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SOME NEW SCALES OF CHARACTERIZATION OF HARDY'S INEQUALITY

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ABSTRACT. Let $1 < p \le q < \infty$. Inspired by some recent results concerning Hardy type inequalities where the equivalence of four scales of integral conditions was proved, we use related ideas to find ten new equivalence scales of integral conditions. By applying our result to the original Hardy type inequality situation we obtain a new proof of a number of characterizations of the Hardy inequality and obtain also some new weight characterizations.

1. INTRODUCTION

We consider the general one-dimensional Hardy inequality

$$\left(\int_0^b \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{1/q} \le C\left(\int_0^b f^p(x)v(x)dx\right)^{1/p} \tag{1.1}$$

with a fixed b, $0 < b \le \infty$, for measurable functions $f \ge 0$,weights u and v and for the parameters p, q satisfying

$$1 < p \le q < \infty.$$

The inequality (1.1) is usually characterized by the (Muckenhoupt) condition

$$A_1 := \sup_{0<x<b} A_M(x) < \infty, \tag{1.2}$$

where

$$A_M(x) := \left(\int_x^b u(t)dt\right)^{1/q}\left(\int_0^x v^{1-p'}(t)dt\right)^{1/p'}. \tag{1.3}$$

Here and in the sequel $p' = p/(p-1)$. Further, let us denote

$$U(x) := \int_x^b u(t)dt, \qquad V(x) := \int_0^x v^{1-p'}(t)dt, \tag{1.4}$$

and assume that $U(x) < \infty$, $V(x) < \infty$ for every $x \in (0, b)$.

In [2] the equivalence of four scales of integral conditions that characterize the inequality (1.1) (with the usual Muckenhoupt condition as a special case) was proved. The proof was carried out by first proving an equivalence theorem

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of independent interest. We will here extend this theorem by finding some additional new scales of conditions.

As it was shown in [6], [2], [8] and [3], the validity of Hardy's inequality (1.1) for all functions $f \geq 0$ in fact can be characterized e.g. by prescribing that any of the following expressions is finite:

(1.5)

$$
\begin{aligned}
A_M &:= \sup_{0<x<b} U^{1/q}(x)V^{1/p'}(x);\\
A_{PS} &:= \sup_{0<x<b} \left(\int_0^x u(t)V^q(t)dt\right)^{1/q} V^{-1/p}(x);\\
A_W(r) &:= \sup_{0<x<b} \left(\int_x^b u(t)V^{q(p-r)/p}(t)dt\right)^{1/q} V^{(r-1)/p}(x), \quad 1<r<p;\\
A^*_{PS} &:= \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t)U^{p'}(t)dt\right)^{1/p'} U^{-1/q'}(x);\\
A^*_W(r) &:= \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t)U^{p'(q'-r)/q'}(t)dt\right)^{1/p'} U^{(r-1)/q'}(x), \quad 1<r<q';\\
A_T &:= \inf_{h>0}\sup_{0<x<b} \left(\frac{1}{h(x)}\int_0^x u(t)(h(t)+V(t))^{\frac{q}{p'}+1}dt\right)^{1/q};\\
A^*_T &:= \inf_{h>0}\sup_{0<x<b} \left(\frac{1}{h(x)}\int_x^b v^{1-p'}(t)(h(t)+U(t))^{\frac{p'}{q}+1}dt\right)^{1/p'}.
\end{aligned}
$$

Here, we will extend this list.

The paper is organized as follows: In Section 2 we formulate an equivalence theorem of independent interest, and in Section 3 we use this equivalence theorem to describe some new scales of weight characterization of the Hardy inequality. The main result is formulated in Theorem 3.1, which includes the results mentioned in (1.5) but gives also ten new weight characterizations. In Section 4 we give some outlines of the proof of the equivalence theorem (Theorem 2.1), whose detailed proof can be found in the research note [1].

2. The equivalence theorem

Theorem 2.1. *For $-\infty \leq a < b \leq \infty$, α, β and s positive numbers and f, g, h measurable functions positive a.e. in (a,b), let us denote*

$$F(x) := \int_x^b f(t)dt, \qquad G(x) := \int_a^x g(t)dt \tag{2.1}$$

and

$$
(2.2)\quad
\begin{aligned}
&B_1(x;\alpha,\beta) &&:= F^{\alpha}(x)G^{\beta}(x);\\
&B_2(x;\alpha,\beta,s) &&:= \left(\int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)dt\right)^{\alpha} G^{s}(x);\\
&B_3(x;\alpha,\beta,s) &&:= \left(\int_a^x g(t)F^{\frac{\alpha-s}{\beta}}(t)dt\right)^{\beta} F^{s}(x);\\
&B_4(x;\alpha,\beta,s) &&:= \left(\int_a^x f(t)G^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\alpha} G^{-s}(x);\\
&B_5(x;\alpha,\beta,s) &&:= \left(\int_x^b g(t)F^{\frac{\alpha+s}{\beta}}(t)dt\right)^{\beta} F^{-s}(x);\\
&B_6(x;\alpha,\beta,s) &&:= \left(\int_x^b f(t)G^{\frac{\beta}{\alpha+s}}(t)dt\right)^{\alpha+s} F^{-s}(x);\\
&B_7(x;\alpha,\beta,s) &&:= \left(\int_a^x g(t)F^{\frac{\alpha}{\beta+s}}(t)dt\right)^{\beta+s} G^{-s}(x);\\
&B_8(x;\alpha,\beta,s) &&:= \left(\int_a^x f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^{s}(x), \quad \alpha>s;\\
&B_9(x;\alpha,\beta,s) &&:= \left(\int_x^b f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^{s}(x), \quad \alpha<s;\\
&B_{10}(x;\alpha,\beta,s) &&:= \left(\int_x^b g(t)F^{\frac{\alpha}{\beta-s}}(t)dt\right)^{\beta-s} G^{s}(x), \quad \beta>s;\\
&B_{11}(x;\alpha,\beta,s) &&:= \left(\int_a^x g(t)F^{\frac{\alpha}{\beta-s}}(t)dt\right)^{\beta-s} G^{s}(x), \quad \beta<s;\\
&B_{12}(x;\alpha,\beta,s;h) &&:= \left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{\alpha} (h(x)+G(x))^{s}, \quad \beta<s;\\
&B_{13}(x;\alpha,\beta,s;h) &&:= \left(\int_a^x g(t)h^{\frac{\alpha-s}{\beta}}(t)dt\right)^{\beta} (h(x)+F(x))^{s}, \quad \alpha<s;\\
&B_{14}(x;\alpha,\beta,s;h) &&:= \left(\int_a^x f(t)(h(t)+G(t))^{\frac{\beta+s}{\alpha}}dt\right)^{\alpha} h^{-s}(x);\\
&B_{15}(x;\alpha,\beta,s;h) &&:= \left(\int_x^b g(t)(h(t)+F(t))^{\frac{\alpha+s}{\beta}}dt\right)^{\beta} h^{-s}(x).
\end{aligned}
$$

The numbers $B_1(\alpha,\beta) := \sup\limits_{a<x<b} B_1(x;\alpha,\beta)$, $B_i(\alpha,\beta,s) = \sup\limits_{a<x<b} B_i(x;\alpha,\beta,s)$ $(i=2,3,\dots,11)$ *and* $B_i(\alpha,\beta,s) = \inf\limits_{h\geq 0}\sup\limits_{a<x<b} B_i(x;\alpha,\beta,s;h)$ $(i=12,13,14,15)$ *are mutually equivalent. The constants in the equivalence relations can depend on* α,β *and* s.

Remark 2.1. *The proof of Theorem 2.1 (see* [1] *and Section 4)is carried out by determining positive constants* c_i *and* d_i *so that*

$$
c_iB_i(\alpha,\beta,s) \leq B_1(\alpha,\beta) \leq d_iB_i(\alpha,\beta,s),\ i=2,3,\dots,15. \tag{2.3}
$$

3. The main result

Theorem 3.1. *Let* $1<p\leq q<\infty$,$0<s<\infty$, *and define, for the weight functions* u, v, *the functions* U *and* V *by* (1.4), *and the functions* $A_i(s)$, $i=$

$1, 2, \ldots, 15$, *as follows*

(3.1)

$$\begin{aligned}
A_1(s) \quad &:= \sup_{0<x<b} \left(\int_x^b u(t) V^{q(\frac{1}{p'}-s)}(t)dt\right)^{1/q} V^s(x), ;\\
A_2(s) \quad &:= \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t) U^{p'(\frac{1}{q}-s)}(t)dt\right)^{1/p'} U^s(x);\\
A_3(s) \quad &:= \sup_{0<x<b} \left(\int_0^x u(t) V^{q(\frac{1}{p'}+s)}(t)dt\right)^{1/q} V^{-s}(x);\\
A_4(s) \quad &:= \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t) U^{p'(\frac{1}{q}+s)}(t)dt\right)^{1/p'} U^{-s}(x);\\
A_5(s) \quad &:= \sup_{0<x<b} \left(\int_x^b u(t) V^{\frac{q}{p'(1+sq)}}(t)dt\right)^{\frac{1+sq}{q}} U^{-s}(x);\\
A_6(s) \quad &:= \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t) U^{\frac{p'}{q(1+sp')}}(t)dt\right)^{\frac{1+sp'}{p'}} V^{-s}(x);\\
A_7(s) \quad &:= \sup_{0<x<b} \left(\int_0^x u(t) V^{\frac{q}{p'(1-sq)}}(t)dt\right)^{\frac{1-sq}{q}} U^s(x), \quad qs<1;\\
A_8(s) \quad &:= \sup_{0<x<b} \left(\int_x^b u(t) V^{\frac{q}{p'(1-sq)}}(t)dt\right)^{\frac{1-sq}{q}} U^s(x), \quad qs>1;\\
A_9(s) \quad &:= \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t) U^{\frac{p'}{q(1-sp')}}(t)dt\right)^{\frac{1-sp'}{p'}} V^s(x), \quad p's<1;\\
A_{10}(s) \quad &:= \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t) U^{\frac{p'}{q(1-sp')}}(t)dt\right)^{\frac{1-sp'}{p'}} V^s(x), \quad p's>1;\\
A_{11}(s) \quad &:= \inf_{h>0} \sup_{0<x<b} \left(\int_x^b u(t) h(t)^{q(\frac{1}{p'}-s)}dt\right)^{1/q} (h(x)+V(x))^s, \quad p's>1;\\
A_{12}(s) \quad &:= \inf_{h>0} \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t) h(t)^{p'(\frac{1}{q}-s)}dt\right)^{1/p'} (h(x)+U(x))^s, \quad qs>1;\\
A_{13}(s) \quad &:= \inf_{h>0} \sup_{0<x<b} \left(\int_0^x u(t)(h(t)+V(t))^{q(\frac{1}{p'}+s)}dt\right)^{1/q} h^{-s}(x);\\
A_{14}(s) \quad &:= \inf_{h>0} \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t)(h(t)+U(t))^{p'(\frac{1}{q}+s)}(t)\right)^{1/p'} h^{-s}(x).
\end{aligned}$$

Then the Hardy inequality (1.1) holds for all measurable functions $f \geq 0$ if and only if any of the quantities $A_i(s)$ is finite. Moreover, for the best constant C in (1.1) we have $C \approx A_i(s)$, $i = 1, 2, 3, \ldots, 14$.

Remark 3.1. *The conditions in (1.5) can be described in the following way:*

$$\begin{aligned}
A_M &= A_1(\tfrac{1}{p'}),\\
A_{PS} &= A_3(\tfrac{1}{p}),\\
A_W(r) &= A_1(\tfrac{r-1}{p}) \text{ with } 1<r<p,\\
A^*_{PS} &= A_4(\tfrac{1}{q'}),\\
A^*_W(r) &= A_2(\tfrac{r-1}{q'}) \text{ with } 1<r<q',\\
A_T &= A_{13}(\tfrac{1}{q}),\\
A^*_T &= A_{14}(\tfrac{1}{p'}).
\end{aligned}$$

Hence, Theorem 3.1 generalizes the corresponding results in [2], [6] *and also all previous results of this type.*

Proof of Theorem 3.1. In (2.1) we put $a=0, f(x)=u(x), g(x)=v^{1-p'}(x)$, so that $F(x)=U(x)$, $G(x)=V(x)$, and choose

$$\alpha=\frac{1}{q},\ \beta=\frac{1}{p'}.$$

Then the assertion follows from the fact that

$$A_i(s)=B_{i+1}(\frac{1}{q},\frac{1}{p'},s), i=1,2,\dots,14.$$

are all equivalent with A_1 from(1.2) according to Theorem 2.1 and the finiteness of A_1 is necessary and sufficient for the inequality (1.1) to hold. Moreover, since for the least constant C in (1.1) we have $C\approx A_1$ it is clear that $C\approx A_i(s)$ and the proof is complete.

Remark 3.2. *The proof of Theorem 2.1 (cf. Remark 2.1) gives us also the possibility to estimate e.g. the quantities* $A_1,\ A_W(r),\ A^*_W(r),\ A_{PS},\ A^*_{PS},\ A^*_T,$ *and* A^*_{PS}*, in terms of each other.*

4. Outlines of the proof of the equivalence theorem

In the proof, which is rather technical, we use - among other tools - the fact that the function F from (2.1) is decreasing and the function of G from (2.1) is increasing, and that

$$f(x)dx=-dF(x),\ g(x)dx=dG(x)$$

so that

$$\int_x^b f(t)F^\lambda(t)dt=\frac{1}{\lambda+1}F^{\lambda+1}(x);\ \int_a^x g(t)G^\kappa(t)dt=\frac{1}{\kappa+1}G^{\kappa+1}(x). \tag{4.1}$$

Moreover, the equivalences

$$B_i(\alpha,\beta,s)\approx B_1(\alpha,\beta),\quad i=2,3,4,5, \tag{4.2}$$

have been proved in [2, Theorem 2.1], so that it is remains to prove the other 10 equivalences.

Here, we will give a detailed proof only for some equivalence, in order to show the typical steps used. As mentioned, full proofs can be found in [1].

1. $B_1(\alpha,\beta) \approx B_6(\alpha,\beta,s)$.
(i) $B_1(\alpha,\beta) \lesssim B_6(\alpha,\beta,s)$:

$$\begin{aligned} B_1(x;\alpha,\beta) =& F^\alpha(x)G^\beta(x) = F^{\alpha+s}(x)F^{-s}(x)G^\beta(x) \\ &= \left(\int_x^b f(t)dt\right)^{\alpha+s} G^\beta(x)F^{-s}(x) \\ &= \left(\int_x^b f(t)G^{\frac{\beta}{\alpha+s}}(x)dt\right)^{\alpha+s} F^{-s}(x) \\ &\le \left(\int_x^b f(t)G^{\frac{\beta}{\alpha+s}}(t)dt\right)^{\alpha+s} F^{-s}(x) = B_6(x;\alpha,\beta,s). \end{aligned}$$

(we have used the fact that G is increasing). Now we take the suprema for $x \in (a,b)$ and have that $B_1(\alpha,\beta) \le B_6(\alpha,\beta,s)$, i.e. in (2.3) it is $d_6 = 1$.
(ii) $B_6(\alpha,\beta,s) \lesssim B_1(\alpha,\beta)$:

$$\begin{aligned} B_6(x;\alpha,\beta,s) &= \left(\int_x^b f(t)G^{\frac{\beta}{\alpha+s}}(t)dt\right)^{\alpha+s} F^{-s}(x) \\ &= \left(\int_x^b f(t)B_1^{\frac{1}{\alpha+s}}(t,\alpha,\beta)F^{-\frac{\alpha}{\alpha+s}}(t)G^{-\frac{\beta}{\alpha+s}}(t)G^{\frac{\beta}{\alpha+s}}(t)dt\right)^{\alpha+s} F^{-s}(x) \\ &\le B_1(\alpha,\beta)\left(\int_x^b f(t)F^{-\frac{\alpha}{\alpha+s}}(t)dt\right)^{\alpha+s} F^{-s}(x) \\ &= B_1(\alpha,\beta)\left(-\frac{\alpha+s}{s}F^{\frac{s}{\alpha+s}}|_x^b\right)^{\alpha+s} F^{-s}(x) \\ &= \left(\frac{\alpha+s}{s}\right)^{\alpha+s} B_1(\alpha,\beta)F^s(x)F^{-s}(x) = \left(\frac{\alpha+s}{s}\right)^{\alpha+s} B_1(\alpha,\beta) \end{aligned}$$

(we have used the fact that $B_1(t,\alpha,\beta) \le B_1(\alpha,\beta)$, and formula (4.1) for F with $\lambda = -\frac{\alpha}{\alpha+s}$). Taking the supremum on the left-hand side, we have that $B_6(\alpha,\beta,s) \le \frac{1}{c_6}B_1(\alpha,\beta)$ with $c_6 = \left(\frac{s}{\alpha+s}\right)^{\alpha+s}$.

2. $B_1(\alpha,\beta) \approx B_8(\alpha,\beta,s)$; $\alpha > s$.
(i) $B_1(\alpha,\beta) \lesssim B_8(\alpha,\beta,s)$: Fix $x \in (a,b)$ and define $y = y(x) \in (x,b)$ so that

$$\int_x^y f(t)dt = \int_y^b f(t)dt. \tag{4.3}$$

Then

$$\begin{aligned} F^\alpha(x) &= \left(\int_x^b f(t)dt\right)^\alpha = \left(\int_x^y f(t)dt + \int_y^b f(t)dt\right)^\alpha = 2^\alpha\left(\int_y^b f(t)dt\right)^\alpha \\ &= 2^\alpha\left(\int_x^y f(t)dt\right)^{\alpha-s}\left(\int_y^b f(t)dt\right)^\alpha = 2^\alpha\left(\int_y^b f(t)dt\right)^{\alpha-s} F^s(y). \end{aligned}$$

and

$$
\begin{aligned}
B_1(x;\alpha,\beta) &= 2^\alpha \left(\int_x^y f(t)dt\right)^{\alpha-s} F^s(y)G^\beta(x)\\
&= 2^\alpha \left(\int_x^y f(t)G^{\frac{\beta}{\alpha-s}}(x)dt\right)^{\alpha-s} F^s(y)\\
&\le 2^\alpha \left(\int_x^y f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(y)\\
&\le 2^\alpha \left(\int_a^y f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(y) = 2^\alpha B_8(y;\alpha,\beta,s)
\end{aligned}
$$

(we have used the fact that G is increasing). Taking the supremum with respect to y (right) and x (left), we have that $B_1(\alpha,\beta) \le 2^\alpha B_8(\alpha,\beta,s)$, i.e. $d_8 = 2^\alpha$.

(ii) $B_8(\alpha,\beta,s) \lesssim B_1(\alpha,\beta)$:

$$
\begin{aligned}
B_8(x;\alpha,\beta,s) &= \left(\int_a^x f(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(x)\\
&= \left(\int_a^x f(t)B_1^{\frac{\alpha}{\alpha-s}}(t;\alpha,\beta)F^{-\frac{\alpha}{\alpha-s}}(t)G^{-\frac{\beta}{\alpha-s}}(t)G^{\frac{\beta}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(x)\\
&\le B_1(\alpha,\beta)\left(\int_a^x f(t)F^{-\frac{\alpha}{\alpha-s}}(t)dt\right)^{\alpha-s} F^s(x).
\end{aligned}
$$

Now (see (4.1)) $\int_a^x f(t)F^{-\frac{\alpha}{\alpha-s}}(t)dt = \frac{\alpha-s}{s}\left(F^{-\frac{s}{\alpha-s}}(x) - F^{-\frac{s}{\alpha-s}}(a)\right)$ $\le \frac{\alpha-s}{s}F^{-\frac{s}{\alpha-s}}(x)$ (even if $F(a)=\infty$, since $-\frac{s}{\alpha-s}<0$).

Hence

$$
\begin{aligned}
B_8(x;\alpha,\beta,s) &\le B_1(\alpha,\beta)\left(\frac{\alpha-s}{s}\right)^{\alpha-s}\left(F^{-\frac{s}{\alpha-s}}(x)\right)^{\alpha-s} F^s(x)\\
&= \left(\frac{\alpha-s}{s}\right)^{\alpha-s} B_1(\alpha,\beta)
\end{aligned}
$$

and taking the supremum, we have

$$
B_8(\alpha,\beta,s) \le \frac{1}{c_8}B_1(\alpha,\beta,) \quad \text{with} \quad c_8 = \left(\frac{1}{\alpha-s}\right)^{\alpha-s}.
$$

3. $B_1(\alpha,\beta) \approx B_{12}(\alpha,\beta,s)$, $\beta < s$.

(i) $B_1(\alpha,\beta) \lesssim B_{12}(\alpha,\beta,s)$: Assume that $B_{12}(\alpha,\beta,s) < \infty$ and denote it for simplicity by B_{12}. Since $\inf_{h>0}\sup_x \left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^\alpha (h(x)+G(x))^s = B_{12}$, there exists a positive function h such that

$$
\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^\alpha (h(x)+G(x))^s \le B_{12},
$$

and consequently

$$
\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt \le B_{12}^{\frac{1}{\alpha}}h^{-\frac{s}{\alpha}}(x), \tag{4.4}
$$

$$\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt \le B_{12}^{\frac{1}{\alpha}} G^{-\frac{s}{\alpha}}(x). \tag{4.5}$$

From (4.4) we obtain, raising both sides to the power $\frac{s-\beta}{s} > 0$, multiplying by $f(x)$ and integrating from y to b, that

$$\int_y^b f(x)\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{\frac{s-\beta}{s}} dx \le B_{12}^{\frac{s-\beta}{s\alpha}} \int_y^b f(x)h^{\frac{\beta-s}{\alpha}}(x)dx. \tag{4.6}$$

Now we use the equivalence relation

$$B_5(1,1,1) \approx B_5(1,1,\frac{\beta}{s}),$$

which holds, since both terms are equivalent to $B_1(1,1)$ (see (4.2)). This relation reads

$$\sup_x \left(\int_x^b g(t)F^2(t)dt\right) F^{-1}(x) \approx \sup_x \left(\int_x^b g(t)F^{1+\frac{\beta}{s}}(t)dt\right) F^{-\frac{\beta}{s}}(x).$$

We use this relation with $\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt$ for $F(x)$ and with $f(x)\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{-\frac{\beta}{s}-1}$ for $g(x)$. Then we have

$$\begin{aligned}
&\sup_x \left(\int_x^b f(t)dt\right)\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{-\frac{\beta}{s}} \\
&\quad\approx \sup_x \left(\int_x^b f(y)\left(\int_y^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{\frac{s-\beta}{s}} dy\right)\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{-1} \\
&\quad\lesssim B_{12}^{\frac{s-\beta}{s\alpha}},
\end{aligned}$$

where the last inequality follows from (4.6). Therefore we get

$$\sup_x \left(\int_x^b f(t)dt\right)^{\alpha}\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{-\frac{\alpha\beta}{s}} \lesssim B_{12}^{\frac{s-\beta}{s}}. \tag{4.7}$$

Taking into account that due to (4.5)

$$G^{\beta}(x) \le B_{12}^{\frac{\beta}{s}} \left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{-\frac{\alpha\beta}{s}},$$

we get from (4.7) that

$$\begin{aligned}
\sup_x F(x)^{\alpha}G^{\beta}(x) &\le B_{12}^{\frac{\beta}{s}} \sup_x F^{\alpha}(x)\left(\int_x^b f(t)h^{\frac{\beta-s}{\alpha}}(t)dt\right)^{-\frac{\alpha\beta}{s}} \\
&\lesssim B_{12}^{\frac{\beta}{s}} B_{12}^{1-\frac{\beta}{s}} = B_{12}.
\end{aligned}$$

Therefore, we have that

$$B_1(\alpha,\beta) \lesssim B_{12}(\alpha,\beta,s).$$

(ii) $B_{12}(\alpha, \beta, s) \lesssim B_1(\alpha, \beta)$: Since for $h(x) = G(x)$, it is

$$\begin{aligned} B_{12}(\alpha, \beta, s, G) &= 2^s \left(\int_x^b f(t) G^{\frac{\beta - s}{\alpha}}(t) dt \right)^\alpha G^s(x) \\ &= 2^s B_2(x; \alpha, \beta, s) \lesssim 2^s B_2(\alpha, \beta, s) \lesssim B_1(\alpha, \beta) \end{aligned}$$

(see (4.2) for $i = 2$), we immedatly obtain that $B_{12}(\alpha, \beta, s) \lesssim B_1(\alpha, \beta)$.

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