ON AN EVOLUTIONARY NONLINEAR FLUID MODEL IN THE LIMITING CASE

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Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. We consider the two-dimesional spatially periodic problem for an evolutionary system describing unsteady motions of the fluid with shear-dependent viscosity under general assumptions on the form of nonlinear stress tensors that includes those with p-structure. The global-in-time existence of a weak solution is established. Some models where the nonlinear operator corresponds to the case $p = 1$ are covered by this analysis.

Keywords: shear-dependent viscosity, incompressible fluid, global-in-time existence, weak solution

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1. Introduction

The main aim of this note is to give a global existence result for an evolutionary nonlinear system occuring in two-dimensional fluid mechanics as the limiting case.

Let $L, T \in (0,\infty)$, $I \equiv (0,T)$. Then the considered problem for the velocity field $v\colon I \times \mathbb{R}^2 \to \mathbb{R}^2$ and the pressure $\pi\colon I \times \mathbb{R}^2 \to \mathbb{R}$ reads[2]

$$
\begin{aligned}
&\frac{\partial v}{\partial t} + \operatorname{div}(v \otimes v) - \operatorname{div}(\Theta(|\mathcal{D}(v)|^2)\,\mathcal{D}(v)) - \nabla\pi = f \quad \text{in } I \times \mathbb{R}^2,\\
&\operatorname{div} v = 0 \quad \text{in } I \times \mathbb{R}^2,\\
&v(0,\cdot) = v_0(\cdot) \text{ in } \mathbb{R}^2, \quad v, \pi \text{ are } L\text{-periodic at } x_i, i = 1,2.
\end{aligned}
\tag{1.1}
$$

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[2] We assume that the fluid is incompressible with a constant density, which is set to be one, for simplicity.

The scalar function $\Theta\colon \mathbb{R}_0^+ \to \mathbb{R}^+$ is given by

$$\Theta(|\mathcal{D}(v)|^2) \equiv (\mu_0 + |\mathcal{D}(v)|^2)^{-\frac{1}{2}}, \ \mu_0 > 0, \tag{1.2}$$

playing the role of a generalized viscosity; $\mathcal{D}(v)$ is the symmetric part of the velocity gradient ∇v.

Model (1.1), (1.2) is the limiting case for a class of shear-dependent fluids where the stress tensor $\mathcal{T}$ is given by

$$\mathcal{T} = \Theta_p(|\mathcal{D}(v)|^2)\mathcal{D}(v)$$

with

$$\Theta_p(|\mathcal{D}(v)|^2) \equiv (\mu_0 + |\mathcal{D}(v)|^2)^{-\frac{p}{2}}, \ \mu_0 \geqslant 0. \tag{1.3}$$

Notice that (1.3) reduces to (1.2) if $p = 1$ and only positive μ_0 are taken into account.

If (1.1) is combined with (1.3) (instead of (1.2)) and $p > 1$ then the global-in-time existence of a solution $v \in L^\infty(I; W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)) \cap L^p(I; W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2))$ is shown in [6], Chpt. 5, Th. 4.21. The proof strongly relies on the "two-dimensional" cancellation saying that for all v^{N} smooth, periodic and divergence-free we have

$$\int_\Omega v_k^{\mathrm{N}} \frac{\partial v_i^{\mathrm{N}}}{\partial x_k} \Delta v_i^{\mathrm{N}} \,\mathrm{d}x = 0, \qquad \Omega \equiv (0, L) \times (0, L). \tag{1.4}$$

Of course, if $p = 1$ then (1.4) holds too, however, the nonlinear elliptic operator brings much less information in comparison with the case $p = 1$. More precisely, if $p > 1$ then the second apriori estimates derived with help of (1.4) give

$$\int_0^T \int_\Omega (1 + |D(v^{\mathrm{N}})|^2)^{\frac{p-2}{2}} |D(\nabla v^{\mathrm{N}})|^2 \,\mathrm{d}x \,\mathrm{d}t \leqslant \mathrm{K}, \tag{1.5}$$

while the approximations v^{N} to the problem (1.1) with (1.2) (i.e. $p = 1$) satisfy

$$\int_0^T \int_\Omega (\mu_0 + |\mathcal{D}(v^{\mathrm{N}})|^2)^{-\frac{3}{2}} |\mathcal{D}(\nabla v^{\mathrm{N}})|^2 \,\mathrm{d}x \,\mathrm{d}t \leqslant \mathrm{K}\,. \tag{1.6}$$

In both cases, in addition to (1.5) or (1.6) we have at our disposal the estimate

$$\limsup_{t \in I} \|\nabla v^{\mathrm{N}}(t)\|_2^2 \leqslant \mathrm{K}\,. \tag{1.7}$$

Our goal is to show that (1.6) and (1.7) are sufficient to establish the global-in-time existence of a weak solution of (1.1), (1.2).

Let us recall that if the viscosity is omitted ($\Theta = 0$), then global existence of weak solutions to the evolutionary Euler equations is well known even for the Dirichlet problem[3], see [5], Chapter 4 for details and further references. Thus it is quite natural to expect that (1.1), (1.2) has a global-in-time solution as the viscosity should help.

However, our answer to this somehow natural expectation is not complete. First of all, we cannot include the case $\mu_0 = 0$ into our analysis since then (1.6) becomes singular. Secondly, we cannot extend the result to the Dirichlet problem since it is not clear at this moment if the second apriori estimates (1.6) and (1.7) hold.

The method of the proof is different from those used before when establishing the global-in-time existence of a solution to (1.1). Although our sample example yields a monotone operator, the only important ingrediences needed in the course of the proof are the estimates (1.6), (1.7) and the validity of the energy equality; the monotonicity is not used. The result of the below stated theorem could be therefore extended to a class of problems of the type (1.1) where the stress tensor $\mathcal{T}$ of the form $\mathcal{T}(\eta) = \Theta(|\eta|^2)\eta$ satisfies

$$\int_0^T \int_\Omega \frac{\partial}{\partial x_k} \mathcal{T}(\eta) \cdot \frac{\partial \eta}{\partial x_k} \, \mathrm{d}x \, \mathrm{d}t \geqslant C \int_0^T \int_\Omega (\mu_0 + |\eta|^2)^{-\frac{3}{2}} \, |\nabla \eta|^2 \, \mathrm{d}x \, \mathrm{d}t.$$

In order to keep this exposition simple we do not deal with such forms in what follows.

The interested reader can compare the approach used in the proof below with other methods applied to (1.1), namely[4]

1. the monotone operator theory combined with compactness arguments for v^{N}, see [3] and [4] ($p \geqslant \frac{3d+2}{d+2}$);
2. the regularity method providing the compactness of ∇v^{N}, see [6] and references quoted therein ($p > \frac{3d}{d+2}$ for $d = 3, 4$ and $p > 1$ if $d = 2$);
3. the method of the truncated test function combined with the strict monotonicity property of $\mathcal{T}$, see [1] ($p > \frac{2d+2}{d+2}$);
4. the construction of the global-in-time $C^{1,\alpha}$-solution, see [2] ($p > \frac{4}{3}$ if $d = 2$).

Concerning the data, we assume here

$$f \in L^2(I; W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)), \quad v_0 \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2). \tag{1.8}$$

As the reader could notice, we use the summation convention and the dependence of the constant $\mathrm{K} = \mathrm{K}(\|v_0\|^2_{1,2}, \int_0^T \|f\|^2_{1,2} \, \mathrm{d}t, T)$ on the data is not explicitly mentioned. We also use the standard notation for function spaces. By V_2 we denote the

[3] It means that $\Omega \subset \mathbb{R}^2$ is a bounded open set and the condition $v \cdot n = 0$ is prescribed at the boundary $\partial\Omega$; n is the outward normal.

[4] In the brackets, we mark the range of parameters p for which the method is known to be successfully applicable to (1.1), d denotes dimension.

space $\{u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2), u$ periodic , $\int_\Omega u \,\mathrm{d}x = 0, \operatorname{div} u = 0\}$, by V_2^* its dual, and the brackets $\langle \cdot, \cdot \rangle$ represent this duality. By $\mu \cdot \xi$ we mean $\mu_{ij}\xi_{ij}$, and $(\cdot, \cdot)$ denotes the scalar product in $L^2(\Omega)$.

2. Main Theorem and its proof

Theorem 2.1. *Let f, v_0 satisfy* (1.8). *Then there exists a weak solution v to* (1.1), (1.2) *such that*

$$\begin{aligned} &v \in L^\infty(I; V_2) \cap C(I; L^2_{\mathrm{loc}}(\mathbb{R}^2)), \\ &\frac{\partial v}{\partial t} \in L^2(I; V_2^*). \end{aligned} \tag{2.2}$$

P r o o f is split into several steps.

S t e p 1. *Galerkin aproximations.* Let $w^r \in V_2$, $r \in \mathbb{N}$, be the eigenvectors of $(\nabla w^r, \nabla \varphi) = \lambda_r (w^r, \varphi)$ for all $\varphi \in V_2$. Clearly w^r are smooth. Then the Galerkin aproximations v^{N} of the form

$$v^{\mathrm{N}}(t, x) = \sum_{i=1}^{\mathrm{N}} c_i^{\mathrm{N}}(t) w^i(x)$$

are determined as solutions of the system

$$\begin{aligned} &\Big(\frac{\mathrm{d}v^{\mathrm{N}}}{\mathrm{d}t}, w^r\Big) + \big(\Theta(|\mathcal{D}(v^{\mathrm{N}})|^2)\mathcal{D}(v^{\mathrm{N}}), \mathcal{D}(w^r)\big) \\ &\qquad\qquad + \big(v^{\mathrm{N}} \otimes v^{\mathrm{N}}, \nabla w^r\big) = (f, w^r) \quad (r = 1, 2, \ldots, \mathrm{N}), \\ &v^{\mathrm{N}}(0) = \mathbb{P}^{\mathrm{N}} v_0, \quad \mathbb{P}^{\mathrm{N}} v_0 \to v_0 \text{ strongly in } L^2(\Omega) \text{ as } \mathrm{N} \to \infty. \end{aligned} \tag{2.3}$$

Here $\mathbb{P}^{\mathrm{N}}$ denotes the projector from V_2 or $L^2(\Omega)$ into the space generated by the first N-eigenvectors.

S t e p 2. *Uniform apriori estimates.* The important apriori estimates are obtained by multiplying the r-th equation in (2.3) by $c_r^{\mathrm{N}}(t)\lambda_r$ (which is equivalent to "testing" by $-\Delta v^{\mathrm{N}}$). Thanks to (1.4), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v^{\mathrm{N}}\|_2^2 + \mu_0 \int_\Omega (\mu_0 + |\mathcal{D}(v^{\mathrm{N}})|^2)^{-\frac{3}{2}} |\mathcal{D}(\nabla v^{\mathrm{N}})|^2 \,\mathrm{d}x \leqslant \|\nabla f\|_2 \|\nabla v^{\mathrm{N}}\|_2.$$

Integration over time and Gronwall inequality imply then (1.6) and (1.7). From (1.6) we see

$$\int_0^T \|\nabla \Theta^{\frac{1}{2}}(|\mathcal{D}(v^{\mathrm{N}})|^2)\|_2^2 \leqslant \mathrm{K}, \tag{2.4}$$

and since $\nabla\Theta = \nabla(\Theta^{\frac12}\Theta^{\frac12}) = 2\Theta^{\frac12}\nabla\Theta^{\frac12}$ and $\Theta^{\frac12} \in L^\infty(I;L^\infty(\Omega))$ we also have

$$\int_0^T \|\Theta(|\mathcal{D}(v^{\mathrm{N}})|^2)\|_{1,2}^2 \,\mathrm{d}t \leqslant \mathrm{K}\,. \tag{2.5}$$

Finally, it follows from (2.3) and (1.7) (cf. [6], p. 230–231 if necessary) that[5]

$$\Big\|\frac{\mathrm{d}v^{\mathrm{N}}}{\mathrm{d}t}\Big\|_{L^2(I;V_2^*)} \leqslant \mathrm{K}\,. \tag{2.6}$$

Step 3. *First consequences of the uniform estimates.* By (1.7), (2.5) and (2.6) there are $\overline{\Theta} \in L^2(I;W^{1,2}(\Omega))$ and $v \in L^\infty(I;V_2)$ such that for a subsequence of $\{v^{\mathrm{N}}\}$ (denoted again by v^{N})

$$\begin{aligned} &\Theta(|\mathcal{D}(v^{\mathrm{N}})|^2) \rightharpoonup \overline{\Theta} \quad \text{weakly in } L^2(I;W^{1,2}(\Omega)),\\ &\Theta(|\mathcal{D}(v^{\mathrm{N}})|^2) \rightharpoonup \overline{\Theta} \quad *\text{-weakly in } L^\infty(I;L^\infty(\Omega)), \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\left.\begin{aligned} D(v^{\mathrm{N}}) &\rightharpoonup D(v)\\ \nabla v^{\mathrm{N}} &\rightharpoonup \nabla v \end{aligned}\right\} \quad *\text{-weakly in } L^\infty(I;W^{1,2}(\Omega)),\\ &\quad \frac{\mathrm{d}v^{\mathrm{N}}}{\mathrm{d}t} \rightharpoonup \frac{\partial v}{\partial t} \quad \text{weakly in } L^2(I;V_2^*), \end{aligned} \tag{2.8}$$

and due to the Aubin-Lions compactness lemma

$$\begin{aligned} &v^{\mathrm{N}} \to v \quad \text{strongly in} \quad L^q(I;L^q(\Omega)) \;\; \forall q \in [1,\infty),\\ &v^{\mathrm{N}} \to v \quad \text{strongly in} \quad C(I;L^2(\Omega)). \end{aligned} \tag{2.9}$$

Step 4. *A key consequence of apriori estimates.* We are going to show that (2.5), (2.7)–(2.9) imply

$$D(v^{\mathrm{N}})\Theta(|\mathcal{D}(v^{\mathrm{N}})|^2) \rightharpoonup D(v)\overline{\Theta} \quad \text{weakly in } L^2(I;L^2(\Omega)). \tag{2.10}$$

Indeed, it is enough to show that for fixed $r,s \in \{1,2\}$ and ψ smooth we have

$$Y \equiv \int_0^T \int_\Omega \Big(\frac{\partial v_r^{\mathrm{N}}}{\partial x_s}\Theta(|\mathcal{D}(v^{\mathrm{N}})|^2) - \frac{\partial v_r}{\partial x_s}\overline{\Theta}\Big)\psi \,\mathrm{d}x\,\mathrm{d}t \to 0 \text{ as } \mathrm{N}\to\infty. \tag{2.11}$$

[5] The estimate (2.6) is sufficient for our purpose, but it is certainly not optimal.

However,

$$
\begin{aligned}
Y &= \int_0^T \int_\Omega \Big(\frac{\partial v_r^{\mathrm N}}{\partial x_s} - \frac{\partial v_r}{\partial x_s}\Big) \Theta(|\mathcal{D}(v^{\mathrm N})|^2)\psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \frac{\partial v_r}{\partial x_s}(\Theta(|\mathcal{D}(v^{\mathrm N})|^2) - \overline{\Theta})\psi \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T \int_\Omega (v_r^{\mathrm N} - v_r)\frac{\partial \Theta(|\mathcal{D}(v^{\mathrm N})|^2)}{\partial x_s}\psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_\Omega (v_r^{\mathrm N} - v_r)\Theta(|\mathcal{D}(v^{\mathrm N})|^2)\frac{\partial \psi}{\partial x_s} \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \frac{\partial v_r}{\partial x_s}(\Theta(|\mathcal{D}(v^{\mathrm N})|^2) - \overline{\Theta})\psi \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

and we see that (2.11) holds due to (2.7), (2.9) and (2.5). Then (2.11) implies (2.10) by density arguments.

Step 5. We have

$$
\int_0^T \int_\Omega |\mathcal{D}(v^{\mathrm N})|^2 \Theta(|\mathcal{D}(v^{\mathrm N})|^2) \,\mathrm{d}x\,\mathrm{d}t \overset{\mathrm{N}\to\infty}{\longrightarrow} \int_0^T \int_\Omega |\mathcal{D}(v)|^2 \overline{\Theta} \,\mathrm{d}x\,\mathrm{d}t. \tag{2.12}
$$

To prove (2.12), we first multiply the r-th equation by $c_r^{\mathrm N}(t)$, sum over $r = 1, 2, \ldots, \mathrm N$ and integrate with respect to t. This leads to the identity

$$
\frac{1}{2}\|v^{\mathrm N}(T)\|_2^2 - \frac{1}{2}\|v^{\mathrm N}(0)\|_2^2 + \int_0^T \int_\Omega |\mathcal{D}(v^{\mathrm N})|^2 \Theta(|\mathcal{D}(v^{\mathrm N})|^2) \,\mathrm{d}x\,\mathrm{d}t = \int_0^T (f, v^{\mathrm N}) \,\mathrm{d}t.
$$

Because of (2.9) and $(2.3)_2$ we observe that

$$
\int_0^T \int_\Omega |\mathcal{D}(v^{\mathrm N})|^2 \Theta(|\mathcal{D}(v^{\mathrm N})|^2) \,\mathrm{d}x\,\mathrm{d}t \to -\frac{1}{2}\|v(T)\|_2^2 + \frac{1}{2}\|v_0\|_2^2 + \int_0^T (f, v) \,\mathrm{d}t. \tag{2.13}
$$

On the other hand, the passage to the limit in (2.3) with help of (2.8)–(2.10) gives

$$
\begin{aligned}
&\int_0^T \Big\langle \frac{\partial v}{\partial t}, w^r \Big\rangle \,\mathrm{d}t + \int_0^T \int_\Omega \overline{\Theta}\, \mathcal{D}(v) \cdot \mathcal{D}(w^r) \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^T \int_\Omega v_k \frac{\partial v_i}{\partial x_k} w_i^r \,\mathrm{d}x\,\mathrm{d}t = \int_0^T (f, w^r) \quad \forall r \in \mathbb{N}.
\end{aligned} \tag{2.14}
$$

Using the density arguments we can conclude that (2.14) holds also for all $\varphi \in L^2(I; V_2)$ (instead of w^r). Taking in particular $\varphi = v$, we see that

$$
\frac{1}{2}\|v(T)\|_2^2 - \frac{1}{2}\|v_0\|_2^2 + \int_0^T \int_\Omega \overline{\Theta}|\mathcal{D}(v)|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_0^T (f, v) \,\mathrm{d}t. \tag{2.15}
$$

Assertion (2.12) thus follows from (2.15) and (2.13).

Step 6. *Strong convergence of* $\mathcal{D}(v^{\mathrm{N}})$. Let us first check that

$$J \equiv \int_0^T \int_\Omega |\mathcal{D}(v^{\mathrm{N}}) - \mathcal{D}(v)|^2 \Theta(|\mathcal{D}(v^{\mathrm{N}})|^2)\,\mathrm{d}x\,\mathrm{d}t \overset{\mathrm{N}\to\infty}{\longrightarrow} 0. \tag{2.16}$$

Indeed,

$$\begin{aligned} J = &\int_0^T \int_\Omega |\mathcal{D}(v^{\mathrm{N}})|^2 \Theta(|\mathcal{D}(v^{\mathrm{N}})|^2)\,\mathrm{d}x\,\mathrm{d}t \\ &- 2\int_0^T \int_\Omega \Theta(|\mathcal{D}(v^{\mathrm{N}})|^2)\mathcal{D}(v^{\mathrm{N}}) \cdot \mathcal{D}(v)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \int_\Omega |\mathcal{D}(v)|^2 \Theta(|D(v^{\mathrm{N}})|^2)\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

and (2.16) follows due to (2.12), (2.10) and $(2.7)_2$.

Next, by the Hölder inequality

$$\begin{aligned} \|\mathcal{D}(v^{\mathrm{N}} - v)\|_{2\beta}^{2\beta} &= \int_\Omega \{|\mathcal{D}(v^{\mathrm{N}} - v)|^2 \Theta(|\mathcal{D}(v^{\mathrm{N}})|^2)\}^\beta \frac{1}{\Theta(|\mathcal{D}(v^{\mathrm{N}})|^2)^\beta}\,\mathrm{d}x \\ &\leqslant \Big(\int_\Omega |\mathcal{D}(v^{\mathrm{N}} - v)|^2 \Theta(|\mathcal{D}(v^{\mathrm{N}})|^2)\,\mathrm{d}x\Big)^\beta \left(\int_\Omega (1 + |\nabla v^{\mathrm{N}}|^2)^{\frac{\beta}{2(1-\beta)}}\right)^{1-\beta}. \end{aligned}$$

Since $\frac{\beta}{2(1-\beta)} = 1$ if $\beta = \frac{2}{3}$, (1.7) implies

$$\|\mathcal{D}(v^{\mathrm{N}} - v)\|_{\frac{4}{3}}^2 \leqslant \mathrm{K} \int_\Omega |\mathcal{D}(v^{\mathrm{N}}) - \mathcal{D}(v)|^2 \Theta(|\mathcal{D}(v^{\mathrm{N}})|^2)\,\mathrm{d}x.$$

Thus, integration with respect to time and (2.16) give

$$\mathcal{D}(v^{\mathrm{N}}) \to \mathcal{D}(v) \quad \text{strongly in } L^2(I; L^{\frac{4}{3}}(\Omega)). \tag{2.17}$$

Step 7. To prove that v is a solution of (1.1), (1.2) with help of (2.17) is now straightforward. The proof of Theorem 2.1 is complete. □

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