{div} \boldsymbol{w}_0)_\Omega + \delta \operatorname{sign}(\fint_\Omega q \,\mathrm{d}\boldsymbol{x})\,(q_0, \operatorname{div} \boldsymbol{\xi})_\Omega + \delta |\fint_\Omega q \,\mathrm{d}\boldsymbol{x}|\,(1, \operatorname{div} \boldsymbol{\xi})_\Omega \\ &\geqslant \beta_0(\nu)\|q_0\|_{\nu'} - \delta \|q_0\|_{\nu'} \|\operatorname{div} \boldsymbol{\xi}\|_\nu + \delta |\fint_\Omega q \,\mathrm{d}\boldsymbol{x}| \\ &\geqslant \frac{\beta_0(\nu)}{1 + |\Omega|^{1/\nu'} \|\operatorname{div} \boldsymbol{\xi}\|_\nu} \|q\|_{\nu'}.\end{aligned}$$

Also, $\boldsymbol{w} \in \boldsymbol{X}^\nu$, and $\|\boldsymbol{w}\|_{1,\nu} \leqslant 1 + \delta \|\boldsymbol{\xi}\|_{1,\nu}$, which finally gives (2.20) with

$$\beta(\nu) = \frac{\beta_0(\nu)}{1 + |\Omega|^{1/\nu'} \|\operatorname{div} \boldsymbol{\xi}\|_\nu + \beta_0(\nu) |\Omega|^{1/\nu'} \|\boldsymbol{\xi}\|_{1,\nu}}. \tag{2.22}$$

This completes the proof. □

REMARK 2.2 There exists a continuous linear operator $\mathscr{B} : \mathrm{L}^\nu_0(\Omega) \to \mathbf{W}^{1,\nu}_0(\Omega)$ referred to as the Bogovskii operator, such that $\operatorname{div}(\mathscr{B} f) = f$ in Ω and $\|\mathscr{B} f\|_{1,\nu} \leqslant C_{\mathrm{div}}(\Omega, \nu)\|f\|_\nu$, see (Bogovskii, 1980; Amrouche & Girault, 1994; Novotný & Straškraba, 2004). In the preceding studies, see (Franta *et al.*, 2005; Lanzendörfer, 2009), the Bogovskii operator was applied directly instead the inf–sup condition. For $|\Gamma_P| = 0$, one observes $C_{\mathrm{div}}(\Omega, 2) \geqslant \beta_0(2)^{-1}$.

For $|\Gamma_P| > 0$, the modified operator $\widetilde{\mathscr{B}} f := \mathscr{B}(f - (\int_\Omega f \,\mathrm{d}\boldsymbol{x}) \operatorname{div} \boldsymbol{\xi}) + (\int_\Omega f \,\mathrm{d}\boldsymbol{x})\boldsymbol{\xi}$ was utilized, see (Lanzendörfer & Stebel, 2008, Lemma 2.4). Note from (2.22) that the corresponding constant $\widetilde{C}_{\mathrm{div}}(\Omega, \Gamma_P, \nu)$ (see ibid.) equals $\beta(\nu)^{-1}$.

REMARK 2.3 Lemma 2.6 reveals, in terms of the spaces $\boldsymbol{X}^p$, Q^p, why the additional constraint (1.5) is requisite to fix the level of pressure if and only if the boundary condition (1.4) is not present.

Below, we require for given $\nu \in (1, \infty)$ that the families of spaces $\{\boldsymbol{X}^\nu_h\}_{h>0}$, $\{Q^\nu_h\}_{h>0}$ satisfy the discrete inf–sup condition:

(IS$^\nu$) *For given $\nu \in (1, \infty)$, there exists a constant $\tilde{\beta}(\nu)$ independent of h such that*

$$0 < \tilde{\beta}(\nu) \leqslant \inf_{q \in Q^\nu_h} \sup_{\boldsymbol{w} \in \boldsymbol{X}^\nu_h} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|q\|_{\nu'} \|\boldsymbol{w}\|_{1,\nu}}.$$

The availability of **(IS$^\nu$)** (and the value of $\tilde{\beta}(\nu)$) depends on the choice of the spaces $\boldsymbol{X}_h$ and Y_h. For the purposes of Theorem 3.3, we also require the following modification of **(IS$^\nu$)**.

(IS$^\nu_0$) *There exists a constant $\tilde{\beta}_0(\nu)$, independent of h, such that*

$$0 < \tilde{\beta}_0(\nu) \leqslant \inf_{q \in Y_h \cap \mathrm{L}^{\nu'}_0(\Omega)} \sup_{\boldsymbol{w} \in \boldsymbol{X}_h \cap \mathbf{W}^{1,\nu}_0(\Omega)} \frac{(q, \operatorname{div} \boldsymbol{w})_\Omega}{\|q\|_{\nu'} \|\boldsymbol{w}\|_{1,\nu}}.$$

REMARK 2.4 If $|\Gamma_P| = 0$, then **(IS$^\nu_0$)** is exactly **(IS$^\nu$)**. In general, **(IS$^\nu_0$)** need not be implied by **(IS$^\nu$)** and vice versa. Let us suppose for a while that both conditions hold true. Since (2.22) in Lemma 2.6 indicates[‡] $\beta_0(\nu) \geqslant \beta(\nu)$ on the continuous level, we can expect $\tilde{\beta}_0(\nu) \geqslant \tilde{\beta}(\nu)$ for typical choices of $\boldsymbol{X}_h, Y_h$. In such a case, the additional requirement of **(IS$^\nu_0$)** will guarantee convergence results for a larger range of γ_0, see (3.7) in Theorem 3.3 and (3.18) in Corollary 3.1.

[‡]We did not prove $\beta_0(\nu) \geqslant \beta(\nu)$; (2.22) merely gives a lower bound for $\beta(\nu)$ which is lower than $\beta_0(\nu)$.

In the sequel we will use $(\mathbf{IS}_0^2)$ in conjunction with the following observation:

REMARK 2.5 Let $(\mathbf{IS}_0^2)$ hold, let $|\Gamma_P| > 0$ and $p \in (1,2)$. For arbitrary $q \in Q_h^p$, we write $q = q_0 + \fint_\Omega q \,\mathrm{d}\boldsymbol{x}$, where[§] $q_0 \in Y_h \cap \mathrm{L}_0^2(\Omega)$. Since $\|q\|_2 \leqslant \|q_0\|_2 + |\Omega|^{1/2}|\fint_\Omega q\,\mathrm{d}\boldsymbol{x}|$, we obtain

$$\tilde{\beta}_0(2)\left(\|q\|_2 - |\Omega|^{1/2}|\fint_\Omega q\,\mathrm{d}\boldsymbol{x}|\right) \leqslant \sup_{\boldsymbol{w}\in\boldsymbol{X}_h^2} \frac{(q,\operatorname{div}\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,2}}, \qquad \forall q \in Q_h^p. \tag{2.23}$$

3. Well-posedness of the problem

In the following we show the existence of discrete solutions to $(\mathbf{pS}_h)$, we discuss the conditions guaranteeing the uniqueness of solutions both to $(\mathbf{pS}_h)$ and **(pS)**, and we finally establish the existence of a weak solution to **(pS)** as the limit of the discrete solutions.

Note that the well-posedness of **(pS)** with a convective term included has already been resolved: For $\Gamma_D = \partial\Omega$ this was published in (Franta *et al.*, 2005; Lanzendörfer, 2009), while the case $|\Gamma_P| > 0$ was conducted in (Lanzendörfer & Stebel, 2009). In these works, the proof was done in a different way than here: First a quasi-compressible approximation to **(pS)** was established (by the Galerkin method), and later it was shown that this approximation converges (on the continuous level) to the "incompressible" solution to **(pS)**. Here, since our concern lies with the finite element discretization, the weak solution is established directly as a limit of discrete solutions, where the discrete solutions satisfy the (discrete) incompressibility constraint (2.18). Many of the estimates used here will be employed also in the next section. Compared to the previous studies, we slightly relax the restriction on γ_0 and—since we neglect convection—our procedure allows for $p \in (1,2)$. We begin with the well-posedness of $(\mathbf{pS}_h)$:

THEOREM 3.1 (Existence of discrete solutions) Let Assumption 2.1 hold. Let $\boldsymbol{X}_h^p$ and Q_h^p fulfil $(\mathbf{IS}^p)$ with $\tilde{\beta}(p) > 0$ arbitrary.

Then there exists a discrete solution to $(\mathbf{pS}_h)$. Moreover, any such solution $(\boldsymbol{v}_h, \pi_h)$ satisfies the a priori estimate

$$\|\boldsymbol{v}_h\|_{1,p} + \|\boldsymbol{S}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h)\|_{p'} + \tilde{\beta}(p)\|\pi_h\|_{p'} \leqslant K, \tag{3.1}$$

with K depending only on $\Omega, \Gamma_D, p, \varepsilon_0, \sigma_0, \sigma_1, \|\boldsymbol{f}\|_{p'}, \|\boldsymbol{b}\|_{(p^\#)';\Gamma_P}$ and $\|\boldsymbol{v}_{0,h}\|_{1,p}$.

Proof. For any $\delta > 0$ (small), we consider the quasi-compressible problem $(\mathbf{pS}_h^\delta)$:

Find $(\boldsymbol{v}_h^\delta, \pi_h^\delta) \in (\boldsymbol{v}_{0,h} + \boldsymbol{X}_h^p) \times Q_h^p$ such that

$$(\boldsymbol{S}(\pi_h^\delta, \boldsymbol{D}\boldsymbol{v}_h^\delta), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\pi_h^\delta, \operatorname{div}\boldsymbol{w}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega - (\boldsymbol{b}, \boldsymbol{w}_h)_{\Gamma_P} \qquad \forall \boldsymbol{w}_h \in \boldsymbol{X}_h^p, \tag{3.2}$$

$$\delta(\pi_h^\delta, q_h)_\Omega + (q_h, \operatorname{div}\boldsymbol{v}_h^\delta)_\Omega = 0 \qquad \forall q_h \in Q_h^p. \tag{3.3}$$

The inserted term $\delta(\pi_h^\delta, q_h)_\Omega$ ensures the coercivity of the equations with respect to the pressure and allows to use the Brouwer fixed-point theorem to establish the solution to $(\mathbf{pS}_h^\delta)$. Indeed, setting $\boldsymbol{w}_h := \boldsymbol{v}_h^\delta - \boldsymbol{v}_{0,h}$ and $q_h := \pi_h^\delta$, summing the equations and using Hölder's and Korn's inequality, $(2.19)_1$, the embedding $\operatorname{tr}(\mathbf{W}^{1,p}(\Omega)) \hookrightarrow \mathbf{L}^{p^\#}(\partial\Omega)$, the estimate

$$(\boldsymbol{S}(\pi_h^\delta, \boldsymbol{D}\boldsymbol{v}_h^\delta), \boldsymbol{D}\boldsymbol{v}_h^\delta - \boldsymbol{D}\boldsymbol{v}_{0,h})_\Omega \geqslant \frac{\sigma_0}{2p}\|\boldsymbol{D}\boldsymbol{v}_h^\delta\|_p^p - \frac{\sigma_1}{p-1}\|\boldsymbol{D}\boldsymbol{v}_h^\delta\|_p^{p-1}\|\boldsymbol{D}\boldsymbol{v}_{0,h}\|_p - \frac{\sigma_0}{2p}|\Omega|\varepsilon^p$$

[§]Here we assume that constants belong to Y_h.

due to (2.6), and Young's inequality, we obtain the a priori bound

$$\delta\|\pi_h^\delta\|_2^2+\|\boldsymbol{v}_h^\delta\|_{1,p}^p+\|\boldsymbol{S}(\pi_h^\delta,\boldsymbol{D}\boldsymbol{v}_h^\delta)\|_{p'}^{p'}\leqslant C,$$

where $C>0$ depends on $\Omega,\Gamma_D,p,\varepsilon_0,\sigma_0,\sigma_1,\|\boldsymbol{f}\|_{p'},\|\boldsymbol{b}\|_{(p^\#)';\Gamma_P}$ and $\|\boldsymbol{v}_{0,h}\|_{1,p}$. In particular, C is independent of δ and h. Therefore, using **(IS^p)** and (2.14), we observe that

$$\tilde{\beta}(p)\|\pi_h^\delta\|_{p'}\leqslant\sup_{\boldsymbol{w}_h\in\boldsymbol{X}_h^p}\frac{(\pi_h^\delta,\operatorname{div}\boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,p}}\leqslant C,$$

with $C>0$ and $\tilde{\beta}(p)>0$ independent of δ and h. The same arguments applied to **(pS$_h$)** prove (3.1).

The uniform bounds above and the fact that $\boldsymbol{X}_h^p$ and Q_h^p are of finite dimension imply that there is $(\boldsymbol{v}_h,\pi_h)\in(\boldsymbol{v}_{0,h}+\boldsymbol{X}_h^p)\times Q_h^p$ such that (for some sequence $\delta_n\searrow 0$)

$$\begin{aligned}
\boldsymbol{v}_h^{\delta_n}&\to\boldsymbol{v}_h && \text{in } \mathbf{W}^{1,p}(\Omega),\\
\pi_h^{\delta_n}&\to\pi_h && \text{in } \mathrm{L}^{p'}(\Omega),\\
\boldsymbol{S}(\pi_h^{\delta_n},\boldsymbol{D}\boldsymbol{v}_h^{\delta_n})&\to\boldsymbol{S}(\pi_h,\boldsymbol{D}\boldsymbol{v}_h) && \text{in } \mathrm{L}^{p'}(\Omega)^{d\times d}.
\end{aligned}$$

Consequently, $(\boldsymbol{v}_h,\pi_h)$ is a solution to **(pS$_h$)**. □

According to Theorem 3.1, discrete solutions exist regardless of Assumption **(A2)**. However, uniqueness of the solution can only be shown by means of **(A2)** under a smallness assumption on γ_0 as depicted by the following theorem:

THEOREM 3.2 (Uniqueness) Let the assumptions of Theorem 3.1 hold. Provided that **(IS^2)** is satisfied and

$$\gamma_0<\tilde{\beta}(2)\varepsilon^{\frac{2-p}{2}}\frac{\sigma_0}{\sigma_0+\sigma_1}, \tag{3.4}$$

the solution to **(pS$_h$)** is determined uniquely.

Similarly, there is at most one solution to **(pS)** if Assumption 2.1 is satisfied and

$$\gamma_0<\beta(2)\varepsilon^{\frac{2-p}{2}}\frac{\sigma_0}{\sigma_0+\sigma_1}.$$

Proof. We prove the uniqueness to **(pS$_h$)**, the other result is analogous. Let $(\boldsymbol{v}_h^i,\pi_h^i)$, $i=1,2$, be two solutions to **(pS$_h$)**. Then

$$(\boldsymbol{S}(\pi_h^1,\boldsymbol{D}\boldsymbol{v}_h^1)-\boldsymbol{S}(\pi_h^2,\boldsymbol{D}\boldsymbol{v}_h^2),\boldsymbol{D}\boldsymbol{w}_h)_\Omega=(\pi_h^1-\pi_h^2,\operatorname{div}\boldsymbol{w}_h)_\Omega\quad\forall\boldsymbol{w}_h\in\boldsymbol{X}_h^p.$$

In particular, choosing $\boldsymbol{w}_h:=\boldsymbol{v}_h^1-\boldsymbol{v}_h^2$ we observe

$$(\boldsymbol{S}(\pi_h^1,\boldsymbol{D}\boldsymbol{v}_h^1)-\boldsymbol{S}(\pi_h^2,\boldsymbol{D}\boldsymbol{v}_h^2),\boldsymbol{D}\boldsymbol{v}_h^1-\boldsymbol{D}\boldsymbol{v}_h^2)_\Omega=0$$

and we thus obtain from (2.8) that

$$d(\boldsymbol{v}_h^1,\boldsymbol{v}_h^2)^2\leqslant\frac{\gamma_0^2}{\sigma_0^2}\|\pi_h^1-\pi_h^2\|_2^2. \tag{3.5}$$

Hence, **(IS^2)** and (2.10) yields:

$$\begin{aligned}\tilde{\beta}(2)\|\pi_h^1-\pi_h^2\|_2 &\leqslant \sup_{\boldsymbol{w}_h\in\boldsymbol{X}_h^2}\frac{(\pi_h^1-\pi_h^2,\operatorname{div}\boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,2}}\\ &\leqslant \|\boldsymbol{S}(\pi_h^1,\boldsymbol{D}\boldsymbol{v}_h^1)-\boldsymbol{S}(\pi_h^2,\boldsymbol{D}\boldsymbol{v}_h^2)\|_2\\ &\leqslant \sigma_1\varepsilon^{\frac{p-2}{2}}d(\boldsymbol{v}_h^1,\boldsymbol{v}_h^2)+\gamma_0\varepsilon^{\frac{p-2}{2}}\|\pi_h^1-\pi_h^2\|_2,\end{aligned}\tag{3.6}$$

which together with (3.5) and (3.4) leads to $\pi_h^1=\pi_h^2$ a.e. in Ω and to $d(\boldsymbol{v}_h^1,\boldsymbol{v}_h^2)=0$. But this completes the proof, because (2.12), (2.13) and the a priori bound (3.1) ensure that $\|\boldsymbol{D}\boldsymbol{v}_h^1-\boldsymbol{D}\boldsymbol{v}_h^2\|_p^2\leqslant C\,d(\boldsymbol{v}_h^1,\boldsymbol{v}_h^2)^2=0$. Since $|\Gamma_D|>0$, Lemma 2.5 yields $\boldsymbol{v}_h^1=\boldsymbol{v}_h^2$ a.e. in Ω. □

THEOREM 3.3 (Convergence of discrete solutions) Let the assumptions of Theorem 3.1 hold, let the discrete spaces $\{(\boldsymbol{X}_h^p,Q_h^p)\}_{h>0}$ satisfy (2.16), and let $\{\boldsymbol{v}_{0,h}\}_{h>0}$ satisfy (2.19). In addition, let **(IS$_0^2$)** hold and let γ_0 fulfill

$$\gamma_0<\tilde{\beta}_0(2)\varepsilon^{\frac{2-p}{2}}\frac{\sigma_0}{\sigma_0+\sigma_1}.\tag{3.7}$$

Then, the discrete solutions to **(pS$_h$)** converge to a weak solution to **(pS)** as follows,

$$(\boldsymbol{v}_{h_n},\pi_{h_n})\to(\boldsymbol{v},\pi)\quad\text{strongly in }\mathbf{W}^{1,p}(\Omega)\times\mathrm{L}^{p'}(\Omega),\quad\text{for some }h_n\searrow 0.\tag{3.8}$$

In addition, if the weak solution to **(pS)** is unique, then the whole sequence $\{(\boldsymbol{v}_h,\pi_h)\}_{h>0}$ tends to $(\boldsymbol{v},\pi)$.

REMARK 3.1 Note that $\tilde{\beta}_0(2)$ appears in (3.7) even in the case $|\Gamma_P|>0$. In general, this guarantees convergence for larger range of γ_0 than, e.g., compared to (3.4), see Remark 2.4.

Proof of Theorem 3.3.

Theorem 3.1 ensures that discrete solutions $(\boldsymbol{v}_h,\pi_h)\in(\boldsymbol{v}_{0,h}+\boldsymbol{X}_h^p)\times Q_h^p$ to **(pS$_h$)** exist and satisfy the a priori estimate (3.1). Hence, there exist $(\boldsymbol{v},\pi)\in(\boldsymbol{v}_0+\boldsymbol{X}^p,Q^p)$ and $\overline{\boldsymbol{S}}\in\mathrm{L}^{p'}(\Omega)^{d\times d}$ such that for a sequence $h_n\searrow 0$ there holds

$$\boldsymbol{v}_{h_n}\rightharpoonup\boldsymbol{v}\qquad\text{weakly in }\mathbf{W}^{1,p}(\Omega),\tag{3.9}$$

$$\pi_{h_n}\rightharpoonup\pi\qquad\text{weakly in }\mathrm{L}^{p'}(\Omega),\tag{3.10}$$

$$\boldsymbol{S}(\pi_{h_n},\boldsymbol{D}\boldsymbol{v}_{h_n})\rightharpoonup\overline{\boldsymbol{S}}\qquad\text{weakly in }\mathrm{L}^{p'}(\Omega)^{d\times d}.\tag{3.11}$$

Obviously, the weak limits satisfy equation (2.15) and

$$(\overline{\boldsymbol{S}},\boldsymbol{D}\boldsymbol{w})_\Omega-(\pi,\operatorname{div}\boldsymbol{w})_\Omega=(\boldsymbol{f},\boldsymbol{w})_\Omega-(\boldsymbol{b},\boldsymbol{w})_{\Gamma_P}\qquad\forall\boldsymbol{w}\in\boldsymbol{X}^p.\tag{3.12}$$

Here, we have used the density (2.16). Subtracting (3.12) and (2.17), we observe

$$(\boldsymbol{S}(\pi_{h_n},\boldsymbol{D}\boldsymbol{v}_{h_n})-\overline{\boldsymbol{S}},\boldsymbol{D}\boldsymbol{w}_{h_n})_\Omega=(\pi_{h_n}-\pi,\operatorname{div}\boldsymbol{w}_{h_n})_\Omega\qquad\forall\boldsymbol{w}_{h_n}\in\boldsymbol{X}_{h_n}^p.\tag{3.13}$$

Then, (3.13) with $\boldsymbol{w}_h:=\boldsymbol{v}_{h_n}-\boldsymbol{v}_{0,h_n}$ implies

$$\begin{aligned}&(\boldsymbol{S}(\pi_{h_n},\boldsymbol{D}\boldsymbol{v}_{h_n})-\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v}),\boldsymbol{D}\boldsymbol{v}_{h_n}-\boldsymbol{D}\boldsymbol{v})_\Omega=(\pi_{h_n}-\pi,\operatorname{div}(\boldsymbol{v}_{h_n}-\boldsymbol{v}_{0,h_n}))_\Omega\\ &\quad+(\overline{\boldsymbol{S}},\boldsymbol{D}\boldsymbol{v}_{h_n}-\boldsymbol{D}\boldsymbol{v}_{0,h_n})_\Omega+(\boldsymbol{S}(\pi_{h_n},\boldsymbol{D}\boldsymbol{v}_{h_n}),\boldsymbol{D}\boldsymbol{v}_{0,h_n}-\boldsymbol{D}\boldsymbol{v})_\Omega-(\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v}),\boldsymbol{D}\boldsymbol{v}_{h_n}-\boldsymbol{D}\boldsymbol{v})_\Omega.\end{aligned}$$

Using (2.19), (2.18), (2.15), and recalling (3.9)–(3.11), we conclude that

$$(\boldsymbol{S}(\pi_{h_n},\boldsymbol{D}\boldsymbol{v}_{h_n})-\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v}),\boldsymbol{D}\boldsymbol{v}_{h_n}-\boldsymbol{D}\boldsymbol{v})_\Omega=o(1),\qquad h_n\searrow 0,\tag{3.14}$$

where $o(1)$ denotes an arbitrary sequence that tends to zero for $h_n\searrow 0$. Furthermore, from (2.13), (3.1), (2.8), and (3.14) we deduce (cf. (3.5))

$$C\,\|\boldsymbol{D}\boldsymbol{v}_{h_n}-\boldsymbol{D}\boldsymbol{v}\|_p^2\leqslant d(\boldsymbol{v}_{h_n},\boldsymbol{v})^2\leqslant\frac{\gamma_0^2}{\sigma_0^2}\|\pi_{h_n}-\pi\|_2^2+o(1)\tag{3.15}$$

for some $C>0$ independent of h_n. We suppose for a while that

$$\tilde\beta_0(2)\|\pi_{h_n}-\pi\|_2\leqslant\|\boldsymbol{S}(\pi_{h_n},\boldsymbol{D}\boldsymbol{v}_{h_n})-\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v})\|_2+o(1).\tag{3.16}$$

Then, combining (3.16) and (2.10), we arrive at

$$\tilde\beta_0(2)\|\pi_{h_n}-\pi\|_2\leqslant\sigma_1\varepsilon^{\frac{p-2}{2}}d(\boldsymbol{v}_{h_n},\boldsymbol{v})+\gamma_0\varepsilon^{\frac{p-2}{2}}\|\pi_{h_n}-\pi\|_2+o(1),\qquad h_n\searrow 0.$$

Using (3.15) and the assumption (3.7), we conclude $\|\pi_{h_n}-\pi\|_2\leqslant o(1)$. Consequently, (3.15) also yields $\|\boldsymbol{D}\boldsymbol{v}_{h_n}-\boldsymbol{D}\boldsymbol{v}\|_p\leqslant o(1)$, which finally implies that

$$\pi_{h_n}\to\pi\text{ a.e. in }\Omega\quad\text{and}\quad\boldsymbol{D}\boldsymbol{v}_{h_n}\to\boldsymbol{D}\boldsymbol{v}\text{ a.e. in }\Omega.$$

This allows us to apply the Vitali's lemma and to identify $\overline{\boldsymbol{S}}$,

$$\int_\Omega\boldsymbol{S}(\pi_{h_n},\boldsymbol{D}\boldsymbol{v}_{h_n}):\boldsymbol{D}\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}\to\int_\Omega\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v}):\boldsymbol{D}\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}=\int_\Omega\overline{\boldsymbol{S}}:\boldsymbol{D}\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}\qquad\forall\boldsymbol{w}\in\boldsymbol{X}^p.$$

Therefore, it only remains to show (3.16). Define $\tilde{\boldsymbol{w}}_{h_n}\in\boldsymbol{X}^2_{h_n}$, $\|\tilde{\boldsymbol{w}}_{h_n}\|_{1,2}=1$, such that

$$\sup_{\boldsymbol{w}_{h_n}\in\boldsymbol{X}^2_{h_n}}\frac{(\pi_{h_n}-\pi,\operatorname{div}\boldsymbol{w}_{h_n})_\Omega}{\|\boldsymbol{w}_{h_n}\|_{1,2}}=(\pi_{h_n}-\pi,\operatorname{div}\tilde{\boldsymbol{w}}_{h_n})_\Omega.$$

Then, there exists $\tilde{\boldsymbol{w}}\in\boldsymbol{X}^2$ such that (for a not-relabelled subsequence) $\tilde{\boldsymbol{w}}_{h_n}-\tilde{\boldsymbol{w}}\rightharpoonup 0$ weakly in $\boldsymbol{X}^2$ and¶ $\|\tilde{\boldsymbol{w}}_{h_n}-\tilde{\boldsymbol{w}}\|_{1,2}\leqslant 1$. Hence, using (3.13) and (3.11) we obtain:

$$\begin{aligned}(\pi_{h_n}-\pi,\operatorname{div}\tilde{\boldsymbol{w}}_{h_n})_\Omega&=(\boldsymbol{S}(\pi_{h_n},\boldsymbol{D}\boldsymbol{v}_{h_n})-\overline{\boldsymbol{S}},\boldsymbol{D}\tilde{\boldsymbol{w}}_{h_n}-\boldsymbol{D}\tilde{\boldsymbol{w}})_\Omega+o(1)\\&=(\boldsymbol{S}(\pi_{h_n},\boldsymbol{D}\boldsymbol{v}_{h_n})-\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v}),\boldsymbol{D}\tilde{\boldsymbol{w}}_{h_n}-\boldsymbol{D}\tilde{\boldsymbol{w}})_\Omega+o(1)\\&\leqslant\|\boldsymbol{S}(\pi_{h_n},\boldsymbol{D}\boldsymbol{v}_{h_n})-\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v})\|_2+o(1),\qquad h_n\searrow 0.\end{aligned}$$

Further, recalling (2.23) and using that $⨍_\Omega\pi_{h_n}-\pi\,\mathrm{d}\boldsymbol{x}\to 0$, we have for any $q_{h_n}\in Q^p_{h_n}$ that

$$\begin{aligned}\tilde\beta_0(2)\|\pi_{h_n}-q_{h_n}\|_2&\leqslant\sup_{\boldsymbol{w}_{h_n}\in\boldsymbol{X}^2_{h_n}}\frac{(\pi_{h_n}-q_{h_n},\operatorname{div}\boldsymbol{w}_{h_n})_\Omega}{\|\boldsymbol{w}_{h_n}\|_{1,2}}+\tilde\beta_0(2)|\Omega|^{1/2}\left|⨍_\Omega\pi_{h_n}-q_{h_n}\,\mathrm{d}\boldsymbol{x}\right|\\&\leqslant\sup_{\boldsymbol{w}_{h_n}\in\boldsymbol{X}^2_{h_n}}\frac{(\pi_{h_n}-\pi,\operatorname{div}\boldsymbol{w}_{h_n})_\Omega}{\|\boldsymbol{w}_{h_n}\|_{1,2}}+\|\pi-q_{h_n}\|_2+C\left|⨍_\Omega\pi_{h_n}-q_{h_n}\,\mathrm{d}\boldsymbol{x}\right|\\&\leqslant\|\boldsymbol{S}(\pi_{h_n},\boldsymbol{D}\boldsymbol{v}_{h_n})-\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v})\|_2+C\,\|\pi-q_{h_n}\|_2+o(1),\qquad h_n\searrow 0,\end{aligned}$$

¶Indeed, $\|\tilde{\boldsymbol{w}}\|_{1,2}^2\leqslant 2(\tilde{\boldsymbol{w}}_{h_n},\tilde{\boldsymbol{w}})_{1,2;\Omega}$ for n large enough, which implies $\|\tilde{\boldsymbol{w}}_{h_n}-\tilde{\boldsymbol{w}}\|_{1,2}^2\leqslant\|\tilde{\boldsymbol{w}}_{h_n}\|_{1,2}^2\ (=1)$.

with $C > 0$ independent of h_n. Using the density of $\{Q^p_{h_n}\}$ in Q^p, we finally assert (3.16):

$$\begin{aligned}\tilde{\beta}_0(2)\|\pi_{h_n}-\pi\|_2 &\leqslant \tilde{\beta}_0(2)\inf_{q_{h_n}\in Q^p_{h_n}}\{\|\pi_{h_n}-q_{h_n}\|_2+\|q_{h_n}-\pi\|_2\}\\ &\leqslant \|\boldsymbol{S}(\pi_{h_n},\boldsymbol{D}\boldsymbol{v}_{h_n})-\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v})\|_2+o(1), \qquad h_n\searrow 0.\end{aligned}$$

This completes the proof. □

Theorem 3.3 guarantees existence of weak solutions to **(pS)** provided that we have a suitable family of discrete spaces $\{\boldsymbol{X}^p_h, Q^p_h\}_{h>0}$. The proper existence result is formulated in Corollary 3.1. In the following lemma we construct such a family of discrete spaces, which satisfies **(IS^p)** and **(IS$^2_0$)** with the constant $\tilde{\beta}_0(2)$ almost equal to $\beta_0(2)$.

LEMMA 3.1 Let Ω, Γ_D, Γ_P and p be as in Assumption 2.1. Then for any $\delta > 0$ (small), there exists a family of finite-dimensional spaces $\{\boldsymbol{X}_{h_n}\}$, $\{Y_{h_n}\}$, $h_n \searrow 0$ that satisfy (2.16) and fulfill **(IS^p)** and **(IS$^2_0$)** with

$$\tilde{\beta}(p) \geqslant \beta(p)-\delta \qquad \text{and} \qquad \tilde{\beta}_0(2)\geqslant \beta_0(2)-\delta. \tag{3.17}$$

Proof. Consider arbitrary $h_n \searrow 0$, $n = 1,\ldots$. Since $\mathbf{W}^{1,2}_0(\Omega)$, $\boldsymbol{X}^p$, Q^p are separable Banach spaces with the bases $\{\bar{\boldsymbol{w}}_n\}_{n=1}^\infty$, $\{\boldsymbol{w}_n\}_{n=1}^\infty$, $\{q_n\}_{n=1}^\infty$, respectively, and since $\mathbf{W}^{1,2}_0(\Omega)\subset \boldsymbol{X}^p$, we can define the Galerkin spaces by $\boldsymbol{X}^m := \operatorname{span}\{\bar{\boldsymbol{w}}_i, \boldsymbol{w}_i\}_{i=1}^m$ and $Y^n := \operatorname{span}\{q_i\}_{i=1}^n$, clearly allowing for (2.16). In order to ensure (3.17), we only need to choose suitable pairs of the spaces, i.e., to any discrete pressure space we have to assign a rich enough discrete velocity space. We show this only for **(IS$^2_0$)** and $(3.17)_2$, the inclusion of **(IS^p)** is obvious.

Due to (2.16) and Lemma 2.6, for any $q \in \mathrm{L}^2_0(\Omega)$ there exists $k(q)$ such that

$$\beta_0(2)-\delta \;\leqslant \sup_{\boldsymbol{w}\in \boldsymbol{X}^{k(q)}\cap \mathbf{W}^{1,2}_0(\Omega)}\frac{(q,\operatorname{div}\boldsymbol{w})_\Omega}{\|q\|_2\|\boldsymbol{w}\|_{1,2}}.$$

(we choose minimal such $k(q)$). For n fixed, define $m(n) := \sup_{\{q\in Y^n\cap \mathrm{L}^2_0(\Omega)\}} k(q)$. It is easy to see that $Y_{h_n} := Y^n$ and $\boldsymbol{X}_{h_n} := \boldsymbol{X}^{m(n)}$ satisfy **(IS$^2_0$)** and (3.17). It remains to prove that $m(n)$ is finite. This is shown by contradiction: Let $m(n)$ be infinite. Then we find a sequence $q_j \in Y^n \cap \mathrm{L}^2_0(\Omega)$, $\|q_j\|_2 = 1$, $j = 1,2,\ldots$, such that $k(q_j) > j$ and

$$\sup_{\boldsymbol{w}\in \boldsymbol{X}^{j}\cap \mathbf{W}^{1,2}_0(\Omega)}\frac{(q_j,\operatorname{div}\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,2}} \;<\; \beta_0(2)-\delta.$$

Since Y^n is of finite dimension, we find some $\tilde{q}\in Y^n\cap \mathrm{L}^2_0(\Omega)$, $\|\tilde{q}\|_2 = 1$, and a subsequence $j_i > i$ such that $\|q_{j_i}-\tilde{q}\|_2 < \delta/2$ for $i = 1,2,\ldots$. But then,

$$\sup_{\boldsymbol{w}\in \boldsymbol{X}^{i}\cap \mathbf{W}^{1,2}_0(\Omega)}\frac{(\tilde{q},\operatorname{div}\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,2}} \;<\; \beta_0(2)-\delta/2$$

holds for any $i = 1,2,\ldots$, which combined with the density (2.16) and Lemma 2.6 gives the contradiction. □

COROLLARY 3.1 (Existence of solutions) Let Assumption 2.1 hold and

$$\gamma_0 < \beta_0(2)\varepsilon^{\frac{2-p}{2}}\frac{\sigma_0}{\sigma_0+\sigma_1}. \tag{3.18}$$

Then there exists a weak solution to **(pS)**. Moreover, any solution to **(pS)** fulfils the a priori estimate

$$\|\boldsymbol{v}\|_{1,p} + \|\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v})\|_{p'} + \beta(p)\|\pi\|_{p'} \leqslant K, \tag{3.19}$$

with K depending only on $\Omega, \Gamma_D, p, \varepsilon, \sigma_0, \sigma_1, \|\boldsymbol{f}\|_{p'}, \|\boldsymbol{b}\|_{(p^\#)';\Gamma_P}$ and $\|\boldsymbol{v}_0\|_{1,p}$.

Proof. The a priori estimate (3.19) follows by the procedure analogous to the proof of (3.1). The existence result follows from Theorems 3.1 and 3.3, and Lemma 3.1. □

4. A priori error estimates

In this section we aim to derive a priori estimates for the error of approximation $\boldsymbol{v}-\boldsymbol{v}_h$ and $\pi-\pi_h$. For the remainder of this paper, let us use the convention that $(\boldsymbol{v},\pi)$ and $(\boldsymbol{v}_h,\pi_h)$ denotes the solution to **(pS)** and **(pS$_h$)**, respectively, whose existence and uniqueness was shown in the previous section. The main results are given by Corollaries 4.1 and 4.2 which state a priori error estimates in the form of a best approximation result.

LEMMA 4.1 Let Assumption 2.1 hold. For each $\delta > 0$ there exists a constant $c_\delta > 0$ such that for all $\boldsymbol{u}_h \in (\boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p)$ and $r_h \in Q_h^p$ there holds

$$d(\boldsymbol{v},\boldsymbol{v}_h) \leqslant c_\delta\big(\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p + \|\pi - r_h\|_{p'}\big) + \Big(\frac{\gamma_0}{\sigma_0} + \delta\Big)\|\pi - \pi_h\|_2,$$

where the constant c_δ also depends on p, ε_0, γ_0, σ_0, σ_1, Γ_D and Ω.

Proof. Let $(\boldsymbol{u}_h, r_h)$ be an arbitrary element of $(\boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p) \times Q_h^p$. From **(pS)** and **(pS$_h$)** it follows that

$$(\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(\pi_h,\boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega = (\pi - \pi_h, \operatorname{div}\boldsymbol{w}_h)_\Omega = (\pi - r_h, \operatorname{div}\boldsymbol{w}_h)_\Omega$$

for all $\boldsymbol{w}_h \in \boldsymbol{V}_h^p$. This, with $\boldsymbol{w}_h := (\boldsymbol{u}_h - \boldsymbol{v}_h) \in \boldsymbol{V}_h^p$, implies

$$\begin{aligned}(\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(\pi_h,\boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}_h)_\Omega &= (\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(\pi_h,\boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h)_\Omega \\ &\quad + (\pi - r_h, \operatorname{div}(\boldsymbol{u}_h - \boldsymbol{v}_h))_\Omega =: I_1 + I_2\end{aligned}$$

Applying (2.8), we conclude

$$\frac{\sigma_0}{2} d(\boldsymbol{v},\boldsymbol{v}_h)^2 \leqslant I_1 + I_2 + \frac{\gamma_0^2}{2\sigma_0}\|\pi - \pi_h\|_2^2. \tag{4.1}$$

It remains to estimate I_1 and I_2. First, we split the term I_1 in the following way,

$$\begin{aligned}I_1 &= (\boldsymbol{S}(\pi,\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(\pi_h,\boldsymbol{D}\boldsymbol{u}_h), \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h)_\Omega \\ &\quad + (\boldsymbol{S}(\pi_h,\boldsymbol{D}\boldsymbol{u}_h) - \boldsymbol{S}(\pi_h,\boldsymbol{D}\boldsymbol{v}_h), \boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h)_\Omega =: I_3 + I_4.\end{aligned}$$

Due to (2.9) and Lemma 2.3, for each $\delta_1 > 0$ there exists $c_{\delta_1} > 0$ such that

$$I_3 \leqslant c_{\delta_1} d(\boldsymbol{v},\boldsymbol{u}_h)^2 + \delta_1\|\pi - \pi_h\|_2^2 \leqslant c_{\delta_1}\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2^2 + \delta_1\|\pi - \pi_h\|_2^2.$$

In order to get an upper bound to I_4, we apply Lemma 2.1 and Young's inequality (2.1) for shifted N-functions, recalling that the Δ_2-constants of φ_a, $(\varphi_a)^*$ only depend on p and do not depend on the shift-parameter $a \geqslant 0$. Hence, for any $\delta_2 > 0$ we obtain

$$\begin{aligned}
I_4 &\leqslant c \int_\Omega \varphi'_{\varepsilon+|\boldsymbol{Du}_h|}(|\boldsymbol{Du}_h - \boldsymbol{Dv}_h|)|\boldsymbol{Dv} - \boldsymbol{Du}_h|\,dx \\
&\leqslant \delta_2 \int_\Omega \varphi_{\varepsilon+|\boldsymbol{Du}_h|}(|\boldsymbol{Du}_h - \boldsymbol{Dv}_h|)\,dx + c_{\delta_2} \int_\Omega \varphi_{\varepsilon+|\boldsymbol{Du}_h|}(|\boldsymbol{Dv} - \boldsymbol{Du}_h|)\,dx \\
&\sim \delta_2 \|\boldsymbol{F}(\boldsymbol{Du}_h) - \boldsymbol{F}(\boldsymbol{Dv}_h)\|_2^2 + c_{\delta_2} \|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Du}_h)\|_2^2 \\
&\leqslant \delta_2 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + c_{\delta_2} \|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Du}_h)\|_2^2,
\end{aligned}$$

where we have also used Lemma 2.3. Collecting the estimates above, we arrive at

$$I_1 \leqslant c_{\delta_1,\delta_2} \|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Du}_h)\|_2^2 + \delta_1 \|\pi - \pi_h\|_2^2 + \delta_2 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2. \tag{4.2}$$

Next, we estimate the term I_2. Using Korn's and Young's inequality, applying Lemma 2.4 with $\nu = p$, we deduce that for each $\delta_3 > 0$ there exists c_{δ_3} such that

$$\begin{aligned}
I_2 &\leqslant \left|(\pi - r_h, \operatorname{div}(\boldsymbol{u}_h - \boldsymbol{v}_h))_\Omega\right| \leqslant c\|\pi - r_h\|_{p'} \|\boldsymbol{Du}_h - \boldsymbol{Dv}_h\|_p \\
&\leqslant \delta_3 \big(\|\boldsymbol{Dv} - \boldsymbol{Du}_h\|_p^2 + \|\boldsymbol{Dv} - \boldsymbol{Dv}_h\|_p^2\big) + c_{\delta_3} \|\pi - r_h\|_{p'}^2 \\
&\leqslant \delta_3 \|\boldsymbol{Dv} - \boldsymbol{Du}_h\|_p^2 + \delta_3 c \|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Dv}_h)\|_2^2 \|\varepsilon + |\boldsymbol{Dv}| + |\boldsymbol{Dv}_h|\|_p^{2-p} + c_{\delta_3} \|\pi - r_h\|_{p'}^2 \\
&\leqslant \delta_3 \|\boldsymbol{Dv} - \boldsymbol{Du}_h\|_p^2 + \delta_3 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + c_{\delta_3} \|\pi - r_h\|_{p'}^2.
\end{aligned} \tag{4.3}$$

Here, we have also used the fact that $\boldsymbol{Dv}$ and $\boldsymbol{Dv}_h$ are uniformly bounded in $\mathrm{L}^p(\Omega)^{d\times d}$. Combining the estimates (4.1), (4.2) and (4.3), we conclude

$$\begin{aligned}
\frac{\sigma_0}{2} d(\boldsymbol{v}, \boldsymbol{v}_h)^2 \leqslant{} & \delta_2 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + \delta_3 c d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + c_{\delta_1,\delta_2} \|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Du}_h)\|_2^2 + \delta_3 \|\boldsymbol{Dv} - \boldsymbol{Du}_h\|_p^2 \\
& + c_{\delta_3} \|\pi - r_h\|_{p'}^2 + \left(\frac{\gamma_0^2}{2\sigma_0} + \delta_1\right) \|\pi - \pi_h\|_2^2.
\end{aligned}$$

Multiplying this with $2/\sigma_0$, taking the square root, we easily complete the proof. □

Lemma 4.1 enables us to estimate the pressure error in the L^2-norm.

THEOREM 4.1 Let Assumption 2.1 hold. Let the discrete spaces fulfil $(\mathbf{IS}^2)$ and let the parameters meet the condition (3.4): $\gamma_0 < \tilde{\beta}(2)\varepsilon^{\frac{2-p}{2}} \frac{\sigma_0}{\sigma_0+\sigma_1}$. Then, there exists a constant $c = c(p, \varepsilon, \gamma_0, \sigma_0, \sigma_1, \tilde{\beta}(2), \Gamma_D, \Omega)$ such that the pressure error is bounded in $\mathrm{L}^2(\Omega)$ by

$$\begin{aligned}
\|\pi - \pi_h\|_2 \leqslant{} & c \inf_{\boldsymbol{u}_h \in \boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p} \big(\|\boldsymbol{F}(\boldsymbol{Dv}) - \boldsymbol{F}(\boldsymbol{Du}_h)\|_2 + \|\boldsymbol{Dv} - \boldsymbol{Du}_h\|_p\big) \\
& + c \inf_{r_h \in Q_h^p} \|\pi - r_h\|_{p'}.
\end{aligned}$$

Proof. Let $(\boldsymbol{u}_h, r_h)$ be an arbitrary element of $(\boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p) \times Q_h^p$. Then, **(pS)**, **($\mathbf{pS}_h$)** imply

$$(r_h - \pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega = (\boldsymbol{S}(\pi, \boldsymbol{Dv}) - \boldsymbol{S}(\pi_h, \boldsymbol{Dv}_h), \boldsymbol{Dw}_h)_\Omega + (r_h - \pi, \operatorname{div} \boldsymbol{w}_h)_\Omega \tag{4.4}$$

for all $\boldsymbol{w}_h \in \boldsymbol{X}_h^p$. Using $(\mathbf{IS}^2)$ and (4.4), we deduce, cf. (3.6),

$$\tilde{\beta}(2)\|r_h - \pi_h\|_2 \leqslant \sup_{\boldsymbol{w}_h \in \boldsymbol{X}_h^2} \frac{(r_h - \pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,2}} \leqslant \|\boldsymbol{S}(\pi, \boldsymbol{Dv}) - \boldsymbol{S}(\pi_h, \boldsymbol{Dv}_h)\|_2 + \|r_h - \pi\|_2.$$

Applying (2.10) and Lemma 4.1, we conclude that for each $\delta > 0$ there exists a constant $c_\delta > 0$ such that

$$\begin{aligned}\tilde{\beta}(2)\|r_h - \pi_h\|_2 &\leqslant \sigma_1 \varepsilon^{\frac{p-2}{2}} d(\boldsymbol{v}, \boldsymbol{v}_h) + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - \pi_h\|_2 + \|r_h - \pi\|_2 \\ &\leqslant \sigma_1 \varepsilon^{\frac{p-2}{2}} c_\delta \big(\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p + \|\pi - r_h\|_{p'}\big) \\ &\quad + \sigma_1 \varepsilon^{\frac{p-2}{2}} \Big(\frac{\gamma_0}{\sigma_0} + \delta\Big) \|\pi - \pi_h\|_2 + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - \pi_h\|_2 + \|r_h - \pi\|_2.\end{aligned}$$

Using Minkowski's inequality and $L^{p'}(\Omega) \hookrightarrow L^2(\Omega)$ for $p \leqslant 2$, we arrive at

$$\begin{aligned}\|\pi - \pi_h\|_2 &\leqslant c_\delta \big(\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p + \|\pi - r_h\|_{p'}\big) \\ &\quad + \tilde{\beta}(2)^{-1} \sigma_1 \varepsilon^{\frac{p-2}{2}} \Big(\frac{\gamma_0}{\sigma_0} + \delta\Big) \|\pi - \pi_h\|_2 + \tilde{\beta}(2)^{-1} \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - \pi_h\|_2.\end{aligned}$$

Recalling (3.4), and choosing $\delta > 0$ sufficiently small, we can absorb all terms, which include the pressure error, in the left-hand side. Hence, we get the desired result. □

COROLLARY 4.1 Let the assumptions of Theorem 4.1 be satisfied. Then, the error of approximation of the velocity field is bounded by

$$\begin{aligned}\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 &\leqslant c \inf_{\boldsymbol{u}_h \in (\boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p)} \big(\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p\big) \\ &\quad + c \inf_{r_h \in Q_h^p} \|\pi - r_h\|_{p'}. \end{aligned} \tag{4.5}$$

Proof. The estimate follows from Lemma 2.3, Lemma 4.1, and Theorem 4.1. □

COROLLARY 4.2 Let the assumptions of Theorem 4.1 hold. In addition, let $(\mathbf{IS}^p)$ hold and

$$\gamma_0 < \tilde{\beta}(p) \varepsilon^{\frac{2-p}{2}}. \tag{4.6}$$

Then, the error of approximation of the pressure field is bounded in $\mathrm{L}^{p'}(\Omega)$ by

$$\|\pi - \pi_h\|_{p'} \leqslant c \|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^{\frac{2}{p'}} + c \inf_{r_h \in Q_h^p} \|r_h - \pi\|_{p'}. \tag{4.7}$$

Proof. The estimate is again based on the inf–sup inequality $(\mathbf{IS}^p)$. Using $(\mathbf{IS}^p)$, Hölder's inequality, (4.4), (2.11) and (2.12), for arbitrary $r_h \in Q_h^p$ we obtain the estimate

$$\begin{aligned}\tilde{\beta}(p)\|r_h - \pi_h\|_{p'} &\leqslant \sup_{\boldsymbol{w}_h \in \boldsymbol{X}_h^p} \frac{(r_h - \pi_h, \operatorname{div} \boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,p}} \\ &\leqslant \|\boldsymbol{S}(\pi, \boldsymbol{D}\boldsymbol{v}) - \boldsymbol{S}(\pi_h, \boldsymbol{D}\boldsymbol{v}_h)\|_{p'} + \|r_h - \pi\|_{p'} \\ &\leqslant c \|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2^{\frac{2}{p'}} + \gamma_0 \varepsilon^{\frac{p-2}{2}} \|\pi - \pi_h\|_{p'} + \|r_h - \pi\|_{p'}.\end{aligned}$$

Due to assumption (4.6), this completes the proof. □

COROLLARY 4.3 Let the assumptions of Theorem 4.1 hold. Then, for all $(\boldsymbol{u}_h, r_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p) \times Q_h^p$ there holds

$$\begin{aligned}\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 + \|\pi - \pi_h\|_2 \leqslant c\,\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 \\ + c\big(\varepsilon_0 + \|\nabla \boldsymbol{u}_h\|_\infty + \|\nabla \boldsymbol{v}_h\|_\infty\big)^{\frac{2-p}{2}} \|\pi - r_h\|_2,\end{aligned} \tag{4.8}$$

where c only depends on p, ε, γ_0, σ_0, σ_1, $\tilde{\beta}(2)$, Γ_D and Ω.

Proof. First, we slightly modify the proof of Lemma 4.1. Let $(\boldsymbol{u}_h, r_h)$ be an arbitrary element of $(\boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p) \times Q_h^p$. Here, we estimate the term I_2 differently. Using (2.13) with $\nu = 2$, Young's inequality, and (2.12), we deduce that for each $\delta_3 > 0$ there exists c_{δ_3} such that

$$\begin{aligned}I_2 &\leqslant \big|(\pi - r_h, \operatorname{div}(\boldsymbol{u}_h - \boldsymbol{v}_h))_\Omega\big| \leqslant c\|\pi - r_h\|_2 \|\boldsymbol{D}\boldsymbol{u}_h - \boldsymbol{D}\boldsymbol{v}_h\|_2 \\ &\leqslant c\|\pi - r_h\|_2 \|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \big(\varepsilon_0 + \|\nabla \boldsymbol{u}_h\|_\infty + \|\nabla \boldsymbol{v}_h\|_\infty\big)^{\frac{2-p}{2}} \\ &\leqslant \delta_3\Big[d(\boldsymbol{v}, \boldsymbol{v}_h)^2 + \|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2^2\Big] + c_{\delta_3}\big(\varepsilon_0 + \|\nabla \boldsymbol{u}_h\|_\infty + \|\nabla \boldsymbol{v}_h\|_\infty\big)^{2-p} \|\pi - r_h\|_2^2.\end{aligned}$$

Following the same arguments as in the proof of Lemma 4.1, we conclude that for each $\delta > 0$ there exists a constant c_δ, which only depends on $p, \varepsilon_0, \gamma_0, \sigma_0, \sigma_1, \Omega$ and δ, such that

$$\begin{aligned}d(\boldsymbol{v}, \boldsymbol{v}_h) \leqslant c_\delta \Big(\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \big(\varepsilon_0 + \|\nabla \boldsymbol{u}_h\|_\infty + \|\nabla \boldsymbol{v}_h\|_\infty\big)^{\frac{2-p}{2}} \|\pi - r_h\|_2\Big) \\ + \Big(\frac{\gamma_0}{\sigma_0} + \delta\Big)\|\pi - \pi_h\|_2.\end{aligned}$$

Adopting the arguments presented in the proof of Theorem 4.1, we arrive at (w.l.o.g. $\varepsilon_0 \geqslant 1$)

$$\|\pi - \pi_h\|_2 \leqslant c\Big(\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \big(\varepsilon_0 + \|\nabla \boldsymbol{u}_h\|_\infty + \|\nabla \boldsymbol{v}_h\|_\infty\big)^{\frac{2-p}{2}} \|\pi - r_h\|_2\Big)$$

provided that condition (3.4) is satisfied. Obviously, the latter estimate implies the desired error estimate for the velocity. □

In practice, one never obtains the discrete solution $(\boldsymbol{v}_h, \pi_h)$ to $(\mathbf{pS}_h)$ exactly. Instead, one obtains its approximation $(\tilde{\boldsymbol{v}}_h, \tilde{\pi}_h) \in (\boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p) \times Q_h^p$, satisfying

$$\begin{aligned}(\boldsymbol{S}(\tilde{\pi}_h, \boldsymbol{D}\tilde{\boldsymbol{v}}_h), \boldsymbol{D}\boldsymbol{w}_h)_\Omega - (\tilde{\pi}_h, \operatorname{div} \boldsymbol{w}_h)_\Omega = (\boldsymbol{f}, \boldsymbol{w}_h)_\Omega - (\boldsymbol{b}, \boldsymbol{w}_h)_{\Gamma_P} + \langle \boldsymbol{e}, \boldsymbol{w}_h\rangle \qquad & \forall \boldsymbol{w}_h \in \boldsymbol{X}_h^p, \\ (\operatorname{div} \tilde{\boldsymbol{v}}_h, q_h)_\Omega = \langle g, q_h\rangle \qquad & \forall q_h \in Q_h^p,\end{aligned}$$

where $\boldsymbol{e} \in (\boldsymbol{X}_h^p)^*$, $g \in (Q_h^p)^*$, and the brackets denote the corresponding duality pairings. Here, $\boldsymbol{e} = \boldsymbol{e}(\tilde{\boldsymbol{v}}_h, \tilde{\pi}_h)$ and $g = g(\tilde{\boldsymbol{v}}_h, \tilde{\pi}_h)$ represent some additional error (which includes, e.g., the residuum associated with the approximate solution to the non-linear algebraic problem, or the error due to numerical integration). However, provided that one is able to estimate $\boldsymbol{e}$ and g, then one can derive estimates for $\boldsymbol{v} - \tilde{\boldsymbol{v}}_h$ and $\pi - \tilde{\pi}_h$ analogous to those derived in this section by following the same procedure. For instance, denoting $|\langle \boldsymbol{e}, \boldsymbol{w}_h\rangle| \leqslant E\,\|\boldsymbol{w}_h\|_{1,p}$ and $|\langle g, q_h\rangle| \leqslant G\,\|q_h\|_2$ (with E, G independent of h and assuming, say, $E, G \leqslant 1$, such that $\|\boldsymbol{D}\tilde{\boldsymbol{v}}_h\|_p$ remains reasonably bounded) one can show (cf. (4.5), (4.7)):

$$\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\tilde{\boldsymbol{v}}_h)\|_2 \leqslant c \inf_{\boldsymbol{u}_h \in (\boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p)} \big(\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{u}_h)\|_2 + \|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{u}_h\|_p\big)$$
$$+ c \inf_{r_h \in Q_h^p} \|\pi - r_h\|_{p'} + c\,(E+G)$$
$$\|\pi - \tilde{\pi}_h\|_{p'} \leqslant c\,\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\tilde{\boldsymbol{v}}_h)\|_2^{\frac{2}{p'}} + c \inf_{r_h \in Q_h^p} \|r_h - \pi\|_{p'} + c\,E.$$

5. Finite element approximation

In this section, we consider some finite element approximations of **(pS)** that satisfy the abstract theory of the previous sections. We assume that, for ease of exposition, Ω is a polygonal/polyhedral domain and that $\mathbb{T}_h$ is a shape regular decomposition of Ω into d-dimensional simplices (or quadrilaterals/hexahedra) so that $\overline{\Omega} = \bigcup_{K \in \mathbb{T}_h} \overline{K}$. By h_K we denote the diameter of a cell $K \in \mathbb{T}_h$; the mesh parameter h represents the maximum diameter of the cells, i.e., $h := \max\{h_K; K \in \mathbb{T}_h\}$. We assume that $\mathbb{T}_h$ is non-degenerate (see (Brenner & Scott, 1994)). Hence, the neighbourhood S_K of $K \in \mathbb{T}_h$, which denotes the union of all elements in $\mathbb{T}_h$ touching K, fulfills $|K| \sim |S_K|$ with constants independent of h. Furthermore, the number of cells in S_K is uniformly bounded with respect to $K \in \mathbb{T}_h$. Let X_h and Y_h be appropriate finite element spaces defined on $\mathbb{T}_h$ that satisfy $X_h \subset \mathrm{W}^{1,\infty}(\Omega)$ and $Y_h \subset \mathrm{L}^\infty(\Omega)$. We recall that the finite element spaces for the velocity and pressure are given by $\boldsymbol{X}_h^p := \boldsymbol{X}_h \cap \boldsymbol{X}^p$, $\boldsymbol{X}_h = [X_h]^d$, and $Q_h^p := Y_h \cap Q^p$. In order to ensure approximation properties and the discrete inf-sup conditions, we need to specify the choice of spaces:

Assumption 5.1 (Approximation property of X_h and Y_h)**.** *We assume that X_h contains the set of linear polynomials on Ω. Moreover, we suppose that there exist a linear projection $\boldsymbol{j}_h : \mathbf{W}^{1,1}(\Omega) \to \boldsymbol{X}_h$ and an interpolation operator $i_h : \mathrm{W}^{1,1}(\Omega) \to Y_h$ such that*

(1) *$\boldsymbol{j}_h$ preserves zero boundary values on Γ_D, such that $\boldsymbol{j}_h(\boldsymbol{X}^p) \subset \boldsymbol{X}_h^p$.*

(2) *$\boldsymbol{j}_h$ is locally $\mathrm{W}^{1,1}$-stable in the sense that there exists $c > 0$ (independent of h) so that*

$$⨍_K |\boldsymbol{j}_h \boldsymbol{w}|\,dx \leqslant c ⨍_{S_K} |\boldsymbol{w}|\,dx + c ⨍_{S_K} h_K |\nabla \boldsymbol{w}|\,dx \qquad \forall \boldsymbol{w} \in \mathbf{W}^{1,1}(\Omega),\ \forall K \in \mathbb{T}_h, \tag{5.1}$$

where S_K denotes a local neighbourhood of K (as defined above).

(3) *$\boldsymbol{j}_h$ preserves divergence[‖] in the Y_h^*-sense, i.e.,*

$$(\operatorname{div} \boldsymbol{w}, q_h)_\Omega = (\operatorname{div} \boldsymbol{j}_h \boldsymbol{w}, q_h)_\Omega \qquad \forall \boldsymbol{w} \in \mathbf{W}^{1,1}(\Omega),\ \forall q_h \in Y_h. \tag{5.2}$$

(4) *i_h preserves mean values, i.e., $i_h(Q^p) \subset Q_h^p$, and, for any $\nu \geqslant 1$, i_h satisfies*

$$\|q - i_h q\|_\nu \leqslant c h \|q\|_{1,\nu} \qquad \forall q \in \mathrm{W}^{1,\nu}(\Omega). \tag{5.3}$$

Later we will suppose that functions in X_h satisfy the following global inverse inequality:

[‖]Note that in case of $|\Gamma_P| > 0$ this implies $\int_{\Gamma_P} \boldsymbol{w} \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{x} = \int_{\Gamma_P} (\boldsymbol{j}_h \boldsymbol{w}) \cdot \boldsymbol{n}\, \mathrm{d}\boldsymbol{x}$; this requires that the triangulation matches Γ_P appropriately.

Assumption 5.2 (Inverse property of X_h)**.** *For* $\nu, \mu \in [1,\infty]$ *and* $0 \leqslant m \leqslant l$ *there holds*

$$\|w_h\|_{l,\nu} \leqslant Ch^{m-l+\min(0,\frac{d}{\nu}-\frac{d}{\mu})}\|w_h\|_{m,\mu} \qquad \forall w_h \in X_h. \tag{5.4}$$

Assumption 5.2 usually requires that the mesh is quasi-uniform (in the sense of (Brenner & Scott, 1994)). Assumption 5.1 is similar to Assumption 2.21 in (Belenki *et al.*, 2010). Clearly, the existence of $\boldsymbol{j}_h$ and i_h as in Assumption 5.1 depends on the choice of the finite element pairing X_h/Y_h:

- The construction of $\boldsymbol{j}_h$, such that it satisfies Assumptions 5.1 (1) – (3), is well-known for some particular finite elements, including the Crouzeix-Raviart and MINI element (see (Belenki *et al.*, 2010)). If $\Gamma_D \neq \partial\Omega$, Assumption 5.1 (1) requires that the triangulation matches Γ_D appropriately (see (Scott & Zhang, 1990)).

- Assumption 5.1 (2) is standard in the context of interpolation in Sobolev-Orlicz spaces (cf. (Diening & Růžička, 2007)). E.g., the Scott-Zhang interpolation operator (see (Scott & Zhang, 1990)) satisfies (5.1). It is crucial that from (5.1) one can derive the local stability result

$$\fint_K \psi(|\nabla \boldsymbol{j}_h \boldsymbol{w}|)\,dx \leqslant c\fint_{S_K} \psi(|\nabla \boldsymbol{w}|)\,dx \qquad \forall \boldsymbol{w} \in \mathbf{W}^{1,\psi}(\Omega) \qquad \forall K \in \mathbb{T}_h, \tag{5.5}$$

  which is valid for arbitrary N-functions ψ with $\Delta_2(\psi) < \infty$. Here, $\mathbf{W}^{1,\psi}(\Omega)$ is the classical Sobolev-Orlicz space and the constant c only depends on $\Delta_2(\psi)$. For details we refer to (Diening & Růžička, 2007).

- For standard finite elements, i_h may be chosen as the L^2-projection onto Y_h:

$$(i_h q, q_h)_\Omega = (q, q_h)_\Omega \qquad \forall q_h \in Y_h \qquad \forall q \in \mathrm{L}^1(\Omega). \tag{5.6}$$

  Indeed, it is shown in (Crouzeix & Thomée, 1987) that the L^2-projection is L^ν-stable and even $\mathrm{W}^{1,\nu}$-stable for any $\nu \in [1,\infty]$, and, consequently, the L^2-projection fulfills (5.3). The results of (Crouzeix & Thomée, 1987) are derived for finite element spaces Y_h based on simplices, $Y_h := \{w \in \mathrm{C}(\overline{\Omega});\, w|_K \in \mathbb{P}_r(K) \text{ for all } K \in \mathbb{T}_h\}$, where $\mathbb{P}_r(K)$ denotes the space of polynomials on K of degree less than or equal to r. Moreover, setting $q_h = 1$ in (5.6), we deduce that i_h preserves mean values. Hence, $i_h(Q^p) \subset Q_h^p$.

Next, we depict important consequences of Assumption 5.1:

LEMMA 5.1 Let there exist a linear projection $\boldsymbol{j}_h$ that satisfies Assumption 5.1 (2). Then, for all $K \in \mathbb{T}_h$ and $\boldsymbol{w} \in \mathbf{W}^{1,p}(\Omega)$ there holds

$$\fint_K |\boldsymbol{F}(\boldsymbol{D}\boldsymbol{w}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{w})|^2\,dx \leqslant ch_K^2 \fint_{S_K} |\nabla \boldsymbol{F}(\boldsymbol{D}\boldsymbol{w})|^2\,dx \tag{5.7}$$

provided that $\boldsymbol{F}(\boldsymbol{D}\boldsymbol{w}) \in \mathrm{W}^{1,2}(\Omega)^{d\times d}$. The constant c only depends on p.

Proof. The proof is based on the Orlicz-stability (5.5). We refer to (Belenki *et al.*, 2010; Hirn, 2010). □

Moreover, the assumptions on $\boldsymbol{j}_h$ imply the discrete versions of the inf-sup inequality:

LEMMA 5.2 Let there exist a linear projection $\boldsymbol{j}_h$ that satisfies Assumption 5.1 (1) – (3). Then, for $\nu \in (1,\infty)$ the discrete inf-sup inequality $(\mathbf{IS}^\nu)$ is satisfied.

Proof. Since $\mathbb{T}_h$ is non-degenerate, the local stability result (5.5) (with $\psi(t) := t^\nu$) leads to the global $\mathrm{W}^{1,\nu}$-stability inequality, $\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,\nu} \leqslant C_s\|\boldsymbol{w}\|_{1,\nu}$ for all $\boldsymbol{w} \in \boldsymbol{X}^\nu$, where $\nu \in (1,\infty)$ and the stability constant C_s does not depend on h. Thus, the continuous inf-sup inequality (2.20) and Assumption 5.1 imply that for arbitrary $q_h \in Q_h^\nu \subset Q^\nu$ it holds

$$\begin{aligned}\|q_h\|_{\nu'} &\leqslant \beta(\nu)^{-1}\sup_{\boldsymbol{w}\in\boldsymbol{X}^\nu}\frac{(q_h,\operatorname{div}\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,\nu}} = \beta(\nu)^{-1}\sup_{\boldsymbol{w}\in\boldsymbol{X}^\nu}\frac{(q_h,\operatorname{div}\boldsymbol{j}_h\boldsymbol{w})_\Omega}{\|\boldsymbol{w}\|_{1,\nu}}\\ &\leqslant \beta(\nu)^{-1}C_s\sup_{\boldsymbol{w}\in\boldsymbol{X}^\nu}\frac{(q_h,\operatorname{div}\boldsymbol{j}_h\boldsymbol{w})_\Omega}{\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,\nu}} \leqslant \tilde{\beta}(\nu)^{-1}\sup_{\boldsymbol{w}_h\in\boldsymbol{X}_h^\nu}\frac{(q_h,\operatorname{div}\boldsymbol{w}_h)_\Omega}{\|\boldsymbol{w}_h\|_{1,\nu}},\end{aligned}$$

where $\tilde{\beta}(\nu) := \beta(\nu)/C_s$ is independent of h. □

REMARK 5.1 Let us briefly discuss the case of unstable discretizations. For instance, one may consider an equal-order discretization, where both X_h and Y_h are based on piece-wise polynomials of the same degree. In this case, the discrete inf-sup condition is violated. For p-Stokes systems, for which the generalized viscosity only depends on the shear-rate, Hirn (Hirn, 2010) proposes a stabilization technique based on the local projection stabilization (LPS) method, that leads to optimal convergence results. Whether the stabilization method can be applied to the equal-order discretization of **(pS)**, is subject of current research.

Next we state our a priori error estimates that quantify the convergence of the finite element method. For this, the regularity $\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) \in \mathrm{W}^{1,2}(\Omega)^{d\times d}$ of the solution $\boldsymbol{v}$ is required (which is equivalent to $(1.6)_1$, see (Berselli *et al.*, 2010)). We mention that (1.6) is available for sufficiently smooth data at least in the space-periodic setting in two space dimensions (see (Bulíček & Kaplický, 2008)).

COROLLARY 5.1 Let the assumptions of Theorem 4.1 hold. We suppose that there exist operators $\boldsymbol{j}_h$ and i_h satisfying Assumption 5.1. Moreover, we assume the additional regularity of the weak solution

$$\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) \in \mathrm{W}^{1,2}(\Omega)^{d\times d} \qquad \text{and} \qquad \pi \in \mathrm{W}^{1,p'}(\Omega)$$

and we set $\boldsymbol{v}_{0,h} := \boldsymbol{j}_h\boldsymbol{v}_0$. Then, the error of approximation is bounded in terms of the maximum mesh size h as follows:

$$\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leqslant C_{\boldsymbol{v}}h, \qquad \|\pi - \pi_h\|_2 \leqslant C_\pi h. \tag{5.8}$$

Assume additionally (4.6): $\gamma_0 < \tilde{\beta}(p)\varepsilon^{\frac{2-p}{2}}$. Then the pressure error in $L^{p'}(\Omega)$ is bounded by

$$\|\pi - \pi_h\|_{p'} \leqslant C'_\pi h^{\frac{2}{p'}}. \tag{5.9}$$

The constants $C_{\boldsymbol{v}}, C_\pi, C'_\pi > 0$ only depend on $\|\nabla\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{1,p'}$, p, ε, σ_0, σ_1, γ_0, $\tilde{\beta}(2)$ (and C'_π additionally depends on $\tilde{\beta}(p)$).

Proof. According to Lemma 5.2, the discrete inf-sup inequalities **($\mathbf{IS}^2$)**, **($\mathbf{IS}^p$)** hold true. Hence, the desired error estimates follow from Theorem 4.1, Corollaries 4.1 and 4.2, and the interpolation properties of $\boldsymbol{j}_h$ and i_h. More precisely, the velocity is given by $\boldsymbol{v} = \boldsymbol{v}_0 + \hat{\boldsymbol{v}}$ for some $\hat{\boldsymbol{v}} \in \boldsymbol{X}^p$. Since $\hat{\boldsymbol{v}}$ is divergence-free, the interpolant $\boldsymbol{j}_h\hat{\boldsymbol{v}}$ fulfills $(\operatorname{div}\boldsymbol{j}_h\hat{\boldsymbol{v}}, q_h)_\Omega = 0$ for all $q_h \in Q_h^p$. Hence, $\boldsymbol{j}_h\hat{\boldsymbol{v}} \in \boldsymbol{V}_h^p$ and $\boldsymbol{j}_h\boldsymbol{v} = \boldsymbol{j}_h\boldsymbol{v}_0 + \boldsymbol{j}_h\hat{\boldsymbol{v}} \in (\boldsymbol{v}_{0,h} + \boldsymbol{V}_h^p)$. Consequently, we can set $\boldsymbol{u}_h := \boldsymbol{j}_h\boldsymbol{v}$ and $r_h := i_h\pi$ in Theorem 4.1 and Corollary 4.1. Using Lemma 2.4 with $\nu := p$, the global $\mathrm{W}^{1,p}$-stability of $\boldsymbol{j}_h$ (which follows from (5.5) with $\psi(t) = t^p$ and the non-degeneracy of $\mathbb{T}_h$), the interpolation properties (5.7) and (5.3), we easily conclude (5.8). Finally, (5.9) follows from Corollary 4.2 and (5.8). □

REMARK 5.2 Using (2.13) and (3.1), we deduce from Corollary 5.1 that

$$\|\boldsymbol{D}\boldsymbol{v}-\boldsymbol{D}\boldsymbol{v}_h\|_p \leqslant c\,\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leqslant ch.$$

Hence, we also obtain an a priori error estimate in $\mathbf{W}^{1,p}(\Omega)$.

The $\mathrm{W}^{1,p'}$-regularity assumption for pressure can be avoided and confined to $\pi \in \mathrm{W}^{1,2}(\Omega)$ only. This is depicted by the following variant of Corollary 5.1. There, the assertion remains the same. The price to pay is twofold: We assume that $\boldsymbol{v} \in \mathbf{W}^{1,\infty}(\Omega)$; the property which we have not been able to show. Moreover, in order to reproduce (5.9) we restrict to $d=2$.

COROLLARY 5.2 Let $d=2$. Let the assumptions of Theorem 4.1 hold and let Assumption 5.2 be satisfied. We suppose that there exist operators $\boldsymbol{j}_h$ and i_h as in Assumption 5.1. Moreover, we assume that the solution $(\boldsymbol{v},\pi)$ satisfies the additional regularity

$$\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) \in \mathrm{W}^{1,2}(\Omega)^{d\times d}, \qquad \boldsymbol{v} \in \mathbf{W}^{1,\infty}(\Omega), \quad \text{and} \qquad \pi \in \mathrm{W}^{1,2}(\Omega).$$

We set $\boldsymbol{v}_{0,h} := \boldsymbol{j}_h\boldsymbol{v}_0$. Then, the error of approximation is bounded as follows:

$$\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})-\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leqslant C_{\boldsymbol{v}}h, \qquad \|\pi-\pi_h\|_2 \leqslant C_\pi h, \tag{5.10}$$

Assume additionally (4.6) and the $\mathrm{W}^{1,2}$-stability of i_h. Then, there holds

$$\|\pi-\pi_h\|_{p'} \leqslant C'_\pi h^{\frac{2}{p'}}. \tag{5.11}$$

The constants $C_{\boldsymbol{v}}, C_\pi, C'_\pi > 0$ only depend on $\|\nabla\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})\|_2$, $\|\pi\|_{1,2}$, $\|\boldsymbol{v}\|_{1,\infty}$, p, ε, σ_0, σ_1, γ_0, $\tilde{\beta}(2)$ (and C'_π additionally depends on $\tilde{\beta}(p)$).

Proof. First, we mention that the projection $\boldsymbol{j}_h$ is $\mathrm{W}^{1,\infty}$-stable. Indeed, similarly as in (Scott & Zhang, 1990) it can be shown that $\boldsymbol{j}_h$ is locally $\mathrm{W}^{1,1}$-stable, i.e., there holds $\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,1;K} \lesssim \|\boldsymbol{w}\|_{1,1;S_K}$ for all $\boldsymbol{w} \in \mathbf{W}^{1,1}(\Omega)$ and $K \in \mathbb{T}_h$. Moreover, since $X_h(K)$ is finite dimensional, there holds $|\nabla^i\boldsymbol{j}_h\boldsymbol{w}(\boldsymbol{y})| \lesssim \fint_K |\nabla^i\boldsymbol{j}_h\boldsymbol{w}|\,\mathrm{d}\boldsymbol{x}$, $i \in \{0,1\}$, for all $\boldsymbol{y} \in K$ and $K \in \mathbb{T}_h$. Due to the non-degeneracy of $\mathbb{T}_h$ it follows $\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,\infty;K} \lesssim \|\boldsymbol{w}\|_{1,\infty;S_K}$ for all $\boldsymbol{w} \in \mathbf{W}^{1,\infty}(\Omega)$. This yields $\|\boldsymbol{j}_h\boldsymbol{w}\|_{1,\infty;\Omega} \lesssim \|\boldsymbol{w}\|_{1,\infty;\Omega}$ for all $\boldsymbol{w} \in \mathbf{W}^{1,\infty}(\Omega)$. Next, we depict that $\boldsymbol{v}_h$ is uniformly bounded in $\mathbf{W}^{1,\infty}(\Omega)$. Using the inverse inequality (5.4) with $d=2$, the $\mathrm{W}^{1,\infty}$-stability of $\boldsymbol{j}_h$, Korn's Lemma 2.5, and Lemma 2.4 with $\nu=2$, we estimate

$$\begin{aligned}
\|\boldsymbol{v}_h\|_{1,\infty} &\leqslant \|\boldsymbol{v}_h-\boldsymbol{j}_h\boldsymbol{v}\|_{1,\infty} + \|\boldsymbol{j}_h\boldsymbol{v}\|_{1,\infty}\\
&\leqslant c\Big[h^{-1}\|\boldsymbol{v}_h-\boldsymbol{j}_h\boldsymbol{v}\|_{1,2} + \|\boldsymbol{v}\|_{1,\infty}\Big]\\
&\leqslant c\Big[h^{-1}\|\boldsymbol{D}\boldsymbol{v}_h-\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v}\|_2 + \|\boldsymbol{v}\|_{1,\infty}\Big]\\
&\leqslant c\Big[h^{-1}\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)-\boldsymbol{F}(\boldsymbol{D}\boldsymbol{j}_h\boldsymbol{v})\|_2\big(\varepsilon_0+\|\nabla\boldsymbol{v}_h\|_\infty+\|\nabla\boldsymbol{v}\|_\infty\big)^{\frac{2-p}{2}} + \|\boldsymbol{v}\|_{1,\infty}\Big].
\end{aligned}$$

Setting $\boldsymbol{u}_h := \boldsymbol{j}_h\boldsymbol{v}$ and $r_h := i_h\pi$ in (4.8), and using the properties of the interpolation operators, we obtain the error estimate (w.l.o.g. $\varepsilon_0 \geqslant 1$)

$$\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)-\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})\|_2 \leqslant Ch\big(\varepsilon_0+\|\nabla\boldsymbol{v}_h\|_\infty+\|\nabla\boldsymbol{v}\|_\infty\big)^{\frac{2-p}{2}}$$

where the constant C depends on $\|\nabla \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})\|_2$, and $\|\pi\|_{1,2}$. Combining the latter inequalities, we easily conclude that

$$\|\boldsymbol{v}_h\|_{1,\infty} \leqslant C = C\big(\|\nabla \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v})\|_2, \|\pi\|_{1,2}, \|\boldsymbol{v}\|_{1,\infty}\big). \tag{5.12}$$

Of course, the constant C in (5.12) also depends on $p, \varepsilon, \varepsilon_0, \gamma_0, \sigma_0, \sigma_1, \tilde{\beta}(2), \Omega$. However, C is independent of h. In view of (5.12), (4.8) yields the desired error estimates (5.10).

It remains to prove the pressure estimate in $\mathrm{L}^{p'}(\Omega)$. Interpolating $\mathrm{L}^{p'}(\Omega)$ between $\mathrm{L}^2(\Omega)$ and $\mathrm{W}^{1,2}(\Omega)$, and using the interpolation property (5.3), and the $\mathrm{W}^{1,2}$-stability of i_h, for $p > \frac{2d}{d+2}$ and $\lambda := \frac{d}{2} - \frac{d}{p'}$ we obtain the estimate

$$\|\pi - i_h\pi\|_{p'} \leqslant c\,\|\pi - i_h\pi\|_{1,2}^{\lambda}\|\pi - i_h\pi\|_2^{1-\lambda} \leqslant c h^{1+\frac{d}{p'}-\frac{d}{2}}\|\pi\|_{1,2}. \tag{5.13}$$

Thus, for $d = 2$ the estimate (5.11) follows from the combination of (4.7), (5.10) and (5.13). This completes the proof. □

REMARK 5.3 Using (2.13) and (5.12), we deduce from Corollary 5.2 that

$$\|\boldsymbol{D}\boldsymbol{v} - \boldsymbol{D}\boldsymbol{v}_h\|_2 \leqslant c\,\|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2 \leqslant ch. \tag{5.14}$$

Hence, we also obtain an a priori error estimate in $\mathbf{W}^{1,2}(\Omega)$.

6. Numerical examples

In this section we present some numerical examples, which illustrate the a priori error estimates of Corollary 5.1. Here, the following model is used:

$$\eta(\pi, |\boldsymbol{D}\boldsymbol{v}|^2) := \eta_0\left(\delta_1 + \delta_2\,(\delta_3 + \exp(\alpha\pi))^{-q} + \delta_4\,|\boldsymbol{D}\boldsymbol{v}|^2\right)^{\frac{p-2}{2}}, \tag{6.1}$$

where $\alpha, q, \delta_1, \ldots, \delta_4 \geqslant 0$.

REMARK 6.1 Similarly as (e.g.) in (Málek *et al.*, 2002), it can be shown that (6.1) satisfies Assumptions **(A1)**–**(A2)** e.g. with $\varepsilon^2 := \delta_1/\delta_4$, $\sigma_0 := \hat{\eta}\varepsilon^{2-p}(1 + \delta_2\delta_3^{-q}/\delta_1)^{(p-2)/2}$, $\sigma_1 := \hat{\eta}\varepsilon^{2-p}(p-1)$, and $\gamma_0 := \alpha\hat{\eta}\varepsilon^{(2-p)/2}\delta_2\delta_3^{-q}\delta_4^{-1/2}q(p-2)/2$, where $\hat{\eta} := \eta_0\delta_1^{(p-2)/2}$ (so that $\eta(\pi, |\boldsymbol{D}\boldsymbol{v}|^2) \leqslant \hat{\eta}$).

Problem **(pS)** was discretized with the following finite elements based on quadrilateral meshes: the first-order $\mathbb{Q}_2/\mathbb{Q}_0$ elements, the second order $\mathbb{Q}_2/\mathbb{Q}_1$ and $\mathbb{Q}_2/\mathbb{P}_{-1}$ elements (see (Gresho & Sani, 2000), or (Sani *et al.*, 1981)), and the bilinear $\mathbb{Q}_1/\mathbb{Q}_1$ elements. The latter element pair is not stable, thus we used the LPS-type stabilization method presented in (Hirn, 2010); it is worth mentioning that in all examples the stabilization method was little sensitive with respect to the stabilization parameter. The algebraic equations were solved by Newton's method, the linear subproblems by the GMRES method. All computations were performed by means of the software package (GASCOIGNE, 2006) and/or the software developed by J. Hron, see e.g. (Hron *et al.*, 2003). In the following numerical experiments we depict the rates of convergence with respect to the number of cells (under global mesh refinement). For ease of presentation, we use the shortcuts $E_{\boldsymbol{v}}^{\boldsymbol{F}} := \|\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) - \boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}_h)\|_2$, $E_{\boldsymbol{v}}^{1,\nu} := \|\boldsymbol{v} - \boldsymbol{v}_h\|_{1,\nu}$, $E_{\boldsymbol{v}}^{\nu} := \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\nu}$, and $E_{\pi}^{\nu} := \|\pi - \pi_h\|_{\nu}$.

Table 1. Numerical experiments on error estimates.

(a) $p = 1.7$, $\mathbb{Q}_2/\mathbb{Q}_0$ | (b) $p = 1.5$, $\mathbb{Q}_2/\mathbb{Q}_0$ | (c) $p = 1.1$, $\mathbb{Q}_2/\mathbb{Q}_0$

#cells	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
4^4	0.98	1.83	0.82	0.74	0.97	1.85	0.82	0.65	0.90	1.90	0.82	0.19
4^5	1.01	1.89	0.85	0.77	1.00	1.91	0.85	0.66	0.95	1.95	0.85	0.19
4^6	1.02	1.92	0.88	0.79	1.00	1.95	0.88	0.67	0.98	1.97	0.88	0.19
4^7	1.01	1.93	0.90	0.80	1.01	1.96	0.90	0.67	0.98	1.99	0.90	0.19
4^8	1.01	1.96	0.91	0.81	1.01	1.96	0.91	0.67	0.98	2.00	0.91	0.19
expected	1	–	1	0.82	1	–	1	0.67	1	–	1	0.18

(d) $p = 1.7$, $\mathbb{Q}_1/\mathbb{Q}_1$ stabilized | (e) $p = 1.3$, $\mathbb{Q}_1/\mathbb{Q}_1$ stabilized | (f) $p = 1.1$, $\mathbb{Q}_1/\mathbb{Q}_1$ stabilized

#cells	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
4^5	1.00	2.17	1.00	0.83	0.99	2.49	1.00	0.46	0.99	2.70	0.99	0.19
4^6	1.00	2.17	1.00	0.83	0.99	2.48	1.00	0.46	0.99	2.66	1.00	0.19
4^7	1.00	2.17	1.00	0.82	0.99	2.45	1.00	0.46	0.99	2.56	1.00	0.19
4^8	1.00	2.16	1.00	0.83	1.00	2.41	1.00	0.47	1.00	2.44	1.00	0.19
4^9	1.00	2.16	1.00	0.83	1.00	2.36	1.00	0.47	1.00	2.30	1.01	0.19

(g) $p = 1.5$, $\mathbb{Q}_1/\mathbb{Q}_1$ stabilized | (h) $p = 1.5$, $\mathbb{Q}_2/\mathbb{Q}_1$ | (i) $p = 1.5$, $\mathbb{Q}_2/\mathbb{P}_{-1}$

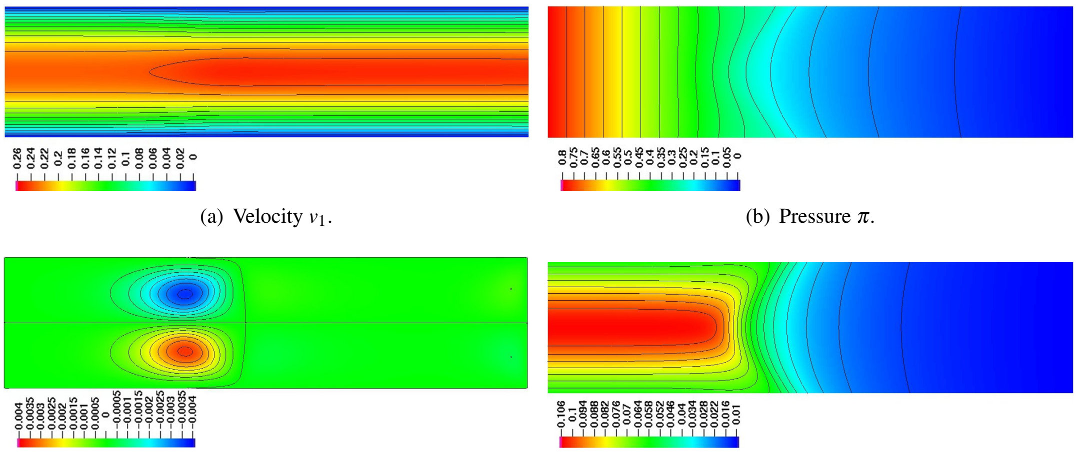
#cells	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$	$E_{\boldsymbol{v}}^{\boldsymbol{F}}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\pi}^{p'}$
4^4	–	–	–	–	1.02	2.33	1.01	0.68	1.02	2.30	1.01	0.68
4^5	1.01	2.33	1.01	0.68	1.01	2.32	1.01	0.68	1.02	2.27	1.01	0.68
4^6	1.01	2.33	1.01	0.67	1.02	2.33	1.01	0.68	1.02	2.26	1.01	0.68
4^7	1.00	2.32	1.01	0.67	1.02	2.30	1.01	0.68	1.01	2.23	1.01	0.68
4^8	1.00	2.31	1.01	0.67	1.02	2.25	1.01	0.68	1.02	2.10	1.01	0.67
4^9	1.00	2.29	1.01	0.67	–	–	–	–	–	–	–	–

EXAMPLE 1: In a square domain $\Omega := (-0.5, 0.5) \times (-0.5, 0.5)$, the exact solution to **(pS)** was given by $\boldsymbol{v}(\boldsymbol{x}) := |\boldsymbol{x}|^{a-1}\binom{x_2}{-x_1}$ and $\pi(\boldsymbol{x}) := |\boldsymbol{x}|^b x_1 x_2$ for $a, b \in \mathbb{R}$. Problem **(pS$_h$)** was then solved** for $\boldsymbol{f} := -\operatorname{div} \boldsymbol{S}(\boldsymbol{D}\boldsymbol{v}) + \nabla\pi$. The parameters a and b were chosen so that $\boldsymbol{F}(\boldsymbol{D}\boldsymbol{v}) \in \mathrm{W}^{1,2}(\Omega)^{d\times d}$ and $\pi \in \mathrm{W}^{1,2}(\Omega)$; this requirement amounts to the conditions $a > 1$ and $b > -2$. Since $\|\nabla\boldsymbol{v}\|_\infty$ is bounded for $a > 1$, according to Corollary 5.2 the requirement $\pi \in \mathrm{W}^{1,2}(\Omega)$ is sufficient to ensure the optimal rate of convergence (note that Corollary 5.1 would require $\pi \in \mathrm{W}^{1,p'}(\Omega)$ with $p' > 2$). We set $a = 1.01$ and $b = -1.99$. Hence, as soon as (3.4) is satisfied, we expect $E_{\boldsymbol{v}}^{\boldsymbol{F}} = \mathscr{O}(h)$, $E_\pi^2 = \mathscr{O}(h)$, and $E_\pi^{p'} = \mathscr{O}(h^{2/p'})$, for finite elements satisfying Assumption 5.1.

The parameters of the model (6.1) were set to $\delta_1 := 10^{-8}$, $q := 2/(2-p)$ and $\eta_0 = \delta_2 = \delta_3 = \delta_4 := 1$ in this example. Then, Remark 6.1 gives the estimate $\gamma_0 \leqslant \alpha\delta_1^{(2-p)/4}$ and hence, (3.4) is ensured at least for (using $\delta_1 \ll 1$) $\alpha \leqslant \tilde{\beta}(2)\delta_1^{(2-p)}/(p-1)$, i.e. for $\alpha \ll 1$. In this particular example, we have numerically observed the expected convergence rates (see below) for $\alpha \in [0, 8]$. For $\alpha > 8$, Newton's

**Both $\Gamma_P = \emptyset$ (with $⨍_\Omega \pi \,\mathrm{d}\boldsymbol{x}$ prescribed) or Γ_P chosen as one of the square edges were tested as the boundary conditions.

FIG. 1. Pressure drop problem, $p = 1.5$.



(a) Velocity v_1. (b) Pressure π.

(c) Velocity v_2. (d) Viscosity η.

method did not converge any more. One may ask, whether the assumption (3.4) could be relaxed[††]; in particular, whether the estimates (5.11) and (5.14) remain valid in the degenerate case $\varepsilon \searrow 0$. Here it is worth noting that in case of Carreau-type models (i.e., $\gamma_0 \equiv 0$), the error estimates similar to (5.11) and (5.14) actually hold true and are numerically validated also for $\varepsilon = 0$ (see (Belenki *et al.*, 2010; Hirn, 2010)). For fluids whose viscosity highly depends on the pressure, though, the behaviour for $\varepsilon \searrow 0$ remains an open question. In what follows, we set $\alpha := 1$.

For the $\mathbb{Q}_2/\mathbb{Q}_0$ elements, which are stable and of the first-order, the convergence rates for different values of $p \in (1,2)$ are presented in Tables 1(a)–1(c). We realize that the numerical results agree with the presented theory very well. In particular, the example reflects that the rate of convergence for the pressure in $\mathrm{L}^{p'}(\Omega)$ depends on the choice of p as predicted by the estimate (5.11). Apart from that, we observed that the experimental order of convergence declines as soon as $a < 1$ or $b < -2$. This indicates that the derived a priori error estimates are optimal with respect to the regularity of the solution. We also observe that the error $E^p_{\boldsymbol{v}}$ behaves like $\mathscr{O}(h^2)$. This raises hope that a duality argument (see (Brenner & Scott, 1994)) may be applicable here. In Tables 1(d)–1(i), we present the observed convergence rates for the element pairs $\mathbb{Q}_1/\mathbb{Q}_1$, $\mathbb{Q}_2/\mathbb{Q}_1$, and $\mathbb{Q}_2/\mathbb{P}_{-1}$. In this example, they basically coincide with those obtained for $\mathbb{Q}_2/\mathbb{Q}_0$.

EXAMPLE 2: PRESSURE DROP PROBLEM. In order to confirm the results in a realistic flow configuration, we consider a planar flow between two steady parallel plates, driven by the difference of pressure between inlet and outlet. Here, $\Omega = (0, 1.64) \times (0, 0.41)$ and the homogeneous Dirichlet boundary condition is prescribed on the upper and lower edge, while we set $\boldsymbol{b} := 0.8\boldsymbol{n}$ on the inflow (left) boundary,

[††]However, this observation does not allow us to *claim* that (3.4) could be relaxed. The solution to Example 1 is given a priori while $\boldsymbol{f}$ is defined accordingly. In particular, the solution always exists, whatever large α and γ_0 is. Moreover, the above estimate for γ_0 takes into account *all* $\pi \in \mathbb{R}$, $|\boldsymbol{Dv}| \geqslant 0$, and may be far from describing the behaviour of viscosity in a neighbourhood of the given solution.

Table 2. Pressure drop problem, $p = 1.5$.

	(a) $\mathbb{Q}_2/\mathbb{Q}_0$			(b) $\mathbb{Q}_2/\mathbb{Q}_1$			(c) $\mathbb{Q}_2/\mathbb{P}_{-1}$			(d) $\mathbb{Q}_1/\mathbb{Q}_1$ stabilized		
#cells	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}	$E_{\boldsymbol{v}}^{1,p}$	$E_{\boldsymbol{v}}^{p}$	E_{π}^{2}
4^4	0.99	1.95	1.00	2.29	3.44	2.19	2.16	3.19	1.92	–	–	–
4^5	0.99	1.98	1.01	2.51	3.78	2.24	2.19	3.15	1.96	1.00	1.97	1.94
4^6	1.02	1.96	1.03	2.46	3.69	2.08	2.14	3.04	1.99	1.00	2.00	2.04
4^7	1.08	2.02	1.16	2.25	3.26	2.06	*	*	*	1.01	2.01	1.98
4^8	–	–	–	–	–	–	–	–	–	1.02	2.06	1.89
expected	1	–	1									

*) In this case we were not able to solve the algebraic problem to the accuracy sufficient to improve the discrete solution on finer meshes. Note that $E_{\boldsymbol{v}}^{p}/\|\boldsymbol{v}\|_p \sim 10^{-7}$ at this level of refinement.

and $\boldsymbol{b} := \mathbf{0}$ on the outflow (right) boundary. Moreover, we additionally require[‡‡] there that $\boldsymbol{v} = (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}$, i.e., the stream lines are orthogonal to the inflow and outflow boundary (cf. (Heywood *et al.*, 1996)). Note that if the viscosity did not vary with the pressure, this setting would lead to a unidirectional flow (Poiseuille flow) of the form $\boldsymbol{v} = (v_1(x_2), 0)$ and $\pi = \pi(x_1)$. Since the viscosity depends on the pressure, however, this needs not be the case; e.g., there is no such unidirectional solution for the Barus model $\eta = \eta_0 \exp(\alpha\pi)$, as was shown in (Hron *et al.*, 2001). Here we consider the model (6.1), provided with $\eta_0 := 0.005$, $p = 1.5$, $q := \frac{2}{2-p}$, $\delta_1 := 5 * 10^{-6}$, $\delta_2 = \delta_3 := 1$, $\delta_4 := 10^{-5}$, and $\alpha := 10$. The resulting velocity, pressure and viscosity fields are shown in Figure 1. For moderate and low pressures (in the middle-length and the right-hand part of the domain) this model approximates the Barus model, while for higher pressures (in the domain left-hand part) the behaviour is that of Carreau model. In Table 2, we present the observed convergence rates for the different finite element pairs. Since the exact solution is unknown, we have used the finite element approximation computed on a grid of 4^{10} cells as the reference solution. In view of Table 2, we observe good agreement with the derived estimates. While E_{π}^{2} behaves as $\mathscr{O}(h)$ in the case of $\mathbb{Q}_2/\mathbb{Q}_0$ discretization, the higher order element pairs, including $\mathbb{Q}_1/\mathbb{Q}_1$ discretization, leads to better convergence rates.

Conclusion

We have shown the convergence of the finite element method in the context of fluids with shear rate and pressure dependent viscosity. The convergence of the method has been quantified by the a priori error estimates of Corollary 5.1. These error estimates have been demonstrated practically by numerical experiments. All results of the present paper also cover the case of Carreau-type models. In this case, the error estimates of Corollary 5.1 coincide with the optimal error estimates for Carreau-type models which have been established in (Hirn, 2010) and (Belenki *et al.*, 2010).

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[‡‡] This requirement is achieved by altering the definition of the space $\boldsymbol{X}^p$, see, e.g., (Lanzendörfer & Stebel, 2008).

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