ON THE OSCILLATION OF CERTAIN DIFFERENCE EQUATIONS

S. R. Grace, Giza, H. A. El-Morshedy, New Damietta

(Received September 25, 1998)

Abstract. In this paper we study the oscillation of the difference equations of the form

$$\Delta^2 x_n + p_n \Delta x_n + f(n,\, x_{n-g},\, \Delta x_{n-h}) = 0,$$

in comparison with certain difference equations of order one whose oscillatory character is known. The results can be applied to the difference equation

$$\Delta^2 x_n + p_n \Delta x_n + q_n |x_{n-g}|^{\lambda} |\Delta x_{n-h}|^{\mu} \operatorname{sgn} x_{n-g} = 0,$$

where λ and μ are real constants, $\lambda > 0$ and $\mu \geqslant 0$.

Keywords: oscillation, delay difference equations, forced equations

MSC 2000: 39A10, 39A12

1. Introduction

Consider the difference equation

$$\Delta^2 x_n + p_n \Delta x_n + f(n,\, x_{n-g},\, \Delta x_{n-h}) = 0, \tag{E}$$

where $\{p_n\}$ is a nonnegative real sequence, $0 \leqslant p_n < 1$, $f\colon\ \mathbb{N} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous for each fixed n, $\mathbb{N} = \{0, 1, 2, \ldots\}$; g and h are in $\mathbb{N}$, Δ is the first order forward difference operator, $\Delta x_n = x_{n+1} - x_n$.

We assume that there exist an eventually positive real sequence $\{q_n\}$ and real numbers $\lambda > 0$ and $\mu \geqslant 0$ such that

$$f(n,\, x,\, y) \operatorname{sgn} x \geqslant q_n |x|^{\lambda} |y|^{\mu} \qquad \text{for} \quad n \geqslant 0 \quad \text{and} \quad x\, y \neq 0. \tag{1}$$

By a solution of Eq. (E), we mean a non-constant sequence $\{x_n\}$ satisfying (E) for $n \geqslant 0$. A solution $\{x_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

In recent years there has been an increasing interest in studying the oscillatory behavior of difference equations of special cases of type (E) when $p_n \equiv 0$ and condition (1) holds with $\mu = 0$. For recent contributions to this study we refer to the papers [4]–[9] and the references cited therein. It seems that very little is known regarding the oscillation of Eq. (E) when f satisfies condition (1) with $\mu \neq 0$ and $p_n \neq 0$. Therefore, the purpose of this paper is to present some new criteria for the oscillation of Eq. (E). Theorems 1 and 2 are concerned with the oscillation of Eq. (E) via its comparison with the oscillatory character of first order difference equations. Theorem 3 deals with the oscillation of a special case of Eq. (E) when condition (1) holds with $\lambda = 1$ and $\mu = 0$ and the condition on $\{p_n\}$ introduced in Theorems 1 and 2 is not required or else violated, and Theorems 4 and 5 are concerned with the oscillatory behavior of the difference of two eventually positive solutions of the difference equation

$$(\mathrm{L}_e) \qquad \Delta^2 x_n + p_n \Delta x_n + q_n g(x_{n-g}) = e_n,$$

where $g(x)x > 0$ for $x \neq 0$, $g'(x) \geqslant k$ and $\{e_n\}$ is a sequence of real numbers. Finally, we remark that this paper is motivated by the analogy between functional differential equations of the form

$$(\mathrm{E}_c) \qquad \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + p(t)\frac{\mathrm{d}x(t)}{\mathrm{d}t} + f\Big(t, x(t-g), \frac{\mathrm{d}x(t-h)}{\mathrm{d}t}\Big) = 0,$$

where $p\colon [t_0,\infty) \to [0,\infty)$ and $f\colon [t_0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and g and h are real constants, and difference equations of type (E). In fact, discrete versions of some of the results in [1]–[3] for second order equations have been developed.

2. Preliminaries

We need the following two lemmas. The first is extracted from Lemma 5 in [8] and the other is Theorem 7.5.1 in [6].

Lemma 1. *Assume $h\colon \mathbb{R} \to \mathbb{R}$ is continuous, $xh(x) > 0$ and $h(x)$ is nondecreasing for $x \neq 0$. Let $\{q_n\}$ be a sequence of nonnegative real numbers and k a positive integer. If the difference inequality*

$$\Delta x_n + q_n h(x_{n-k}) \leqslant 0$$

has an eventually positive solution, then the difference equation

$$\Delta x_n + q_n h(x_{n-k}) = 0$$

has an eventually positive solution.

Lemma 2. *Suppose that $\{a_n\}$ is a nonnegative sequence of real numbers and let k be a positive integer. Then*

$$\liminf_{n\to\infty} \left[\frac{1}{k}\sum_{i=n-k}^{n-1} a_i\right] > \frac{k^k}{(1+k)^{1+k}}$$

is a sufficient condition for every solution of the equation

$$\Delta x_n + a_n x_{n-k} = 0$$

to be oscillatory.

3. Main results

Now, we are ready to establish the following criterion for the oscillation of Eq. (E):

Theorem 1. *Let condition* (1) *hold, let*

$$\lim_{n\to\infty} \sum_{k=n_0\geqslant 0}^{n-1} \left(\prod_{i=n_0}^{k-1} (1-p_i)\right) = \infty \tag{2}$$

and

$$\sum_{i=n+1}^{n+h} q_i > 0 \qquad \text{for sufficiently large } n. \tag{3}$$

If for every $\nu > 0$ the equation

$$\Delta w_n + \nu q_n |w_{n-h}|^{\mu} \operatorname{sgn} w_{n-h} = 0, \tag{4}$$

is oscillatory, then Eq. (E) *is oscillatory.*

P r o o f. Let $\{x_n\}$ be a nonoscillatory solution of Eq. (E), say $x_n > 0$ for $n \geqslant n_0 \geqslant 1$. First, we claim that $\{\Delta x_n\}$ is eventually of one sign. To this end, we assume that $\{\Delta x_n\}$ is oscillatory. There exists $N \geqslant n_0 + \max\{h, g\}$ such that $\Delta x_N < 0$. Let $n = N$ in Eq. (E) and then multiply the resulting equation by Δx_N to obtain

$$\begin{aligned}\Delta^2 x_N \Delta x_N &= -p_N(\Delta x_N)^2 - f(N,\ x_{N-g},\ \Delta x_{N-h})\Delta x_N \\ &\geqslant -p_N(\Delta x_N)^2\end{aligned}$$

or

$$\Delta x_{N+1}\Delta x_N \geqslant (1-p_N)(\Delta x_N)^2 > 0,$$

which implies that

$$\Delta x_{N+1} < 0$$

By induction, we obtain $\Delta x_n < 0$ for $n \geqslant N$, contradicting the assumption that $\{\Delta x_n\}$ is oscillatory.

Next, suppose there exists $N_1 \geqslant n_0 + \max\{h, g\}$ such that $\Delta x_{N_1} = 0$. Then setting $n = N_1$ in Eq. (E) leads to

$$\Delta^2 x_{N_1} = -f(N_1,\ x_{N_1-g},\ \Delta x_{N_1-h}) \leqslant 0,$$

which implies that

$$\Delta x_{N_1+1} \leqslant \Delta x_{N_1} = 0.$$

As in the above case, we have seen that this contradicts the assumption that $\{\Delta x_n\}$ is oscillatory.

Now, we consider the following two cases:

(I) $\Delta x_n < 0$ eventually, (II) $\Delta x_n > 0$ eventually.

(I) Suppose that $\Delta x_n < 0$ for $n \geqslant n_1 \geqslant \max\{N,\ N_1\}$. From Eq. (E) it follows that

$$\Delta^2 x_n + p_n \Delta x_n \leqslant 0 \quad \text{for} \quad n \geqslant n_1.$$

Set $z_n = -\Delta x_n$ for $n \geqslant n_1$. Then

$$\Delta z_n + p_n z_n \geqslant 0$$

or

$$z_{n+1} \geqslant (1-p_n)z_n \geqslant \prod_{i=n_1}^{n-1}(1-p_i)z_{n_1},$$

where z_{n_1} is an arbitrary constant. Thus,

$$-\Delta x_k \geqslant \prod_{i=n_1}^{k-1}(1-p_i)z_{n_1}.$$

Summing this inequality from n_1 to $n-1$, we get

$$x_{n_1} - x_n \geqslant z_{n_1} \sum_{k \geqslant n_1}^{n-1} \Big(\prod_{i=n_1}^{k-1} (1-p_i) \Big) \to \infty \quad \text{as} \quad n \to \infty,$$

which is a contradiction. Next, we consider the other case
(II) Suppose that $\Delta x_n > 0$ for $n \geqslant n_1 \geqslant \max\{N,\ N_1\}$. There exist $n_2 \geqslant n_1$ and $\alpha > 0$ such that

$$x_{n-g} \geqslant \alpha \quad \text{for} \quad n \geqslant n_2. \tag{5}$$

Using conditions (1) and (5) in Eq. (E) we obtain

$$\Delta^2 x_n + \alpha^\lambda q_n (\Delta x_{n-h})^\mu \leqslant 0 \quad \text{for} \quad n \geqslant n_2. \tag{6}$$

Set $z_n = \Delta x_n$, $n \geqslant n_2$. Then (6) assumes the form

$$\Delta z_n + \alpha^\lambda q_n (z_{n-h})^\mu \leqslant 0, \qquad n \geqslant n_2.$$

Therefore, by Lemma 1, Eq. (4) has an eventually positive solution, which is a contradiction. This completes the proof. □

Next, we present an oscillation theorem for Eq. (E).

Theorem 2. *Let conditions* (1) *and* (2) *hold and let*

$$\sum_{i=n+1}^{n+\tau} q_i > 0 \qquad \textit{for all sufficiently large } n, \tag{7}$$

where $\tau = \min\{g,\ h\}$. *If the equation*

$$\Delta V_n + \Big(\frac{n-g}{2}\Big)^\lambda q_n |V_{n-\tau}|^{\lambda+\mu} \operatorname{sgn} V_{n-\tau} = 0 \tag{8}$$

is oscillatory, then Eq. (E) *is oscillatory.*

P r o o f. Let $\{x_n\}$ be a nonoscillatory solution of Eq. (E), say $x_n > 0$ for $n \geqslant n_0 \geqslant 1$. As in the proof of Theorem 1, we see that $\{\Delta x_n\}$ is eventually of one sign and case (I) is impossible. Next, we consider
Case (II). Suppose that $\Delta x_n > 0$ for $n \geqslant n_1 \geqslant n_0$. From the fact that Δx_n is nonincreasing, we see that

$$x_n - x_{n_1} = \sum_{k=n_1}^{n-1} \Delta x_k \geqslant (n-n_1)\Delta x_{n-1},$$

which implies that

$$x_n \geqslant \frac{n}{2}\Delta x_n \quad \text{for} \quad n \geqslant n_2 \geqslant 2n_1 + 1.$$

Then

$$x_{n-g} \geqslant \left(\frac{n-g}{2}\right)\Delta x_{n-g} \geqslant \left(\frac{n-g}{2}\right)\Delta x_{n-\tau} \quad \text{for} \quad n \geqslant n_2 + g. \tag{9}$$

Using conditions (1) and (9) in Eq. (E) yields

$$\Delta y_n + \left(\frac{n-g}{2}\right)^{\lambda} q_n |y_{n-\tau}|^{\lambda+\mu} \leqslant 0 \quad \text{for} \quad n \geqslant n_2 + g,$$

where $y_n = \Delta x_n$, $n \geqslant n_2 + g$. The rest of the proof is similar to that of Theorem 1 (II) and hence is omitted. □

As an application, we apply Lemma 2 to the equations (4) and (8) appearing in Theorems 1 and 2 respectively and obtain the following immediate corollaries:

Corollary 1. *Let conditions* (1)–(3) *hold. If*

(i) *for every constants* $\nu > 0$, $h > 1$ *we have*

$$\liminf_{n\to\infty} \left[\frac{\nu}{h}\sum_{i=n-h}^{n-1} q_i\right] > \frac{h^h}{(1+h)^{1+h}} \qquad \textit{when} \quad \mu = 1 \quad \textit{and} \quad \lambda > 0$$

or

(ii)

$$\sum^{\infty} q_i = \infty \qquad \textit{when} \quad 0 < \mu < 1 \quad \textit{and} \quad \lambda > 0,$$

then Eq. (E) *is oscillatory.*

Corollary 2. *Let conditions* (1), (2) *and* (7) *hold. If*

(i) $\tau = \min\{g,\ h\} > 1$, *and*

$$\liminf_{n\to\infty} \left[\frac{1}{\tau}\sum_{i=n-\tau}^{n-1} \left(\frac{i-g}{2}\right)^{\lambda} q_i\right] > \frac{\tau^{\tau}}{(1+\tau)^{1+\tau}} \qquad \textit{when} \quad \lambda + \mu = 1$$

or

(ii)

$$\sum^{\infty} \left(\frac{i-g}{2}\right)^{\lambda} q_i = \infty \qquad \textit{when} \quad 0 < \lambda + \mu < 1,$$

then Eq. (E) *is oscillatory.*

The following theorem is concerned with the oscillation of a special case of the equation

$$\Delta^2 x_n + p_n \Delta x_n + q_n |x_{n-g}|^\lambda |\Delta x_{n-h}|^\mu \operatorname{sgn} x_{n-g} = 0$$

when $\mu = 0$ and $\lambda = 1$, namely, the linear difference equation

$$\text{(L)} \qquad \Delta^2 x_n + p_n \Delta x_n + q_n x_{n-g} = 0,$$

provided condition (2) is not required.

Theorem 3. *Let $\Delta p_n \leqslant 0$ for $n \geqslant n_0 \geqslant 0$, $g > 1$, and*

$$\text{(10)} \qquad \sum_{i=n+1}^{n+g} Q_i > 0 \quad \text{and} \quad \sum_{i=n+1}^{n+g} (i-g) q_i > 0 \qquad \text{for all large } n,$$

where $Q_n = \left(\sum\limits_{i=n-g}^{n-1} q_i \right) - p_{n-g}$. If the equation

$$\text{(11)} \qquad \Delta z_n + c_n z_{n-g} = 0,$$

where

$$\text{(12)} \qquad c_n = \min \left\{ Q_n,\ \frac{n-g}{2} q_n \right\},$$

is oscillatory, then Eq. (L) is oscillatory.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of Eq. (L), say $x_n > 0$ for $n \geqslant n_0 \geqslant 1$. As in the proof of Theorem 1, we see that $\{\Delta x_n\}$ is eventually of one sign. Next we consider the two cases (I) and (II) as in Theorem 1.

(I) Suppose that $\Delta x_n < 0$ for $n \geqslant n_1 \geqslant n_0$. Summing both sides of Eq. (L) from $n-g$ to $n-1$, we obtain

$$\Delta x_n - \Delta x_{n-g} + \sum_{i=n-g}^{n-1} p_i \Delta x_i + \sum_{i=n-g}^{n-1} q_i x_{i-g} = 0,$$

or

$$\Delta x_n + \left[p_n x_n - p_{n-g} x_{n-g} - \sum_{i=n-g}^{n-1} x_{i+1} \Delta p_i \right] + x_{n-g} \sum_{i=n-g}^{n-1} q_i \leqslant 0 \quad \text{for} \quad n \geqslant n_1.$$

Since $\Delta p_n \leqslant 0$, we have

$$\Delta x_n + \left[\left(\sum_{i=n-g}^{n-1} q_i \right) - p_{n-g} \right] x_{n-g} \leqslant 0, \qquad n \geqslant n_1,$$

and hence, by (12), we get

$$\Delta x_n + c_n x_{n-g} \leqslant 0, \qquad n \geqslant n_1.$$

The rest of the proof is similar to that of Theorem 1 Case II, and hence will be omitted.

(II) Suppose that $\Delta x_n > 0$ for $n \geqslant n_1 \geqslant n_0$. Then Eq. (L) assumes the form

$$\Delta^2 x_n + q_n x_{n-g} \leqslant 0, \qquad n \geqslant n_1. \tag{13}$$

As in the proof of Theorem 2 Case II, there exists an $n_2 \geqslant n_1$ such that (9) holds for $n \geqslant n_2$. Using (9) in (13), we have

$$\Delta y_n + c_n y_{n-g} \leqslant \Delta y_n + \Big(\frac{n-g}{2}\Big) q_n y_{n-g} \leqslant 0 \quad \text{for} \quad n \geqslant n_2$$

where $y_n = \Delta x_n$, $n \geqslant n_2$. The rest of the proof is similar to the proof of the above case and hence is omitted. This completes the proof. $\square$

Finally, we present results for the forced difference equations of the form (L_e).

Theorem 4. *Let the conditions of Theorem 3 hold with q_n being replaced by $k\,q_n$. If $\{u_n\}$ and $\{v_n\}$ are eventually positive solutions of Eq. (L_e), then $\{u_n - v_n\}$ is oscillatory.*

Proof. Let $\{u_n\}$ and $\{v_n\}$ be two positive solutions of Eq. (L_e) for $n \geqslant n_0 \geqslant 1$, and let $w_n = u_n - v_n$ for $n \geqslant n_0$. From Eq. (L_e) we can obtain

$$\Delta^2 w_n + p_n \Delta w_n + q_n \left[g(u_{n-g}) - g(v_{n-g})\right] = 0.$$

To show that $\{w_n\}$ is oscillatory we will assume that $\{w_n\}$ is eventually positive. The negative case follows analogously.

So, let us suppose that $w_n > 0$ for $n \geqslant n_0 \geqslant 1$. The Mean Value Theorem implies that

$$\Delta^2 w_n + p_n \Delta w_n + k\; q_n \Delta w_{n-g} \leqslant 0.$$

The rest of the proof is similar to that of Theorem 3 and hence we omit the details. $\square$

In the case when condition (2) is satisfied, we have the following immediate result.

Theorem 5. *Let condition (2) hold and assume that Eq. (8) is oscillatory for $\lambda = 1$, $\mu = 0$, $g = \tau$ and q_n is replaced by $k\,q_n$. If $\{u_n\}$ and $\{v_n\}$ are two eventually positive solutions of Eq. (L_e), then $\{u_n - v_n\}$ is oscillatory.*

Proof. The proof of this theorem follows the lines of proofs of Theorems 4, 3 and 1, and hence is omitted. $\square$

R e m a r k 1. The results of this paper remain valid when $p_n \equiv 0$. On the other hand, if $p_n \equiv p$ is a positive constant, the series in condition (2) is a convergent geometric series and hence condition (2) is violated. In this case we are (only) able to describe the oscillatory behavior of the linear difference equation (L) which is a special case of Eq. (E).

As an application, we present the following criteria for the oscillation of Eq. (L) when $\{p_n\}$ and $\{q_n\}$ are constant sequences, i.e., for the difference equation

$$\Delta^2 x_n + p\Delta x_n + q x_{n-g} = 0 \tag{L_c}$$

where $p \geqslant 0$ and $q > 0$ are real constants, $p < 1$ and g is a positive integer, $g > 1$.

Corollary 3. *If*

$$g\,q - p > \frac{g^g}{(1+g)^{1+g}}, \tag{14}$$

then Eq. (L_c) *is oscillatory.*

Corollary 4. *If condition* (14) *holds,* $\{u_n\}$ *and* $\{v_n\}$ *are two eventually positive solutions of Eq.* (L_c)*, then* $\{u_n - v_n\}$ *is oscillatory.*

R e m a r k 2. From Corollary 3 we see that the characteristic equation associated with Eq. (L_c), namely

$$(m-1)^2 + p\,(m-1) + q\,m^{-g} = 0 \tag{15}$$

has no positive roots provided that condition (14) holds.

R e m a r k 3. It would be interesting to obtain results similar to Theorems 1 and 2 without imposing condition (2). Also, to extend Theorems 3–5 to more general equations of type (E).

A c k n o w l e d g m e n t. The authors are very grateful to the referee for his valuable suggestions.

References

[1] *S. R. Grace*: Oscillatory and asymptotic behavior of delay differential equations with a nonlinear damping term. J. Math. Anal. Appl. *168* (1992), 306–318.

[2] *S. R. Grace*: Oscillation theorems of comparison type of delay differential equations with a nonlinear damping term. Math. Slovaca *44* (1994), no. 3, 303–314.

[3] *S. R. Grace, B. S. Lalli*: An oscillation criterion for certain second order strongly sublinear differential equations. J. Math. Anal. Appl. *123* (1987), 584–586.

[4] *S. R. Grace, B. S. Lalli*: Oscillation theorems for second order delay and neutral difference equations. Utilitas. Math. *45* (1994), 197–211.

[5] *S. R. Grace, B. S. Lalli*: Oscillation theorems for forced neutral difference equations. J. Math. Anal. Appl. *187* (1994), 91–106.

[6] *I. Györi, G. Ladas*: Oscillation Theory of Delay Differential Equations with Applications. Oxford Univ., Oxford, 1991.

[7] *J. W. Hooker, W. T. Patula*: A second order nonlinear difference equations: Oscillation and asymptotic behavior. J. Math. Anal. Appl. *91* (1983), 9–29.

[8] *G. Ladas, C. Qian*: Comparison results and linearized oscillations for higher order difference equations. Internt. J. Math. Math. Sci. *15* (1992), 129–142.

[9] *B. G. Zhang*: Oscillation and asymptotic behavior of second order difference equations. J. Math. Anal. Appl. *173* (1993), 58–68.

Authors' addresses: *S. R. Grace*, Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Orman, Giza 12000, Egypt; *H. A. El-Morshedy*, Department of Mathematics, Damietta Faculty of Science, New Damietta 34517, Egypt.