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WEAKLY WANDERING VECTORS AND INTERPOLATION THEOREMS FOR POWER BOUNDED OPERATORS

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ABSTRACT. Let $\mathbf{c} = \{c_{p,q}\} \subset \mathbb{C}, p, q \in \mathbb{N}, p > q$, be such that $\mathrm{D} - \lim_{p\to\infty} c_{p,q} = 0$ for each $q \in \mathbb{N}$, and let T be a power bounded operator on a Hilbert space H with infinite peripheral spectrum and empty point peripheral spectrum. We prove that $\mathbf{c}$ can be interpolated by the orbits of T in the sense that the set of x's from H with $\langle T^{n_k}x, T^{n_{k'}}x\rangle = c_{n_k,n_{k'}}$ for a certain increasing sequence $\{n_k\} \subset \mathbb{N}$ (depending on x) and all $k, k' \in \mathbb{N}, k > k'$, is dense in H. In particular, the set of weakly wandering vectors for such T is dense in H. This extends previous similar results known only in the context of unitary representations of groups. Our results are optimal as far as spectral conditions are concerned. Moreover, our technique allows to treat operators whose sequence of powers is unbounded.

1. INTRODUCTION

A classical result going back to Krengel [13] says that if U is a unitary operator on a Hilbert space H then U has continuous spectrum if and only if for every $x \in H$ and $\epsilon > 0$ there exists $y \in H$ and an infinite increasing sequence $\{n_i\} \subset \mathbb{N}$ such that $\|x-y\| < \epsilon$ and the vectors $\{U^{n_i}y\}$ are mutually orthogonal.

The Krengel result was extended for unitary representations of various types of groups by Bergelson, Mityagin, and Kornfeld, Leibman, Gracham, del Junco and Begun, see [8], [3], [4], [2] [6], [1]. Their approaches relied on a) a kind of involved inductive construction of $U^{n_i}y$ with strictly decaying deviation from orthogonality (Krengel), b) a tricky application of a Banach fixed point theorem on an appropriate subset of $l_2(\mathbb{Z})$ (Bergelson, Mityagin, Kornfeld, Leibman), c) a spectral theorem for unitary operators (representations) thus reducing the problem to a certain result on Fourier transforms of finite measures (del Junco) or Fourier coefficients of orthonormal systems in L^2 (Begun) requiring in turn nevertheless an application of a fixed point

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theorem again. In all of these approaches the assumption of unitarity (isometricity) was indispensable and the corresponding reasonings in b) and c) could not be even started otherwise.

We treat the problem of weak wandering from the point of view of abstract operator theory. The purpose of the paper is to create a general framework for the study of weakly wandering vectors of operators being in general far from unitary ones.

Our result on weak wandering is a consequence of a (formally) more general theorem on interpolation of weak orbits of power bounded operators on Hilbert spaces. The study of weak orbits for power bounded operators attracted a considerable attention last years in view of similarity problems, see, for instance, [22], [23], [5]. There the fact that weak orbits of contractions and power bounded operators have different "interpolation properties" was exploited, in particular, to produce new examples of power bounded operators not similar to contractions. On the other hand, while the former activity concentrated on *constructing* operators with a weak orbit interpolating a fixed sequence, in this paper we are interested in interpolation of a fixed sequence by the orbits of a *fixed* operator. This seems to be much more difficult task. Given a power bounded operator T subject to certain necessary spectral conditions, we show, in particular, that any sequence in $c_0(\mathbb{N})$ contains a subsequence that can be interpolated by the corresponding subsequence of a weak orbit of T, see Theorem 4.1. However, the price we have to pay is that we have no, in general, control over the distribution of "interpolation nodes".

2. Weak wandering and peripheral spectrum

Let H be a complex Hilbert space, and let $B(H)$ be the space of bounded linear operators on H. As usual, denote by $\sigma(T)$ and $\sigma_p(T)$ the spectrum and the point spectrum of $T \in B(H)$ respectively.

As our arguments will depend on the notion of convergence in density we will give some basic definitions and properties of this type of convergence. Let $A \subset \mathbb{N}$. The lower and upper density of A are defined by

$$\underline{\mathrm{Dens}}\, A = \liminf_{n\to\infty} n^{-1}\mathrm{card}\,\{a \in A : a \le n\}$$

and

$$\overline{\mathrm{Dens}}\, A = \limsup_{n\to\infty} n^{-1}\mathrm{card}\,\{a \in A : a \le n\},$$

respectively. If the lower and upper densities of A coincide, then the common value is called the density of A and denoted by $\mathrm{Dens}\, A$.

A sequence $\{x_n\}$ of elements of a normed space converges to x in density if there exists a subset $A \subset \mathbb{N}, \mathrm{Dens}(A) = 0$, such that $\lim_{n\to\infty, n\notin A} x_n = x$. Equivalently, $\mathrm{D} - \lim_{n\to\infty} x_n = x$ if and only if there exists a subset $A \subset \mathbb{N}$ with $\mathrm{Dens}\, A = 0$ with the property that for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|x_n - x\| < \varepsilon$, $n \ge n_0$, $n \notin A$.

Recall that

$$\mathrm{D} - \lim_{n\to\infty} x_n = x \Longleftrightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \|x_k - x\| = 0.$$

We will need the following 'density' version of the famous de Leeuw-Glicksberg theorem. Let $\mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

Theorem 2.1. *Let H be a Hilbert space and let T be a power bounded operator on H. Then*

$$\begin{aligned} H &= H_0 \oplus H_1, \quad \textit{where} \\ H_1 &:= \textit{c.l.s}\{x \in H : Tx = \lambda\, x,\ \lambda \in \mathbb{T}\}; \\ H_0 &:= \{x \in H : 0 \textit{ belongs to the weak closure of } \{T^n x : n \geq 0\}\} \\ &= \{x \in H : \mathrm{D} - \lim_{n\to\infty} \langle T^n x, y\rangle = 0 \textit{ for every } y \in H\}. \end{aligned}$$

A simple proof of this statement can be found in [9]. Alternatively, see [14, p.108-109]. Thus, if $T \in B(H)$ is a power bounded operator then

$$\sigma_p(T) \cap \mathbb{T} = \emptyset \Longleftrightarrow \mathrm{D} - \lim_{n\to\infty} \langle T^n x, y\rangle = 0 \quad \text{for all } x, y \in H.$$

This property will be an important tool in our construction.

Note also that if $\{N_k : k \geq 1\}$ is a sequence of subsets of $\mathbb{N}$ of density 1 then there exists $N_0 \subset \mathbb{N}$ of density 1 such that $N_0 \setminus N_k$ is finite for every k, see, in particular, [10, Lemma 1], [14, p. 102], and also [21, Lemma 9.1]. Hence if H is separable, then the subspace H_0 can be characterized by the next condition. There exists $M \subset \mathbb{N}$, $\mathrm{Dens}(M) = 0$, such that

$$w - \lim_{\substack{k\to\infty,\\ n_k \notin M}} T^{n_k} x = 0 \qquad \text{for every } x \in H_0.$$

So for separable H the D-limit in Theorem 2.1 can be chosen along a subsequence $\{n_k\}$ independent of x and y in H. This fact, however, will not be used in the sequel.

The second important tool in our considerations is the 'fine' spectral theory. Let $T \in B(H)$ be an operator with spectral radius $r(T)$ equal to 1. Suppose that $\sigma_p(T) \cap \mathbb{T} = \emptyset$, and let $\lambda \in \sigma(T) \cap \mathbb{T}$. Since $\sigma(T) \subset \{z \in \mathbb{C} : |z| \leq 1\}$, we have $\lambda \in \partial\sigma(T)$ and consequently, λ is an element of the approximate point spectrum of T. Moreover, $\lambda \in \sigma_e(T) = \{z \in \mathbb{C} : T - z \text{ is not Fredholm}\}$. Indeed, if $\lambda \in \sigma(T) \setminus \sigma_e(T)$ then $\operatorname{ind}(T - \lambda) = 0$, and so $\lambda \in \sigma_p(T)$, a contradiction. Hence $\lambda \in \partial\sigma_e(T) \subset \sigma_{\pi e}(T)$, see [19, Theorem III.16.8 and Proposition III.19.1], where $\sigma_{\pi e}(T)$ denotes the essential approximate point spectrum of T. This means that for every $\varepsilon > 0$ and every subspace $M \subset H$ of finite codimension there exists a unit vector $x \in M$ called approximate eigenvector such that $\|(T - \lambda)x\| < \varepsilon$. It is also easy to see that for each $n_0 \in \mathbb{N}$ we can even find a unit vector $x \in M$ such that $\|T^n x - \lambda^n x\| < \varepsilon$ for all $n \leq n_0$. Moreover, since the choice of M is arbitrary, we are able to keep control over orthogonality relations defined in terms of finite number of approximate eigenvectors and their images under finite number of iterates of T. See, in particular, [15]-[18] and [19] where this idea was used for the study of orbits of operators.

Now we recall the notion of a weakly wandering vector.

Definition 2.2. *A vector $x \in H$ is called weakly wandering for T if there exists an increasing sequence of positive integers $\{n_k\}$ such that $\langle T^{n_k}x, T^{n_s}x\rangle = 0$ for all $k, s \in \mathbb{N}$, $k \neq s$.*

Equivalently, x is a weakly wandering vector for T if the orbit $\{T^n x\}_{n\in\mathbb{N}}$ of T at x contains infinitely many mutually orthogonal vectors.

The notion of weak wandering corresponds to the 'trivial' case of subsequence interpolation by the orbits of power bounded operators.

To motivate our results we start with the two observations.

Observation 1. If x is a weakly wandering vector for power bounded T, then $x \in H_0$. Indeed, without loss of generality we can assume then $\inf_i \|T^{n_i}x\| > 0$. Then by the orthogonality of $\{T^{n_i}x\}$ and the Bessel inequality:

$$\sum_{i\geq 1}\Big|\Big\langle \frac{T^{n_i}x}{\|T^{n_i}x\|}, y\Big\rangle\Big|^2 \leq \|y\|^2$$

for every $y \in H$, so $T^{n_i}x \to 0, i \to \infty$, weakly.

If, moreover, the set of weakly wandering vectors of T is dense then by Theorem 2.1, we have $H_1 = \{0\}$, or, equivalently $\sigma_p(T) \cap \mathbb{T} = \emptyset$. In other words, the latter condition is *necessary* for density of weakly wandering vectors of T.

In case of unitary T this was the only spectral condition in the Krengel theorem to guarantee weak wandering property for a dense set of vectors. However for some power bounded T this assumption does not suffice. Let us consider several instructive examples.

Example 2.3. *Let $T = \operatorname{diag}(\frac{n}{n+1} : n = 1, 2, \dots)$. Then T is a selfadjoint operator, $T^k = \operatorname{diag}((\frac{n}{n+1})^k : n = 1, 2, \dots)$ and the numerical range $W(T^k) = \operatorname{conv}\sigma(T^k) = [2^{-k}, 1]$.*

If x is any non-zero vector, $\|x\| = 1$, and $k, l \in \mathbb{N}$, then $\langle T^k x, T^l x\rangle = \langle T^{k+l}x, x\rangle \in W(T^{k+l}) \not\ni 0$. Hence no orbit of T for a non-zero x contains two orthogonal vectors. Note that $\sigma_p(T) \cap \mathbb{T} = \emptyset$ and $\operatorname{card}(\sigma(T) \cap \mathbb{T}) = 1$.

Example 2.4. *Let $k \in \mathbb{N}$ and let H be the Hilbert space with an orthonormal basis $e_{js}, j = 0, \dots, k-1, s \in \mathbb{N}$. Define $T \in B(H)$ by $Te_{js} = \frac{s}{s+1}\eta^j e_{js}$, where $\eta = e^{2\pi i/k}$. Then the orbit of any non-zero vector contains at most k mutually orthogonal vectors. Indeed, let $x \neq 0$, $x = \sum \alpha_{js}e_{js}$. Suppose that there are $n_1 < n_2 < \dots < n_{k+1}$ such that the vectors $T^{n_1}x, \dots, T^{n_{k+1}}x$ are mutually orthogonal. Then there are $s, r \in \mathbb{N}$ such that $s < r$, $s = r \pmod k$ and $\langle T^r x, T^s x\rangle = 0$. Then*

$$0 = \langle T^r x, T^s x\rangle = \sum_{j=0}^{k-1}\sum_{s=1}^{\infty} |\alpha_{js}|^2 \big(\frac{s}{s+1}\big)^{s+r}.$$

Hence $\alpha_{js} = 0$ *for all* j, s *and* $x = 0$.

Clearly $\sigma_p(T) \cap \mathbb{T} = \emptyset$ *and* $\operatorname{card}(\sigma(T) \cap \mathbb{T}) = k$.

The previous example can give the impression that if the peripheral spectrum $\sigma(T) \cap \mathbb{T}$ contains at least k points then there are orbits containing at least k mutually orthogonal vectors. The next example shows that the situation is more complicated.

Example 2.5. *Let* T *be the operator constructed in Example 2.3 and let* $S = T \oplus \alpha T \oplus \alpha^3 T$, *where* $\alpha = e^{2\pi i/7}$. *Then no non-zero orbit of* T *contains two orthogonal vectors. Indeed, since* T *is normal, then as in the first example,* $W(T^k) = \operatorname{conv} \sigma(T^k)$ *for every* $k \in \mathbb{N}$, *and, moreover,*

$$\operatorname{conv} \sigma(T^k) = \operatorname{conv} \Big\{1, \alpha^k, \alpha^{3k}, \frac{1}{2^k}, \frac{\alpha^k}{2^k}, \frac{\alpha^{3k}}{2^k}, \, k \in \mathbb{N}\Big\}.$$

Since according [11] *(see also* [24], [12]*),* $\operatorname{conv}\{1, \alpha^k, \alpha^{3k}\} \not\ni 0, k \in \mathbb{N}$, *geometric considerations show that* $\operatorname{conv} \sigma(T^k) \not\ni 0, k \in \mathbb{N}$, *and therefore* $W(T^k) \not\ni 0$, $k \in \mathbb{N}$. *It remains to observe that for* $r \geq s$

$$\begin{aligned} \{\langle T^r x, T^s x\rangle : x \in H \setminus \{0\}\} &= \{\langle T^{r-s} T^s x, T^s x\rangle : x \in H \setminus \{0\}\} \\ &= \{W(T^{r-s}) \|T^s x\|^2 : x \in H \setminus \{0\}\}. \end{aligned}$$

Since $W(T^{r-s}), r \geq s$, *does not contain zero, the latter set does not contain zero either. So no orbit of a non-zero vector* x *contains two orthogonal vectors. Note that in this case* $\sigma_p(T) \cap \mathbb{T} = \emptyset$ *and* $\operatorname{card}(\sigma(T) \cap \mathbb{T}) = 3$.

Observation 2. Thus, to prove the existence of weakly wandering vectors for a general power bounded T it is reasonable to assume that the peripheral spectrum of T is infinite.

Now Observations 1 and 2 suggest that if T is power bounded then the strongest result on weak wandering we may hope to obtain for T is that the conditions $\sigma(T) \cap \mathbb{T}$ is infinite and $\sigma_p(T) \cap \mathbb{T} = \emptyset$ imply the existence of a dense subset of weakly wandering vectors for T. This is one of the main results of the paper. Moreover, weakly wandering vectors exist and are dense even for a large class of operators whose powers might grow arbitrarily fast.

Three more natural questions arise regarding weakly wandering vectors.

The first question is how large the set of weakly wandering vectors can be. If T is a nilpotent operator then obviously every vector is weakly wandering. On the other hand, if T is not nilpotent then weakly wandering vectors form necessarily a set of first category. In fact, even the set of all vectors whose orbit contains at least two orthogonal vectors is of first category.

Proposition 2.6. *Let* $T \in B(H)$ *be a non-nilpotent operator. Then the set of all vectors* x *whose orbit* $\{T^n x : n = 0, 1, \dots\}$ *contains at least two orthogonal vectors is of first category. Consequently, the set of all weakly wandering vectors for* T *is of first category.*

Proof. Fix non-negative integers m, n, $m > n$. We show that the set $M_{m,n} = \{x \in H : T^m x \perp T^n x\}$ is of first category.

Clearly $M_{m,n}$ is closed. Suppose on the contrary that $M_{m,n}$ contains a non-empty open set U. Let $x \in U$ and let $y \in H$ be arbitrary. Consider the function f defined by $f(t) = \langle T^m((1-t)x+ty), T^n((1-t)x+ty)\rangle, t \in [0,1]$. Then f is a quadratic function. For all $t \in [0,1]$ small enough we have $(1-t)x + ty \in U$, and so $f(t) = 0$. Hence f is identically equal to 0. In particular, $\langle T^m y, T^n y\rangle = f(1) = 0$.

Thus $\langle T^{m-n}u, u\rangle = 0$ for each u from the range $R(T^n)$ of T^n, and so for each $u \in \overline{R(T^n)}$. Therefore $T^{m-n}|_{\overline{R(T^n)}} = 0$, and so $T^m = 0$, a contradiction. Hence $M_{m,n}$ is of first category for all $m, n,\ m > n$. Consequently, the set $\bigcup_{m>n} M_{m,n}$ of all vectors whose orbit contains at least two orthogonal vectors is of first category. $\square$

Thus, in particular, the set of weakly wandering vectors is not closed whenever it is dense.

Note that the set of all weakly wandering vectors may contain a dense linear manifold. (Consider the unilateral shift on $l_2(\mathbb{N})$ and the vectors in $l_2(\mathbb{N})$ with finite support.) It is also possible that the set of all weakly wandering vectors contains an infinite-dimensional closed subspace. (Consider the unilateral shift on $l^2(\mathbb{N}, H), \dim H = \infty$ and the subspace $\{(x, 0, 0, \dots) : x \in H\}$ of $l_2(\mathbb{N}, H)$.)

Secondly, for applications, it is important to know how sparse the subsequences $\{n_k\}$ along which weak wandering takes place can be (see Definition 2.2). Let us consider the case of unitary T. Recall that the set $\mathcal{U}$ of unitary operators T on H such that $T^{m_k} \to I, k \to \infty$, strongly for some subsequence $\{m_k\} \subset \mathbb{N}$ (such operators are called rigid), and $\sigma_p(T) \cap \mathbb{T} = \emptyset$ is residual in the group of unitary operators with a natural metric making it complete metric space; use for instance, [20, Theorem 8.25] combined with an easy argument from the proof of [25, Theorem 5.2, (3)⇒(1)].

If $T \in \mathcal{U}$ then for any $x \in H$ there is a subsequence of $\{n_k\}$ denoted by the same symbol such that $\|T^{m_k - m'_k}x - x\| < \frac{1}{2}\|x\|$, for all $k, k' \in \mathbb{N}$. By Krengel's theorem every $T \in \mathcal{U}$ has a dense set of weakly wandering vectors. On the other hand, if x wanders along a subsequence $\{m_k\}$ of positive upper density then $\|T^{n_k - n_{k'}}x - x\| = \sqrt{2}\|x\|,\ k, k' \in \mathbb{N}$. Recall that the set of differences $D - D$ of any set D of positive upper density intersects nontrivially any set of differences of any infinite subset of $\mathbb{N}$ (since the latter set is a *recurrence set*, and such sets are *intersective*, see [7, p.74-76, p.176-177], [26, Proposition IV.20], for this statement, terminology and more details.) Hence for some k we must have $n_k - n'_k = m_k - m'_k$, a contradiction. Thus, even for the most of unitary operators, we cannot expect, in general, that wandering takes place along "massive" subsets of $\mathbb{N}$. The fact that necessarily $\overline{\text{Dens}}\,(\{n_k\}) = 0$ if T is a rigid unitary operator was noted (with less details) in [3, p. 1133].

Note that by, for example, [20, Theorem] one has $\sigma(T) = \mathbb{T}$ if a unitary T as above is induced by a measure preserving transformation of a finite measure space so that the size of peripheral spectrum of T does not influence, in general, how sparse the subsequence $\{m_k\}$ is. (But it does help to prove the density of the set of weakly wandering vectors as it was remarked above.)

Third, observe finally that the set of weakly wandering vectors is not in general a linear set. Consider, for instance, a unitary operator $(Tf)(z) = zf(z)$ on $L^2(\mathbb{T}, \rho(z)dz)$, where

$$\rho(z) = 3 + \sum_{n\in\mathbb{N}\cup\{0\}} \frac{1}{2^n}(z^{2n+1} + z^{-2n-1}), \quad |z| = 1.$$

Then the vectors 1 and z are weakly wandering for T but a routine computation shows that their sum, $1 + z$, is not.

3. Approximation lemmas

We start with several purely combinatorial lemmas.

The first lemma is known in a greater generality but we provide a simple argument available in our particular situation.

Lemma 3.1. *Let $\lambda_1, \dots, \lambda_k \in \mathbb{T}$, $\varepsilon > 0$. Then there exists $r \in \mathbb{N}$ such that for each $n_0 \in \mathbb{N}$ there exists n, $n_0 \le n \le n_0 + r$ with $|\lambda_j^n - 1| < \varepsilon, \quad j = 1, \dots, k$.*

Proof. Let $\lambda_j = e^{2\pi i t_j}, j = 1, \dots, k$, where $0 \le t_j < 1$. Let $\theta_0 = 1$ and let $\theta_1, \dots \theta_m$ be real numbers linearly independent over the field $\mathbb{Q}$ of rational numbers such that the numbers t_j can be expressed as $t_j = \sum_{s=0}^m \alpha_{js}\theta_s$, where $\alpha_{js} \in \mathbb{Q}$.

Let $d \in \mathbb{N}$ satisfy $d\alpha_{js} \in \mathbb{Z}$ for all j, s. Let

$$c = \max\{|d\alpha_{js}| : j = 1, \dots, k, s = 0, \dots, m\}.$$

By the Kronecker theorem, there exists $r_0 \in \mathbb{N}$ such that

$$\{(e^{2\pi i b\theta_1}, \dots, e^{2\pi i b\theta_m}) : b = 1, \dots, r_0\}$$

is an $\frac{\varepsilon}{c(m+1)}$-net in $\mathbb{T}^m$ with the ℓ_∞ metric.

There exists $n' \in \mathbb{N}$, $\frac{n_0}{d} \le n' \le \frac{n_0}{d} + r_0$ such that $|e^{2\pi i n'\theta_j} - 1| < \frac{\varepsilon}{c(m+1)}$ for $j = 1, \dots, m$. Let $r = dr_0$ and $n = dn'$. Then $n_0 \le n \le n_0 + r$ and

$$\begin{aligned} |\lambda_j^n - 1| &= \left| e^{2\pi i \sum_{s=0}^m \alpha_{js} d n' \theta_s} - 1 \right| \\ &\le \left| \sum_{s=0}^m d\alpha_{js} \right| \cdot |e^{2\pi i n'\theta_s} - 1| \\ &\le (m+1)c \cdot \frac{\varepsilon}{c(m+1)} \\ &= \varepsilon \end{aligned}$$

for $j = 1, \dots, k$. □

Denote by A' the set of all accumulation points of a set $A \subset \mathbb{C}$.

Lemma 3.2. *Let $A \subset \mathbb{T}$, $1 \in A'$, $\lambda_1, \dots, \lambda_k \in A$, $n \in \mathbb{N}$, $\varepsilon > 0$ and $\mu \in \mathbb{T}$. Let $M \subset \mathbb{N}$ be a set with* $\operatorname{Dens} M = 1$. *Then there exist $m \in M$, $m > n$ and $\lambda_{k+1} \in A$ such that*

$$\begin{array}{rcll} |\lambda_j^m - \lambda_j^n| & < & \varepsilon, & j = 1, \dots, k, \\ |\lambda_{k+1}^{n'} - 1| & < & \varepsilon, & n' \le n, \\ |\lambda_{k+1}^m - \mu| & < & \varepsilon. & \end{array} \tag{3.1}$$

Proof. By Lemma 3.1, there exists $r \in \mathbb{N}$ such that for each $n_1 \in \mathbb{N}$ there exists $s, n_1 \le s \le n_1 + r$ with $|\lambda_j^s - 1| < \varepsilon, j = 1, \dots, k$.

Find $\lambda_{k+1} \in A$ such that $0 < |\lambda_{k+1} - 1| < \frac{\varepsilon}{2(n+r)}$. Let $n_2 \in \mathbb{N}$ satisfy $n_2 \ge \frac{2\pi}{|\lambda_{k+1} - 1|}$.

Let $n_0 \in \mathbb{N}$. There exists s, $n_0 \le s \le n_0 + n_2$, such that $|\lambda_{k+1}^s - \mu| < \frac{\varepsilon}{2(n+r)} \le \varepsilon/2$.

There exists $s' \in \mathbb{N}$, $s \le s' \le s + r$ such that $|\lambda_j^{s'} - 1| < \varepsilon, \quad j = 1, \dots, k$. Then

$$|\lambda_j^{s'+n} - \lambda_j^n| = |\lambda_j^{s'} - 1| < \varepsilon, \qquad j = 1, \dots, k,$$

$|\lambda_{k+1}^{n'} - 1| < \frac{\varepsilon n'}{2(r+n)} < \varepsilon \quad (n' \le n)$ and

$$\begin{array}{rcl} |\lambda_{k+1}^{s'+n} - \mu| & \le & |\lambda_{k+1}^{s'+n} - \lambda_{k+1}^s| + |\lambda_{k+1}^s - \mu| \\ & < & |\lambda_{k+1}^{s'-s+n} - 1| + \varepsilon/2 \\ & \le & \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{array}$$

We have $n_0 \le s' + n \le n_0 + n_2 + r + n$. Hence the set of all $m \in \mathbb{N}$ satisfying (3.1) has a positive lower density. Consequently, it is possible to find $m \in M$, $m > n$, satisfying (3.1). □

Lemma 3.3. *Let $A \subset \mathbb{T}$, $1 \in A'$, let $m \in \mathbb{N}$, $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_{m+1} > 0$, $\lambda_{kj} \in A, k = 1, \dots, m, j = 1, \dots, 4$, $n_1 < n_2 < \cdots < n_m$ be such that*

$$\begin{array}{rcll} |\lambda_{kj}^{n_s} - 1| & < & \varepsilon_k, & 1 \le k, s \le m, k \ne s, j = 1, \dots, 4, \\ |\lambda_{kj}^{n_k} - i^j| & < & \varepsilon_k, & 1 \le k \le m, j = 1, \dots, 4, \end{array}$$

where i denotes the imaginary unit. Let $M \subset \mathbb{N}$, $\operatorname{Dens} M = 1$. *Then there exist $n_{m+1} \in M$, $n_{m+1} > n_m$ and $\lambda_{m+1,j} \in A$, $j = 1, \dots 4$, such that*

$$\begin{array}{rcll} |\lambda_{kj}^{n_s} - 1| & < & \varepsilon_k, & 1 \le k, s \le m+1, k \ne s, j = 1, \dots, 4, \\ |\lambda_{kj}^{n_k} - i^j| & < & \varepsilon_k, & 1 \le k \le m+1, j = 1, \dots, 4. \end{array}$$

Proof. By Lemma 3.1, there exists $n_0' > n_m$ such that $|\lambda_{kj}^{n_0'} - 1| < \varepsilon_{m+1}/5, k \le m, j = 1, \dots, 4$. By Lemma 3.2, there exist $n_1' > n_0'$ and $\lambda_{m+1,1} \in A$ such that $|\lambda_{kj}^{n_1'} - \lambda_{kj}^{n_0'}| < \varepsilon_{m+1}/5, k \le m, j = 1, \dots, 4, |\lambda_{m+1,1}^t - 1| < \varepsilon_{m+1}/5, t \le n_0'$, and $|\lambda_{m+1,1}^{n_1'} - i| < \varepsilon_{m+1}/5$.

In the same way find $\lambda_{m+1,2}, \lambda_{m+1,3}, \lambda_{m+1,4} \in A$ and $n_2' > n_1'$, $n_3' > n_2'$, $n_4' > n_3'$, $n_4' \in M$ such that

$$\begin{aligned}
|\lambda_{kj}^{n_{s+1}'} - \lambda_{kj}^{n_s'}| &< \varepsilon_{m+1}/5, \qquad k \le m, j = 1, \ldots, 4, s = 0, \ldots, 3, \\
|\lambda_{m+1,j}^{n_{s+1}'} - \lambda_{m+1,j}^{n_s'}| &< \varepsilon_{m+1}/5, \qquad 1 \le j \le s \le 3, \\
|\lambda_{m+1,j}^{l} - 1| &< \varepsilon_{m+1}/5, \qquad l \le n_{j-1}', \\
|\lambda_{m+1,j}^{n_j'} - i^j| &< \varepsilon_{m+1}/5, \qquad j = 1, \ldots, 4.
\end{aligned}$$

Set $n_{m+1} = n_4'$. Then $n_{m+1} \in M$, $n_{m+1} > n_m$. It is easy to see that $\lambda_{m+1,1}, \ldots, \lambda_{m+1,4}$ satisfy all the required properties. Indeed, for $k \le m, j = 1, \ldots, 4$ we have

$$\begin{aligned}
|\lambda_{kj}^{n_{m+1}} - 1| &\le |\lambda_{kj}^{n_4'} - \lambda_{kj}^{n_3'}| + |\lambda_{kj}^{n_3'} - \lambda_{kj}^{n_2'}| \\
&+ |\lambda_{kj}^{n_2'} - \lambda_{kj}^{n_1'}| + |\lambda_{kj}^{n_1'} - \lambda_{kj}^{n_0'}| + |\lambda_{kj}^{n_0'} - 1| \\
&< \varepsilon_{m+1} \le \varepsilon_k
\end{aligned}$$

and

$$|\lambda_{m+1,j}^{n_k} - 1| < \varepsilon_{m+1}/5 < \varepsilon_{m+1}.$$

Finally,

$$\begin{aligned}
|\lambda_{m+1,j}^{n_{m+1}} - i^j| &\le |\lambda_{m+1,j}^{n_4'} - \lambda_{m+1,j}^{n_3'}| + \cdots + |\lambda_{m+1,j}^{n_{j+1}'} - \lambda_{m+1,j}^{n_j'}| + |\lambda_{m+1,j}^{n_j'} - i^j| \\
&< \varepsilon_{m+1}.
\end{aligned}$$

□

Let $\mathbf{u} \in \mathbb{C}^m$. We write $\mathbf{u} = (u(1), \ldots, u(m))$ and $\|\mathbf{u}\|_\infty = \max\{|\mathbf{u}(t)| : 1 \le t \le m\}$.

Lemma 3.4. *Let $m \in \mathbb{N}$, $\mathbf{w}, \mathbf{w}_{kj} \in \mathbb{C}^m, 1 \le k \le m, j = 1, \ldots, 4$. Suppose that*

$$\begin{aligned}
|\mathbf{w}_{kj}(t) - 1| &< 2^{-k-2}, \qquad 1 \le k, t \le m, t \ne k, j = 1, \ldots, 4, \\
|\mathbf{w}_{kj}(k) - (-i)^j| &< 2^{-k-2}, \qquad 1 \le k \le m, j = 1, \ldots, 4, \\
|\mathbf{w}(t) + 1| &< 2^{-m-2}m^{-1}, \qquad 1 \le t \le m.
\end{aligned}$$

Let $\mathbf{c} \in \mathbb{C}^m$. Then there are $\alpha, \alpha_{kj} \ge 0$ such that

$$\alpha \mathbf{w} + \sum_{k,j} \alpha_{kj} \mathbf{w}_{kj} = c$$

and

$$\alpha + \sum_{k,j} \alpha_{kj} \le 8m\|\mathbf{c}\|_\infty.$$

Proof. Let $\tilde{\mathbf{w}} = (-1, \ldots, -1)$ and $\tilde{\mathbf{w}}_{kj} = (\underbrace{1, \ldots, 1}_{k-1}, (-i)^j, 1, \ldots, 1)$.

Let

$$\begin{aligned}
\gamma \qquad &= 2(m-1)\|\mathbf{c}\|_\infty, \\
\gamma_{k,1} \qquad &= \frac{\|\mathbf{c}\|_\infty - \operatorname{Im}\mathbf{c}(k)}{2}, \\
\gamma_{k,2} \qquad &= \frac{\|\mathbf{c}\|_\infty - \operatorname{Re}\mathbf{c}(k)}{2}, \\
\gamma_{k,3} \qquad &= \frac{\|\mathbf{c}\|_\infty + \operatorname{Im}\mathbf{c}(k)}{2}, \\
\gamma_{k,4} \qquad &= \frac{\|\mathbf{c}\|_\infty + \operatorname{Re}\mathbf{c}(k)}{2}.
\end{aligned}$$

Then $\gamma\tilde{\mathbf{w}} + \sum_{k,j}\gamma_{kj}\tilde{\mathbf{w}}_{kj} = \mathbf{c}$. Indeed, for $1 \le t \le m$ we have

$$\begin{aligned}
\gamma\tilde{\mathbf{w}}(t) + \sum_{k,j}\gamma_{kj}\tilde{\mathbf{w}}_{kj}(t) \quad &= \quad -2(m-1)\|\mathbf{c}\|_\infty + \sum_{k\ne t,j}\gamma_{kj}\tilde{\mathbf{w}}_{kj}(t) + \sum_j \gamma_{tj}\tilde{\mathbf{w}}_{tj}(t) \\
&= \quad -2(m-1)\|\mathbf{c}\|_\infty + 2(m-1)\|\mathbf{c}\|_\infty \\
&- \quad i\gamma_{t,1} - \gamma_{t,2} + i\gamma_{t,3} + \gamma_{t,4} \\
&= \quad \mathbf{c}(t).
\end{aligned}$$

For $1 \le t \le m$ we have

$$\begin{aligned}
|\gamma\mathbf{w}(t) + \sum_{k,j}\gamma_{kj}\mathbf{w}_{kj}(t) - \mathbf{c}| \quad &= \quad \Big|\gamma(\mathbf{w}(t) - \tilde{\mathbf{w}}(t)) + \sum_{k,j}\gamma_{kj}(\mathbf{w}_{kj}(t) - \tilde{\mathbf{w}}_{kj}(t))\Big| \\
&\le \quad 2(m-1)2^{-m-2}m^{-1}\|\mathbf{c}\|_\infty + \sum_k 2^{-k-2}\cdot 2\|\mathbf{c}\|_\infty \\
&\le \quad \|\mathbf{c}\|_\infty/2.
\end{aligned}$$

So $\|\gamma\mathbf{w} - \sum_{k,j}\gamma_{kj}\mathbf{w}_{kj} - \mathbf{c}\|_\infty \le \|\mathbf{c}\|_\infty/2$. Furthermore, $\gamma + \sum_{k,j}\gamma_{k,j} \le 4m\|\mathbf{c}\|_\infty$.

Repeat this approximation for $\mathbf{c} - \mathbf{w} - \sum_{k,j}\gamma_{kj}\mathbf{w}_{kj}$ instead of $\mathbf{c}$ and so on. On the n-th step, we obtain an approximation $\gamma^{(n)}, \gamma^{(n)}_{k,j} \ge 0, \gamma^{(1)} := \gamma, \gamma^{(1)}_{k,j} := \gamma_{k,j}$, to the solution α, α_{kj} of the equation $\alpha\mathbf{w} + \sum_{k,j}\alpha_{k,j}\mathbf{w}_{kj} = \mathbf{c}$ such that

$$\|\gamma^{(n)}\mathbf{w} - \sum_{k,j}\gamma^{(n)}_{kj}\mathbf{w}_{kj} - \mathbf{c}\|_\infty \le 2^{-n}\|\mathbf{c}\|_\infty$$

and

$$\gamma^{(n)} + \sum_{k,j}\gamma^{(n)}_{kj} \le 4m\Big(\sum_{k=1}^{n}\frac{1}{2^{k-1}}\Big)\|\mathbf{c}\|_\infty.$$

Then in the limit we get numbers $\alpha, \alpha_{kj} \ge 0$ satisfying $\alpha\mathbf{w} + \sum_{k,j}\alpha_{k,j}\mathbf{w}_{kj} = \mathbf{c}$ and $\alpha + \sum_{k,j}\alpha_{kj} \le 8m\|\mathbf{c}\|_\infty$. $\square$

4. Interpolation theorem and weak wandering

Theorem 4.1. *Let $T \in B(H)$ be power bounded, $\sigma_p(T) \cap \mathbb{T} = \emptyset$ and let $\sigma(T) \cap \mathbb{T}$ be infinite. Let $c_{p,q}, p, q \in \mathbb{N}, p > q$, be complex numbers satisfying $\mathrm{D} - \lim_{p\to\infty} c_{p,q} = 0$ for each $q \in \mathbb{N}$. Let $x \in H$ and $\varepsilon > 0$ be fixed. Then there exist a vector $y \in H$ and positive integers $n_1 < n_2 < \cdots$ such that $\|y - x\| \le \varepsilon$ and $\langle T^{n_s} y, T^{n_{s'}} y\rangle = c_{n_s,n_{s'}}$ for all $s, s' \in \mathbb{N}, s > s'$.*

Proof. Since $\sigma(T) \cap \mathbb{T}$ is infinite, it has an accumulation point. Without loss of generality we can assume that $1 \in (\sigma(T) \cap \mathbb{T})'$. Indeed, if $\alpha \in (\sigma(T) \cap \mathbb{T})'$ then consider the operator $\alpha^{-1}T$ and the numbers $\alpha^{q-p}c_{p,q}$.

Since T is power bounded and $\sigma_p(T) \cap \mathbb{T} = \emptyset$, Theorem 2.1 implies that $\mathrm{D} - \lim_{n\to\infty}\langle T^n u, v\rangle = 0$ for all $u, v \in H$.

Without loss of generality we may assume that $\|x\| = 1$. Indeed, we may assume that $x \ne 0$. Consider then the vector $\frac{x}{\|x\|}$ and the numbers $\frac{c_{p,q}}{\|x\|^2}$.

Let $\varepsilon_1 > \varepsilon_2 > \cdots$ be a sufficiently rapidly decreasing sequence of positive numbers. More precisely, let $\varepsilon_1 < \min\{\varepsilon/4, 10^{-5}\}$ and $\frac{\varepsilon_m}{\varepsilon_{m-1}} < 2^{-8}m^{-2}$ for all $m \ge 2$.

Let $n_0 = 0$. We construct inductively positive integers $n_1 < n_2 < \cdots$, elements $\lambda_{kj} \in \sigma(T) \cap \mathbb{T}, k \in \mathbb{N}, j = 1, \dots, 4$, and unit vectors $x_{kj}^{(m)} \in H, m \in \mathbb{N}, 1 \le k \le m, j = 1, \dots, 4$, such that

$$|\lambda_{kj}^{n_s} - 1| < \varepsilon_k, \qquad k, s \in \mathbb{N}, s \ne k, j = 1, \dots, 4, \tag{4.1}$$

$$|\lambda_{kj}^{n_k} - i^j| < \varepsilon_k, \qquad k \in \mathbb{N}, j = 1, \dots, 4, \tag{4.2}$$

$$|\langle T^{n_m} y, T^{n_s} y'\rangle| < \frac{\varepsilon_m^3}{2m^2} \tag{4.3}$$

for $m, s \in \mathbb{N},\ s < m,\ y, y' \in \{x, x_{kj}^{(t)},\ k \le t < m, j = 1, \dots, 4\}$,

$$\|T^{n_s} x_{kj}^{(m)} - \lambda_{kj}^{n_s} x_{kj}^{(m)}\| < \varepsilon_m \tag{4.4}$$

for $s, k \le m, j = 1, \dots, 4$,

$$\text{the vectors } x, x_{kj}^{(s)} \text{ are mutually orthogonal,} \tag{4.5}$$

$$T^{n_s} x_{kj}^{(m)} \perp T^{n_{s'}} x \tag{4.6}$$

for $s, s' \le m, k \le m, j = 1, \dots, 4$,

$$T^{n_s} x_{kj}^{(m)} \perp T^{n_{s'}} x_{k'j'}^{(m')} \tag{4.7}$$

for $m' < m, s, s' \le m, k \le m, k' \le m', j, j' = 1, \dots, 4$,

$$T^{n_s} x_{kj}^{(m)} \perp T^{n_{s'}} x_{k'j'}^{(m)} \tag{4.8}$$

for $s, s' \le m, (k, j) \ne (k', j')$.

We construct the numbers n_m, elements $\lambda_{mj} \in \sigma(T) \cap \mathbb{T}$ and vectors $x_{kj}^{(m)}$ by induction on m.

Let $m \in \mathbb{N}$ and suppose that the numbers $n_1 < n_2 < \cdots < n_m$, elements $\lambda_{kj} \in \sigma(T) \cap \mathbb{T}$, $k \le m, j = 1, \dots, 4$, and vectors $x_{kj}^{(s)}, k \le s \le m, j = 1, \dots, 4$, satisfying (4.1)–(4.8) have already been constructed.

Let M be the set of all $r \in \mathbb{N}$ satisfying

$$|\langle T^r y, T^{n_s} y' \rangle| < \frac{\varepsilon_{m+1}^3}{2(m+1)^2}$$

for all $s \le m, y, y' \in \{x, x_{kj}^{(t)}, k \le t \le m, j = 1, \dots, 4\}$ and

$$|c_{r,n_s}| < \varepsilon_{m+1}^2, \quad s = 1, \dots, m.$$

Then $\operatorname{Dens} M = 1$.

By Lemma 3.3, there are $n_{m+1} \in M$ and $\lambda_{m+1,j} \in \sigma(T) \cap \mathbb{T}, j = 1, \dots, 4$ satisfying (4.1), (4.2). Since $n_{m+1} \in M$, we also have (4.3).

Since $\lambda_{kj} \in \partial\sigma(T), k \le m+1, j = 1, \dots, 4$, there are unit vectors $x_{kj}^{(m+1)}$ satisfying (4.4). Moreover, since $\lambda_{kj} \in \partial\sigma_e(T)$, the vectors $x_{kj}^{(m+1)}$ satisfying (4.4) can be found in any subspace of H of finite codimension. Thus we can find the vectors $x_{kj}^{(m+1)}$ (for one pair (k, j) after the other) such that (4.5)–(4.8) are also satisfied.

Indeed, consider the lexicographic order $\prec$ on the set $A := \{(k, j) : 1 \le m, j = 1, \dots, 4\}$. Let $(k, j) \in A$ and suppose that the vectors $x_{k'j'}^{(m)}$ satisfying (5)-(8) have already been constructed for all $(k', j') \in A, (k', j') \prec (k, j)$. Note that the set of all vectors $x_{kj}^{(m)}$ satisfying the conditions (4.5)–(4.8) is a subspace of finite codimension (since $\operatorname{codim} L < \infty$ implies $\operatorname{codim} T^{-n}(L) < \infty$ for every $n \in \mathbb{N}$ and since for subspaces $L_1, L_2 \subset H$ with $\operatorname{codim} L_i < \infty, i = 1, 2$, one has $\operatorname{codim}(L_1 \cap L_2) < \infty$). So using $\lambda_{kj} \in \partial\sigma_e(T)$ we may find a unit vector $x_{k,j}^{(m+1)}$ satisfying (4.4)–(4.8).

If we continue in the way described above, we can construct numbers n_m, elements $\lambda_{mj} \in A$ and vectors $x_{kj}^{(m)} \in H$ satisfying (4.1)–(4.8).

Note that for $m \in \mathbb{N}$, $s' < s \le m, k \le m, j = 1, \dots, 4$ we have

$$\begin{aligned}
& \left| \langle T^{n_s} x_{kj}^{(m)}, T^{n_{s'}} x_{kj}^{(m)} \rangle - \lambda_{kj}^{n_s - n_{s'}} \right| \\
\le\ & \left| \langle (T^{n_s} - \lambda_{kj}^{n_s}) x_{kj}^{(m)}, T^{n_{s'}} x_{kj}^{(m)} \rangle \right| + \left| \langle \lambda_{kj}^{n_s} x_{kj}^{(m)}, (T^{n_{s'}} - \lambda_{kj}^{n_{s'}}) x_{kj}^{(m)} \rangle \right| \\
\le\ & \varepsilon_m (\|T^{n_{s'}} x_{kj}^{(m)}\| + 1) \\
\le\ & \varepsilon_m (\|T^{n_{s'}} x_{kj}^{(m)} - \lambda_{kj}^{n_{s'}} x_{kj}^{(m)}\| + 2) \\
\le\ & 3\varepsilon_m.
\end{aligned}$$

Consequently, for $s,k<m$, $s\neq k$ we have

$$\begin{aligned} & \left|\left\langle T^{n_m}x_{kj}^{(m)},T^{n_s}x_{kj}^{(m)}\right\rangle -1\right| \\ \leq\ & \left|\left\langle T^{n_m}x_{kj}^{(m)},T^{n_s}x_{kj}^{(m)}\right\rangle -\lambda_{kj}^{n_m-n_s}\right| + |\lambda_{kj}^{n_m-n_s}-1| \\ \leq\ & 3\varepsilon_m + |(\lambda_{kj}^{n_m}-1)\bar\lambda_{kj}^{n_s}| + |\bar\lambda_{kj}^{n_s}-1| \qquad (4.9)\\ \leq\ & 3\varepsilon_m+2\varepsilon_k \\ \leq\ & 5\varepsilon_k. \end{aligned}$$

Similarly, for $k<m, j=1,\dots,4$ we have

$$\left|\left\langle T^{n_m}x_{kj}^{(m)},T^{n_k}x_{kj}^{(m)}\right\rangle -(-i)^j\right| \leq 5\varepsilon_k. \tag{4.10}$$

Note also that (4.3) implies that

$$\left|\left\langle T^{n_m}u,T^{n_s}u'\right\rangle\right| \leq \varepsilon_m^3\|u\|\cdot\|u'\|$$

for all $u,u'\in\bigvee\{x,x_{kj}^{(t)},k\leq t<m,s<m,j=1,\dots,4\}$.

The required vector y close to x will be constructed as x perturbed by an infinite linear combination of vectors $x_{kj}^{(m)}$. First we construct a vector y close to x such that $\langle T^{n_s}y,T^{n_{s'}}y\rangle = c_{n_s,n_{s'}}$ for all $s'<s\leq m$.

For $s'<s$ and $u\in H$ we write for short $d_{s,s'}(u)=\langle T^{n_s}u,T^{n_{s'}}u\rangle$. For $s\geq 2$ let $D_s(u)=\max_{s'<s}|d_{s,s'}(u)-c_{n_s,n_{s'}}|$.

Claim. Let $m\in\mathbb{N}$, $m\geq 2$, let

$$u = x+\sum_{s=1}^{m-1}\sum_{k,j}\beta_{kj}^{(s)}x_{kj}^{(s)},$$

where

$$\begin{aligned} \beta_{kj}^{(s)} &\geq \varepsilon_s \quad \text{for all} \quad k,j,s,\ 1\leq k\leq s\leq m-1,\ j=1,\dots,4,\\ \|u\|^2 &\leq 2 \quad\text{and}\quad D_s(u)=0,\ 2\leq s\leq m-1. \end{aligned}$$

Then there exists a vector

$$v = x+\sum_{s=1}^{m}\sum_{k,j}\gamma_{kj}^{(s)}x_{kj}^{(s)}$$

such that

$$\begin{aligned} \gamma_{kj}^{(s)} &\geq \beta_{kj}^{(s)}, \quad \text{for all} \quad k,j,s,\ 1\leq k\leq s\leq m-1, \gamma_{kj}^{(m)}\geq\varepsilon_m,\\ D_s(v) &= 0,\ 2\leq s\leq m, \quad\text{and}\quad \|v-u\|\leq 2\varepsilon_m^{1/8}. \end{aligned}$$

Proof. Set $\beta_{kj}^{(m)}=\varepsilon_m$ for all $k\leq m, j=1,\dots,4$. Let

$$a = x+\sum_{s=1}^{m}\sum_{k,j}\beta_{kj}^{(s)}x_{kj}^{(s)} = u+\sum_{k,j}\varepsilon_m x_{kj}^{(m)}.$$

Then $\|a-u\| = \sqrt{4m\varepsilon_m^2} = 2m^{1/2}\varepsilon_m$.

Let M be the set of all vectors b of the form

$$b = x + \sum_{s=1}^{m} \sum_{k,j} \eta_{kj}^{(s)} x_{kj}^{(s)}, \tag{4.11}$$

where $\eta_{kj}^{(s)} \geq \beta_{kj}^{(s)}$ and $\sum_{k,j}((\eta_{kj}^{(s)})^2 - (\beta_{kj}^{(s)})^2) \leq 2\varepsilon_s$ for all $s \leq m$.

Clearly M is a compact set.

Let $Q : M \to \langle 0, \infty)$ be the function defined for b of the form (4.11) by

$$Q(v) = \sum_{s=2}^{m} \varepsilon_s^{-3/2} D_s(v) + \sum_{s=1}^{m} \varepsilon_s^{-1} \sum_{k,j} \bigl((\eta_{kj}^{(s)})^2 - (\beta_{kj}^{(s)})^2\bigr).$$

Clearly Q is a continuous function. Let

$$v = x + \sum_{s=1}^{m} \sum_{k,j} \gamma_{kj}^{(s)} x_{kj}^{(s)}$$

be a point in M where Q attains its minimum.

First of all we calculate the value of $Q(a)$. For $s' < s < m$ we have

$$\begin{aligned} d_{s,s'}(a) &= \langle T^{n_s} u, T^{n_{s'}} u\rangle + \langle T^{n_s}(a-u), T^{n_{s'}} u\rangle \\ &+ \langle T^{n_s} u, T^{n_{s'}}(a-u)\rangle + \langle T^{n_s}(a-u), T^{n_{s'}}(a-u)\rangle, \end{aligned}$$

where $\langle T^{n_s} u, T^{n_{s'}} u\rangle = c_{n_s, n_{s'}}$ by assumption and

$$\langle T^{n_s}(a-u), T^{n_{s'}} u\rangle = \langle T^{n_s} u, T^{n_{s'}}(a-u)\rangle = 0$$

by (4.7). Thus

$$\begin{aligned} |d_{s,s'}(a) - c_{n_s,n_{s'}}| &= |d_{s,s'}(a-u)| \\ &\leq \sum_{k,j} \varepsilon_m^2 |d_{s,s'}(x_{kj}^{(m)})| \\ &\leq 4m\varepsilon_m^2 \max_{k,j} |d_{s,s'}(x_{kj}^{(m)})| \\ &\leq 8m\varepsilon_m^2, \end{aligned}$$

since

$$\begin{aligned} |d_{s,s'}(x_{kj}^{(m)})| &= |\langle T^{n_s} x_{kj}^{(m)}, T^{n_{s'}} x_{kj}^{(m)}\rangle| \\ &\leq \|T^{n_s} x_{kj}^{(m)}\| \cdot \|T^{n_{s'}} x_{kj}^{(m)}\| \\ &\leq (1+\varepsilon_m)^2 \\ &\leq 2. \end{aligned}$$

Hence $D_s(a) \leq 8m\varepsilon_m^2$ for each $s < m$.

For $s < m$ we have

$$\begin{aligned}
|d_{m,s}(a)| &\le \big|\langle T^{n_m}u, T^{n_s}u\rangle\big| + \big|\langle T^{n_m}(a-u), T^{n_s}u\rangle\big| \\
&+ \big|\langle T^{n_m}u, T^{n_s}(a-u)\rangle\big| + \big|\langle T^{n_m}(a-u), T^{n_s}(a-u)\rangle\big| \\
&\le \varepsilon_m^3\|u\|^2 + 0 + 0 + \sum_{k,j}\varepsilon_m^2|d_{m,s}(x_{k,j}^{(m)})| \\
&\le 2\varepsilon_m^2 + 8m\varepsilon_m^2 \\
&\le 10m\varepsilon_m^2.
\end{aligned}$$

Thus

$$D_m(a) \le \max_{s<m}(|d_{m,s}(a)| + |c_{n_m,n_s}|) \le 10m\varepsilon_m^2 + \varepsilon_m^2 \le 11m\varepsilon_m^2.$$

Hence

$$\begin{aligned}
Q(a) &= \sum_{s\le m}\varepsilon_s^{-3/2}D_s(a) \\
&\le \sum_{s\le m}\varepsilon_s^{-3/2}11m\varepsilon_m^2 \\
&\le 11m^2\varepsilon_m^{1/2} \\
&\le \varepsilon_m^{1/4}.
\end{aligned}$$

Since $Q(v) \le Q(a) \le \varepsilon_m^{1/4}$, we have in particular

$$\sum_{k,j}((\gamma_{kj}^{(s)})^2 - (\beta_{kj}^{(s)})^2) \le \varepsilon_s\varepsilon_m^{1/4}$$

for each $s \le m$.

We show that v is the required vector. Clearly $\gamma_{kj}^{(s)} \ge \varepsilon_s$. For all k,j we have

$$\gamma_{kj}^{(s)} - \beta_{kj}^{(s)} = \frac{(\gamma_{kj}^{(s)})^2 - (\beta_{kj}^{(s)})^2}{\gamma_{kj}^{(s)} + \beta_{kj}^{(s)}} \le \frac{\varepsilon_s\varepsilon_m^{1/4}}{2\varepsilon_s} = \frac{\varepsilon_m^{1/4}}{2}.$$

Hence

$$\|v-a\| = \Big(\sum_s\sum_{k,j}(\gamma_{kj}^{(s)} - \beta_{kj}^{(s)})^2\Big)^{1/2} \le \sqrt{4m^2\frac{\varepsilon_m^{1/2}}{4}} = m\varepsilon_m^{1/4} \le \varepsilon_m^{1/8}$$

and

$$\|v-u\| \le \|v-a\| + \|a-u\| \le \varepsilon_m^{1/8} + 2m^{1/2}\varepsilon_m \le 2\varepsilon_m^{1/8}.$$

Furthermore

$$\|v\| \le \|v-u\| + \|u\| \le 2\varepsilon_m^{1/8} + \sqrt{2} \le 2.$$

Suppose on the contrary that $D_s(v) \ne 0$ for some $s \le m$. We show that in this case we can perturb v a little so that we obtain a vector $w \in M$ with $Q(w) < Q(v)$; this would be a contradiction.

In the following we fix $s \le m$ such that $D_s(v) \ne 0$. Note that $D_s(v) \le \varepsilon_s^{3/2}$ since $Q(v) \le Q(a) \le 1$.

Lemma 4.2. *Suppose that $D_s(v)\neq 0$. Then there are $\delta_{kj}\geq\gamma_{kj}^{(s)}$ such that the vector*

$$w=v+\sum_{k,j}(\delta_{kj}-\gamma_{kj}^{(s)})x_{kj}^{(s)}$$

satisfies $w\in M$ and $Q(w)<Q(v)$.

Proof. Consider Lemma 3.4 for $m=s-1$. For $t\leq s-1$ write

$$\begin{aligned}\mathbf{c}(t)\quad&=c_{n_s,n_t}-d_{s,t}(v),\\ \mathbf{w}_{kj}(t)\quad&=\langle T^{n_s}x_{kj}^{(s)},T^{n_t}x_{kj}^{(s)}\rangle,\\ \mathbf{w}(t)\quad&=\langle T^{n_s}x_{s,2}^{(s)},T^{n_t}x_{s,2}^{(s)}\rangle.\end{aligned}$$

By (4.9) and (4.10), the conditions of Lemma 3.4 are satisfied. So there are numbers $\alpha,\alpha_{kj}\geq 0,k\leq s-1,j=1,\dots,4$ such that

$$\alpha\mathbf{w}+\sum_{kj}\alpha_{kj}\mathbf{w}_{kj}=\mathbf{c}$$

and

$$\alpha+\sum\alpha_{kj}\leq 8(s-1)\|\mathbf{c}\|_\infty=8(s-1)D_s(v).$$

Set

$$\begin{aligned}\delta_{kj}\quad&=\sqrt{\alpha_{kj}+(\gamma_{kj}^{(s)})^2},\quad k\leq s-1,\\ \delta_{s,2}\quad&=\sqrt{\alpha+(\gamma_{s,2}^{(s)})^2},\\ \delta_{s,j}\quad&=\gamma_{s,j}^{(s)},\qquad j\neq 2.\end{aligned}$$

Let

$$w=v+\sum_{k,j}(\delta_{kj}-\gamma_{kj}^{(s)})x_{kj}^{(s)}.$$

Then for $t<s$ we have

$$\begin{aligned}\sum_{k,j}(\delta_{kj}^2-(\gamma_{kj}^{(s)})^2)d_{s,t}(x_{kj}^{(s)})\quad&=\quad\sum_{k,j}\alpha_{kj}w_{kj}(t)+\alpha w(t)\\ &=\quad c_{n_s,n_t}-d_{s,t}(v),\end{aligned}$$

and

$$\sum_{k,j}(\delta_{kj}^2-(\gamma_{kj}^{(s)})^2)\leq 8(s-1)D_s(v)\leq\varepsilon_s. \tag{4.12}$$

Thus

$$\sum_{k,j}\bigl(\delta_{kj}^2-(\gamma_{kj}^{(s)})^2\bigr)+\sum_{k,j,s}\bigl((\gamma_{kj}^{(s)})^2-(\beta_{kj}^{(s)})^2\bigr)\leq 2\varepsilon_s$$

and so $w\in M$.

It remains to show that $Q(w)<Q(v)$.

For $t\in\mathbb{N}$ let P_t be the orthogonal projection onto $\bigvee\{x_{kj}^{(t)}:k\leq t,j=1,\dots,4\}$. Denote by P_0 the orthogonal projection onto the one-dimensional subspace generated by x.

For $s' < s$ we have by (4.7)

$$\begin{aligned}
d_{s,s'}(w) &= \langle T^{n_s}v, T^{n_{s'}}v\rangle + \langle T^{n_s}(w-v), T^{n_{s'}}v\rangle \\
&+ \langle T^{n_s}v, T^{n_{s'}}(w-v)\rangle + \langle T^{n_s}(w-v), T^{n_{s'}}(w-v)\rangle \\
&= d_{s,s'}(v) + \langle T^{n_s}P_s(w-v), T^{n_{s'}}P_s v\rangle \\
&+ \langle T^{n_s}P_s v, T^{n_{s'}}P_s(w-v)\rangle + \langle T^{n_s}P_s(w-v), T^{n_{s'}}P_s(w-v)\rangle \\
&= d_{s,s'}(v) + \langle T^{n_s}P_s w, T^{n_{s'}}P_s w\rangle - \langle T^{n_s}P_s v, T^{n_{s'}}P_s v\rangle \\
&= d_{s,s'}(v) + d_{s,s'}(P_s w) - d_{s,s'}(P_s v) \\
&= d_{s,s'}(v) + \sum_{k,j}(\delta_{kj}^2 - (\gamma_{kj}^{(s)})^2) d_{s,s'}(x_{kj}^{(s)}) \\
&= c_{n_s,n_{s'}}.
\end{aligned}$$

Hence $D_s(w) = 0$.

For $t' < t < s$ we have

$$\begin{aligned}
& |d_{t,t'}(w) - d_{t,t'}(v)| \\
= & \left|\langle T^{n_t}(w-v), T^{n_{t'}}v\rangle + \langle T^{n_t}v, T^{n_{t'}}(w-v)\rangle + \langle T^{n_t}(w-v), T^{n_{t'}}(w-v)\right| \\
= & \left|\langle T^{n_t}P_s(w-v), T^{n_{t'}}P_s v\rangle + \langle T^{n_t}P_s v, T^{n_{t'}}P_s(w-v)\rangle + \langle T^{n_t}P_s(w-v), T^{n_{t'}}P_s(w-v)\right| \\
= & \left|\langle T^{n_t}P_s w, T^{n_{t'}}P_s w\rangle - \langle T^{n_t}P_s v, T^{n_{t'}}P_s v\rangle\right| \\
\leq & \sum_{k,j}(\delta_{kj}^2 - (\gamma_{kj}^{(s)})^2)|d_{t,t'}(x_{kj}^{(s)})| \\
\leq & 16 s D_s(v),
\end{aligned}$$

where we again used the estimate $|d_{t,t'}(x_{kj}^{(s)})| \leq 2$ and (4.12). Hence

$$D_t(w) \leq D_t(v) + 16 s D_s(v)$$

for each $t < s$.

Finally, for $t > s, t' < t$ we have

$$\begin{aligned}
& |d_{t,t'}(w) - d_{t,t'}(v)| \\
= & \Big|\langle T^{n_t}(w-v), T^{n_{t'}}v\rangle + \langle T^{n_t}v, T^{n_{t'}}(w-v)\rangle + \langle T^{n_t}(w-v), T^{n_{t'}}(w-v)\Big| \\
\le & \big|\langle T^{n_t}P_s(w-v), T^{n_{t'}}\sum_{l=0}^{t-1}P_l v\rangle\big| + \big|\langle T^{n_t}\sum_{l=0}^{t-1}P_l v, T^{n_{t'}}P_s(w-v)\rangle\big| \\
+ & \big|\langle T^{n_t}(w-v), T^{n_{t'}}(w-v)\big| \\
\le & \sum_{k,j}\Big(\delta_{kj} - \gamma_{kj}^{(s)}\Big)\Big|\langle T^{n_t}x_{kj}^{(s)}, T^{n_{t'}}\sum_{l=0}^{t-1}P_l v\rangle\Big| + \sum_{k,j}\Big(\delta_{kj} - \gamma_{kj}^{(s)}\Big)\Big|\langle T^{n_t}\sum_{l=0}^{t-1}P_l v, T^{n_{t'}}x_{kj}^{(s)}\rangle\Big| \\
+ & \varepsilon_t^3\|w-v\|^2 \\
\le & 2\sum_{k,j}\Big(\delta_{kj} - \gamma_{kj}^{(s)}\Big)\varepsilon_t^3\Big\|\sum_{l=0}^{t-1}P_l v\Big\| + \varepsilon_t^3\sum_{k,j}\Big(\delta_{kj} - \gamma_{kj}^{(s)}\Big)^2 \\
\le & 4\varepsilon_t^3\sum_{k,j}\frac{\delta_{kj}^2 - (\gamma_{kj}^{(s)})^2}{\delta_{kj} + \gamma_{kj}^{(s)}} + \varepsilon_t^3\sum_{k,j}\left(\frac{\delta_{kj}^2 - (\gamma_{kj}^{(s)})^2}{\delta_{kj} + \gamma_{kj}^{(s)}}\right)^2 \\
\le & \frac{4\varepsilon_t^3}{2\varepsilon_s}\sum_{k,j}\Big(\delta_{kj}^2 - (\gamma_{kj}^{(s)})^2\Big) + \frac{\varepsilon_t^3}{4\varepsilon_s^2}\sum_{k,j}\Big(\delta_{kj}^2 - (\gamma_{kj}^{(s)})^2\Big)\cdot\max_{k,j}\{\delta_{kj}^2 - (\gamma_{kj}^{(s)})^2\} \\
\le & 8sD_s(v)\Big(\frac{4\varepsilon_t^3}{2\varepsilon_s} + \frac{\varepsilon_t^3}{4\varepsilon_s}\Big) \\
= & \frac{18s\varepsilon_t^3}{\varepsilon_s}D_s(v) \\
\le & \varepsilon_t^2 D_s(v).
\end{aligned}$$

Thus

$$D_t(w) \le D_t(v) + \varepsilon_t^2 D_s(v)$$

for $s < t \le m$. Hence

$$\begin{aligned}
Q(w) - Q(v) \quad \le & \quad \sum_{t<s}16sD_s(v)\varepsilon_t^{-3/2} - \varepsilon_s^{-3/2}D_s(v) \\
+ & \quad \sum_{t>s}\varepsilon_t^{1/2}D_s(v) + \varepsilon_s^{-1}\sum_{k,j}\Big(\delta_{kj}^2 - (\gamma_{kj}^{(s)})^2\Big) \\
\le & \quad D_s(v)\big(16s^2\varepsilon_{s-1}^{-3/2} - \varepsilon_s^{-3/2} + 2\varepsilon_{s+1}^{1/2} + 8s\varepsilon_s^{-1}\big) \\
< & \quad 0.
\end{aligned}$$

This proves lemma. $\square$

Now the Claim follows.

Let $y_1 = x + \sum_{j=1}^{4} \varepsilon_1 x_{1,j}^{(1)}$. Then $\|y_1 - x\| = \sqrt{4\varepsilon_1^2} = 2\varepsilon_1$. By Claim for $m = 2$, $u = y_1$ there exists $y_2 \in \bigvee\{x, x_{kj}^{(s)}, s \le 2\}$ such that $D_2(y_2) = 0$ and $\|y_2 - y_1\| \le 2\varepsilon_2^{1/8}$.

Since y_2 was constructed in such a way that it also satisfies the conditions of Claim, there exists $y_3 \in \bigvee\{x, x_{kj}^{(s)}, s \le 3\}$ such that $D_2(y_3) = D_3(y_3) = 0$ and $\|y_3 - y_2\| \le 2\varepsilon_3^{1/8}$.

If we continue to construct vectors y_m in this way, we obtain a sequence $\{y_m\} \subset H$ such that $D_s(y_m) = 0$ for all $s \le m$ and $\|y_m - y_{m-1}\| \le 2\varepsilon_m^{1/8}$. So the sequence $\{y_m\}$ is convergent. Denote its limit by y. Then $D_s(y) = 0$ for all s, i.e., $\langle T^{n_s} y, T^{n_{s'}} y\rangle = c_{n_s, n_{s'}}$ for all $s, s' \in \mathbb{N}, s' < s$. Moreover,

$$\|y - x\| \le \|y_1 - x\| + \sum_{m=2}^{\infty} \|y_m - y_{m-1}\| \le 2\varepsilon_1 + 2\sum_{m=2}^{\infty} \varepsilon_m^{1/8} < \varepsilon.$$

This proves Theorem 4.1. □

The following corollary of Theorem 4.1 is immediate.

Corollary 4.3. *Let $T \in B(H)$ be a power bounded operator such that $\sigma(T) \cap \mathbb{T}$ is infinite. The following two conditions are equivalent:*

(i) $\sigma_p(T) \cap \mathbb{T} = \emptyset$;

(ii) there exists a dense subset of H consisting of weakly wandering vectors.

Proof. (i)⇒(ii): Set $c_{p,q} = 0$ in the previous theorem.

(ii)⇒(i): We give a simple proof of this implication not depending of the de Leuuw-Glicksberg theorem. Suppose that $\sigma_p(T) \cap \mathbb{T} \ne \emptyset$. Let $\lambda \in \sigma_p(T)$, $|\lambda| = 1$. Let x be a corresponding eigenvector of norm 1. Let $K = \sup_n \|T^n\|$ and let $y \in H$ satisfy $\|y - x\| < \frac{1}{2K^2}$. We show that the orbit of T at y does not contain two orthogonal vectors, and so y is not weakly wandering for T.

Let $m, n \in \mathbb{N}$, $m \ne n$. Then

$$\begin{aligned} &\big|\langle T^m y, T^n y\rangle - \lambda^{m-n}\big| = \big|\langle T^m y, T^n y\rangle - \langle T^m x, T^n x\rangle\big| \\ &\le \big|\langle T^m y, T^n (y - x)\rangle\big| + \big|\langle T^m (y - x), T^n x\rangle\big| \le 2K^2\|y - x\| < 1. \end{aligned}$$

Thus $\langle T^m y, T^n y\rangle \ne 0$ and y is not weakly wandering for T. Hence the set of weakly wandering vectors is not dense. □

In fact neither the power boundedness of T nor the condition that $\sigma_p(T) \cap \mathbb{T} = \emptyset$ was used in the proof of Theorem 4.1. The crucial property was that $\mathrm{D} - \lim_{n\to\infty}\langle T^n x, y\rangle = 0$ for all $x, y \in H$, or at least for all x, y in a dense subset of H invariant for T. Thus the theorem can be formulated as follows.

Theorem 4.4. *Let $T \in B(H)$ satisfy the following two conditions:*

(i) *$\sigma_{\pi e}(T) \cap \mathbb{T}$ is infinite;*

(ii) *there exists a dense subset $M \subset H$ such that $T(M) \subset M$ and $\mathrm{D} - \lim_{n\to\infty}\langle T^n u, v\rangle = 0$ for all $u, v \in M$.*

Then there is a dense subset of H consisting of weakly wandering vectors for T.

In particular, the first condition is satisfied if $r(T) = 1$ and $\sigma(T) \cap \mathbb{T}$ is infinite.

Corollary 4.5. *Let $T \in B(H)$, $S \in B(H)$ be power bounded operators, $\sigma_p(T) \cap \mathbb{T} = \emptyset = \sigma_p(S) \cap \mathbb{T}$. Let $\sigma(S) \cap \mathbb{T}$ be infinite. Then for every $x \in H$ there exists $y \in H$ such that $x \oplus y$ is weakly wandering for $T \oplus S$.*

Proof. For $p, q \in \mathbb{N}, p > q$, set $c_{p,q} = -\langle T^p x, T^q x\rangle$. Then the numbers $c_{p,q}$ satisfy the conditions of Theorem 4.1. Thus there exists $y \in K$ and an increasing sequence $\{n_k\}$ such that $\langle S^{n_k} y, S^{n_s} y\rangle = c_{n_k,n_s}$ for all $k, s \in \mathbb{N}, k > s$. Hence $x \oplus y$ is a weakly wandering vector for $T \oplus S$. ☐

Corollary 4.6 (Krengel)**.** *Let $T \in B(H)$ be an isometry such that $\sigma_p(T) = \emptyset$. Then there exists a dense set of weakly wandering vectors for T.*

Proof. Let $T = U \oplus S$ be the Wold decomposition of T, i.e., U is a unitary operator and S a unilateral shift (of some multiplicity).

If the shift part is non-trivial, then $\sigma(T) \cap \mathbb{T} = \mathbb{T}$ and the condition of Theorem 4.4 is satisfied.

If the shift part of T is trivial then T is a unitary operator without eigenvalues, and so $\sigma(T) \cap \mathbb{T} = \sigma(T)$ has no isolated points. So we can again apply Theorem 4.4. ☐

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