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A proposal for the modification of the notion of quasiconvexity

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Abstract This note proposes to modify the definition of quasiconvexity of a function $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ on the space $\mathbb{M}^{m\times n}$ of $m \times n$ matrices in such a way that (i) the polyconvexity implies quasiconvexity without any additional measurability or continuity assumption on f and (ii) the pointwise supremum of any family of quasiconvex functions is a quasiconvex function. Property (ii) allows one to define the quasiconvex envelope f^{qc} of any $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ as the largest quasiconvex minorant of f; this, in turn, makes it possible to establish the formula similar to that in Dacorogna [4, 6] for f^{qc}: If $E \subset \mathbb{R}^n$ is a nonempty bounded open set with $|\partial E| = 0$ then for any $\boldsymbol{A} \in \mathbb{M}^{m\times n}$ we have

$$f^{qc}(\boldsymbol{A}) = \inf\Big\{|E|^{-1}\int_E f(\boldsymbol{A} + D\boldsymbol{u})\,d\boldsymbol{x}\Big\}$$

where the infimum is taken over all $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$ such that the integral $\int_E f(\boldsymbol{A}+D\boldsymbol{u})\,d\boldsymbol{x}$ is well defined and there exists a partition of E into a set of measure 0 and a finite number of open sets such that $D\boldsymbol{u}$ is essentially constant on each of these open sets. The definition of quasiconvexity coincides with the original definition of Morrey [12], [13; Section 4.4] and Ball [1] if f is finite valued. If $f : \mathbb{M}^{m\times n} \to [-\infty, \infty)$ then the definition coincides with those in [3, 10–11, 14]; if f has ∞ in its range then there are functions quasiconvex in the present sense but not quasiconvex in the sense of [3, 10–11, 14]. An example is given to show this.

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1 Introduction

Morrey's quasiconvexity [12], [13; Section 4.4] is one of the central notions in the calculus of variations. With appropriate growth conditions it is equivalent to the weak sequential lowersemicontinuity of the multiple integrals of the calculus of variations and thus opens the way to the direct methods of proofs of the existence of minimizers. The quasiconvexity is expressed by the inequality (5) below in which f is the integrand and $\boldsymbol{u}$ a Sobolev function. Clearly, conditions are needed to ensure that the integral makes sense as a Lebesgue integral. For example, one can assume that f is Borel measurable to ensure the Lebesgue measurability of the compound function $f(\boldsymbol{A}+\boldsymbol{Du}(\cdot))$ for any $\boldsymbol{u}$ etc. The literature contains a number of definitions of quasiconvexity which express the same basic idea of Morrey but differ in the technical details imposed on f. The existing definitions do not guarantee that the supremum of any family of quasiconvex functions is quasiconvex so that the quasiconvex envelope can be defined only in certain special (albeit broad) cases. Also, with the existing definitions the convexity or polyconvexity does not unconditionally imply quasiconvexity since there are non-Borelian convex or polyconvex functions. These facts may be interpreted as esthetically displeasing.

The purpose of this note is to give a definition of quasiconvexity which does not involve any measurability or continuity assumptions on f. The idea is to let $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ be arbitrary but to require that $\boldsymbol{u}$ in (5) be piecewise affine in the sense that the essential range of $\boldsymbol{Du}$ contains only a finite number of points realized on open subsets of E. Then $f(\boldsymbol{A} + \boldsymbol{Du}(\cdot))$ is Lebesgue measurable for any f and one may postulate (5) for every piecewise affine $\boldsymbol{u}$ for which the integral makes sense. This makes the quasiconvexity condition "algebraic" since the integral reduces to the obvious sum. Under this definition, (i) the polyconvexity implies quasiconvexity without any additional measurability or continuity assumption on f and (ii) the pointwise supremum of any family of quasiconvex functions is a quasiconvex function. The present definition coincides with the existing definitions for a broad class of integrands f which includes the finite valued functions but there are circumstances when the present definition is less restrictive than those given hitherto. These matters are discussed in detail below in this introduction.

Let m, n be positive integers and let $\mathbb{M}^{m\times n}$ denote the set of all $m \times n$ matrices. If E is an open bounded subset of $\mathbb{R}^n$ we say that a function $\boldsymbol{u} : E \to \mathbb{R}^m$ is Q-piecewise affine if $\boldsymbol{u}$ is Lipschitz continuous and E can be partitioned into a set of measure 0 and a finite number of open sets such that $\boldsymbol{Du}$ is essentially constant on each of these open sets. We say that a function $\boldsymbol{u} : E \to \mathbb{R}^m$ is piecewise affine if $\boldsymbol{u}$ is Lipschitz continuous and E can be partitioned into a set of measure 0 and a finite number of open sets such that $\boldsymbol{u}$ is affine on each of these open sets. Thus every piecewise affine function is Q-piecewise affine but not conversely. We denote by $Q(E, \mathbb{R}^m)$ the set of all Q-piecewise affine functions $\boldsymbol{u}$ on E whose extension by $\boldsymbol{0}$ on $\mathbb{R}^n \sim E$ is Lipschitz continuous. If $\boldsymbol{A} \in \mathbb{M}^{m\times n}$ we denote by $\mathbf{M}(\boldsymbol{A}) \in \mathbb{R}^{s(m,n)}$ the collection of all minors (of all orders) of $\boldsymbol{A}$ ordered in some definite way; here $s(m,n)$ is the number of all possible minors of an $m \times n$ matrix. Note that minors of order 1 constitute the elements of the matrix. If $t \in \bar{\mathbb{R}}$ we denote by $[t]_\pm \in [0, \infty]$ the positive and negative parts of t. If $m : E \to \bar{\mathbb{R}}$ is a function we say that the integral $\int_E m\, d\boldsymbol{x}$ is well defined if m is Lebesgue measurable and either $\int_E [m]_+\, d\boldsymbol{x} < \infty$ or $\int_E [m]_-\, d\boldsymbol{x} < \infty$; we then define the integral in the usual way. If $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ and

$$\boldsymbol{A}_i \in \mathbb{M}^{m\times n}, \quad \alpha_i \ge 0, \quad i = 1, \dots, p, \quad \text{and} \quad \sum_{i=1}^{p} \alpha_i = 1 \tag{1}$$

where p is a positive integer, we say that the sum $\sum_{i=1}^p \alpha_i f(\boldsymbol{A}_i)$ is well defined if either $\sum_{i=1}^p \alpha_i [f(\boldsymbol{A}_i)]_+ < \infty$ or $\sum_{i=1}^p \alpha_i [f(\boldsymbol{A}_i)]_- < \infty$; we then define the sum in the usual way.

Definition 1.1. If $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ and $\boldsymbol{A} \in \mathbb{M}^{m\times n}$ we say that

(i) f is convex at $\boldsymbol{A}$ if

$$f(\boldsymbol{A}) \le \sum_{i=1}^{p} a_i f(\boldsymbol{A}_i) \tag{2}$$

for any collections as in (1) such that the sum in (2) is well defined and

$$\boldsymbol{A} = \sum_{i=1}^{p} a_i \boldsymbol{A}_i; \tag{3}$$

(ii) f is polyconvex at $\boldsymbol{A}$ if (2) holds for any collections as in (1) such that the sum in (2) is well defined and

$$\mathbf{M}(\boldsymbol{A}) = \sum_{i=1}^{p} a_i \mathbf{M}(\boldsymbol{A}_i); \tag{4}$$

(iii) f is quasiconvex at $\boldsymbol{A}$ if

$$|E| f(\boldsymbol{A}) \le \int_E f(\boldsymbol{A} + D\boldsymbol{u})\, d\boldsymbol{x} \tag{5}$$

for every nonempty bounded open set $E \subset \mathbb{R}^n$ with $|\partial E| = 0$ and for every $\boldsymbol{u} \in Q(E, \mathbb{R}^m)$ for which the integral in (5) is well defined;

(iv) f is rank 1 convex at $\boldsymbol{A}$ if

$$f(\boldsymbol{A}) \le (1-\alpha) f(\boldsymbol{A} + \alpha \boldsymbol{a} \otimes \boldsymbol{b}) + \alpha f(\boldsymbol{A} - (1-\alpha)\boldsymbol{a} \otimes \boldsymbol{b}) \tag{6}$$

for every $\alpha \in [0,1]$ and every $\boldsymbol{a} \in \mathbb{R}^m, \boldsymbol{b} \in \mathbb{R}^n$ such that the sum in (6) is well defined.

We say that $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ is convex (polyconvex, quasiconvex, rank 1 convex) if f is convex (polyconvex, quasiconvex, rank 1 convex) at every point of $\mathbb{M}^{m\times n}$.

In (iii) the function $m := f(\boldsymbol{A} + D\boldsymbol{u}(\cdot))$ is always Lebesgue measurable on E. The definition of convexity is the standard one [16, 7]. A more usual definition of polyconvexity is to assume the existence of a convex function $g : \mathbb{R}^{s(m,n)} \to \bar{\mathbb{R}}$ such that $f = g \circ \mathbf{M}$. Under the present definition one can define g as the convex envelope of the function $h : \mathbb{R}^{s(m,n)} \to \bar{\mathbb{R}}$ defined by

$$h(\mathfrak{A}) = \begin{cases} f(\boldsymbol{A}) & \text{if} \quad \mathfrak{A} = \mathbf{M}(\boldsymbol{A}) \ \text{ for some } \ \boldsymbol{A} \in \mathbb{M}^{m\times n}, \\ \infty & \text{otherwise,} \end{cases} \tag{7}$$

$\mathfrak{A} \in \mathbb{R}^{s(m,n)}$. Indeed the condition of polyconvexity ensures that the convex envelope g of h satisfies $f = g \circ \mathbf{M}$.

The main point of this note is the definition of quasiconvexity in (iii). Morrey's original definition [12], [13; Section 4.4] requires that f be finite valued and continuous and that (5) holds for any bounded open set $E \subset \mathbb{R}^n$, any $\boldsymbol{A} \in \mathbb{M}^{m\times n}$, and any $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$. Here $W_0^{1,\infty}(E, \mathbb{R}^m)$ is the space of all Lipschitz continuous functions $\boldsymbol{u} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\boldsymbol{u} = \mathbf{0}$ on $\mathbb{R}^n \sim E$. We note that the continuity of f implies that the function $m := f(\boldsymbol{A} + D\boldsymbol{u}(\cdot))$ is Lebesgue integrable on E. We shall see below that in the class of finite valued functions the present definition is equivalent to Morrey's. If f is a general function and $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$ then a restriction on f or $\boldsymbol{u}$ is needed to make the function m Lebesgue measurable and the integral in (5) well defined. Thus Ball & Murat [3] assume additionally that f is a

Borel function, bounded from below, DACOROGNA [6] that f is a finite valued Borel function, locally bounded, and HÜSSEINOV [10–11], MÜLLER [14] postulate (5) for a general $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ and only for those $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$ for which the integral in (5) is well defined.

Since (4) implies (3), the definitions above give

$$f \text{ convex} \quad \Rightarrow \quad f \text{ polyconvex}.$$

We shall also see below that

$$f \text{ polyconvex} \quad \Rightarrow \quad f \text{ quasiconvex}, \tag{8}$$

and

$$\text{the supremum of any family of quasiconvex functions is quasiconvex.} \tag{9}$$

We consider it desirable to have (8) to hold unconditionally for the economy of thought (as well as to have the definition of quasiconvexity applicable to any function). The assertion (9) is parallel to the similar assertions for convex, polyconvex, and rank 1 convex functions; the assertion (9) implies that

$$\text{every } f \text{ has the largest quasiconvex minorant.}$$

By a minorant of $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ we mean any $g : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ such that $g \le f$ on $\mathbb{M}^{m\times n}$. The largest quasiconvex minorant is clearly unique, it is denoted by f^{qc} and called the quasiconvex envelope of f. Such a function exists since by (9) one can put

$$f^{qc}(\boldsymbol{A}) = \sup \{ g(\boldsymbol{A}) : g \text{ a quasiconvex minorant of } f \}, \tag{10}$$

$\boldsymbol{A} \in \mathbb{M}^{m\times n}$. (Without (9) it is not clear whether the right hand side of (10) is quasiconvex.)

Clearly, the implication (8) and the assertion (9) are not satisfied by the definition in [13]. Since there are non–Borelian convex functions and since there is a family of Borelian convex functions whose supremum is not Borelian (see Example 2.4), the definition in [3] does not satisfy (8), (9) and there exists a function $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ which does not have a quasiconvex envelope under the definition of quasiconvexity in [3]. With the definition of quasiconvexity in [10–11] and [14] we have (8) (see below) but it is not clear if (9) holds (see Remark 2.3).

We now employ Definition 1.1 and use standard arguments in the theory of quasiconvex functions to establish the following results (see Section 2 for the proofs). First, we use the Vitali covering argument of [3] to prove

Proposition 1.2. *If* (5) *holds for one nonempty bounded open set* $E \subset \mathbb{R}^n$, *some* $\boldsymbol{A} \in \mathbb{M}^{m\times n}$ *and all* $\boldsymbol{u} \in Q(E, \mathbb{R}^m)$ *then* f *is quasiconvex at* $\boldsymbol{A}$.

We note that this proposition is no longer true if (5) is postulated to hold only for piecewise affine functions $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$; indeed this is the reason why (5) is postulated for functions from $Q(E, \mathbb{R}^m)$.

Proposition 1.3. *Let* $E \subset \mathbb{R}^n$ *be a nonempty bounded open set and let* $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ *be finite-valued.*

(i) *If* f *is continuous then* f *is quasiconvex at* $\boldsymbol{A} \in \mathbb{M}^{m\times n} \Leftrightarrow$ (5) *holds for all* $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m) \Leftrightarrow$ (5) *holds for all* $\boldsymbol{u} \in C_0^\infty(E, \mathbb{R}^m)$.

(ii) *f is quasiconvex* $\Leftrightarrow$ *f is continuous and* (5) *holds for all $\boldsymbol{A}$ and all $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$* $\Leftrightarrow$ *f is continuous and* (5) *holds for all $\boldsymbol{A}$ and all $\boldsymbol{u} \in C_0^\infty(E, \mathbb{R}^m)$.*

Item (i) above is proved by the density of piecewise affine functions in $W_0^{1,\infty}(E, \mathbb{R}^m)$ (see Proposition 2.2, below). By (ii), within the class of finite valued functions the present definition is equivalent to the original definition of Morrey and also to the definition in [1]. In particular, the lower semicontinuity and relaxation theorems, which are available only for finite valued functions, continue to hold also with the present definition. The main point in (ii) is the continuity as a consequence of the global quasiconvexity; this is proved by showing that f is rank 1 convex by an argument by Fonseca [9] and by invoking the result [13; Theorem 4.4.1] that each finite valued rank 1 convex function is locally Lipschitz continuous.

Proposition 1.4. *If $f : \mathbb{M}^{m\times n} \to [-\infty, \infty)$ is quasiconvex then either $f \equiv -\infty$ on $\mathbb{M}^{m\times n}$ or f is finite valued and hence* (5) *holds for all nonempty bounded open subsets E of $\mathbb{R}^n$, all $\boldsymbol{A}$ and all $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$.*

Hence the present definition of quasiconvexity is equivalent to that given in [10–11, 14] if $f < \infty$ on $\mathbb{M}^{m\times n}$. For functions f taking the value ∞ the present definition of quasiconvexity is less restrictive than those given in [3, 10–11, 14] as the following example, based on [2], shows.

Example 1.5. If $m \geq n \geq 3$ or $n = 2$ and $m \geq 4$ then there exists a quasiconvex and rank 1 convex lower semicontinuous function $f : \mathbb{M}^{m\times n} \to \{0, \infty\}$ such that $f(\boldsymbol{B}) = \infty$ for some $\boldsymbol{B} \in \mathbb{M}^{m\times n}$ and yet

$$\int_F f(\boldsymbol{B} + D\boldsymbol{v})\, d\boldsymbol{x} = 0$$

for some nonempty bounded open set F with $|\partial F| = 0$ and some $\boldsymbol{v} \in W_0^{1,\infty}(F, \mathbb{R}^m)$. See Section 2 for the justification.

Proposition 1.6. *Let $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ and $\boldsymbol{A} \in \mathbb{M}^{m\times n}$. Then*

(i) *we have*

$$f \text{ convex at } \boldsymbol{A} \Rightarrow f \text{ polyconvex at } \boldsymbol{A} \Rightarrow \begin{cases} f \text{ quasiconvex at } \boldsymbol{A}, \\ f \text{ rank 1 convex at } \boldsymbol{A}; \end{cases} \tag{11}$$

in fact if f is polyconvex at $\boldsymbol{A}$ then we have (5) *for any nonempty bounded open set $E \subset \mathbb{R}^n$ and any $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$ such that the integral in* (5) *is well defined;*

(ii) *if $f < \infty$ in some neighborhood of $\boldsymbol{A}$ then*

$$f \text{ quasiconvex at } \boldsymbol{A} \Rightarrow f \text{ rank 1 convex at } \boldsymbol{A}.$$

The second part of (i) shows that the implications (11) hold also with the definitions of quasiconvexity in [10–11] and [14]. This second part of (i) is proved using minors relations and a version of Jensen's inequality for possibly non–Borelian convex functions. Item (ii) follows from the fact that under the present definition, (5) holds in particular for each piecewise affine function from $W_0^{1,\infty}(E, \mathbb{R}^m)$, and this is enough to infer the rank 1 convexity by an argument of Fonseca [9] mentioned above.

The main results of this note are the following two assertions:

Theorem 1.7. *A pointwise supremum of any family of quasiconvex functions is quasiconvex.*

This allows us to define the quasiconvex envelope f^{qc} of $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ as the largest quasiconvex minorant of f, i.e., by (10). We have

Theorem 1.8. *If $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ and $E \subset \mathbb{R}^n$ is a nonempty bounded open set then*

(i) *if* $|\partial E| = 0$, *we have for any* $\boldsymbol{A} \in \mathbb{M}^{m\times n}$ *the formula*

$$f^{qc}(\boldsymbol{A}) = \inf\left\{|E|^{-1}\int_E f(\boldsymbol{A}+D\boldsymbol{u})\,d\boldsymbol{x}\right\} \tag{12}$$

where the infimum is taken over all $\boldsymbol{u} \in Q(E, \mathbb{R}^m)$ *such that the integral in* (12) *is well defined;*

(ii) *if* $f < \infty$ *on* $\mathbb{M}^{m\times n}$ *then* (12) *also holds with the infimum over all* $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$ *such that the integral in* (12) *is well defined;*

(iii) *if* f *is finite valued and continuous then* (12) *holds with the infimum over all* $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$.

We interpret Item (i) as an analogue of the formulas for the convex, polyconvex and rank 1 convex envelopes, f^c, f^{pc}, f^{rc}, defined as the largest minorants of f having the corresponding convexity property. Namely, letting s stand for c, pc, or rc, we have, for any $\boldsymbol{A} \in \mathbb{M}^{m\times n}$,

$$f^s(\boldsymbol{A}) = \inf\left\{\sum_{i=1}^{p} \alpha_i f(\boldsymbol{A}_i)\right\} \tag{13}$$

where the infimum is is taken over all collections $\boldsymbol{A}_i, \alpha_i$ such that the sum in (13) is well defined and (1) and (3) hold in case of $s = c$, (1) and (4) hold in case $s = pc$ and (1) and (3) hold and $\boldsymbol{A}_i, \alpha_i$ satisfy (H_p) condition [5] in case $s = rc$. These formulas hold without any restrictions on f. Item (iii) is the original construction by Dacorogna [4]; Item (ii) was established in [10] under the additional assumption that f be locally bounded on $\mathbb{M}^{m\times n}$.

2 Proofs

***Proof of Proposition* 1.2** Let F be a nonempty open bounded subset of $\mathbb{R}^n$ with $|\partial F| = 0$ and prove that

$$|F| f(\boldsymbol{A}) \le \int_F f(\boldsymbol{A}+D\boldsymbol{v})\,d\boldsymbol{x} \tag{14}$$

for any $\boldsymbol{v} \in Q(F, \mathbb{R}^m)$ for which the integral in (14) is well defined. This is clear if $\int_F f(\boldsymbol{A}+D\boldsymbol{v})\,d\boldsymbol{x} = \infty$; hence assume that $\int_F f(\boldsymbol{A}+D\boldsymbol{v})\,d\boldsymbol{x} < \infty$, i.e., $f(\boldsymbol{A}+D\boldsymbol{v}(\cdot)) < \infty$ for a.e. point of F. Consider the family of closed sets of the form $\boldsymbol{a} + \varepsilon\bar{F}$ contained in E, where $a \in \mathbb{R}^n, \varepsilon > 0$. This family clearly covers E in the sense of Vitali, and hence there exists a finite or countable disjoint sequence $\boldsymbol{a}_i + \varepsilon_i\bar{F}$ of subsets of E such that

$$|E \sim \bigcup_i (\boldsymbol{a}_i + \varepsilon_i \bar{F})| = 0.$$

Extend the function $\boldsymbol{v}$ to a Lipschitz function on $\mathbb{R}^n$ by setting $\boldsymbol{v} = \boldsymbol{0}$ on $\mathbb{R}^n \sim E$ and denote the extended function by $\boldsymbol{v}$ again. Define $\boldsymbol{u} : \mathbb{R}^n \to \mathbb{R}^m$ by

$$\boldsymbol{u}(\boldsymbol{x}) = \begin{cases} \varepsilon_i \boldsymbol{v}\Big(\dfrac{\boldsymbol{x}-\boldsymbol{a}_i}{\varepsilon_i}\Big) & \text{if} \quad \boldsymbol{x} \in \boldsymbol{a}_i + \varepsilon_i F, \\ \boldsymbol{0} & \text{otherwise,} \end{cases}$$

$\boldsymbol{x} \in \mathbb{R}^n$. It is easy to see that $\boldsymbol{u} \in Q(E, \mathbb{R}^m)$ and $f(\boldsymbol{A} + D\boldsymbol{u}) < \infty$ for a.e. point of E; hence (5) holds. For scaling reasons we have

$$|E|^{-1} \int_E f(\boldsymbol{A} + D\boldsymbol{u})\, d\boldsymbol{x} = |F|^{-1} \int_F f(\boldsymbol{A} + D\boldsymbol{v})\, d\boldsymbol{x}$$

and thus (5) implies (14). □

Lemma 2.1. *Let m be a probability measure on a measure space $(Y, \mathfrak{M})$, let $b : Y \to X$ be an m integrable map into a finite dimensional linear space X and let $f : X \to \bar{\mathbb{R}}$. Then*

(i) *$\int_Y b\, dm$ belongs to the convex hull of the range of b;*

(ii) *if f is convex at $\int_Y b\, dm$ then*

$$f\Big(\int_Y b\, dm\Big) \le \int_Y f \circ b\, dm \tag{15}$$

provided the integral on the right hand side is well defined.

We here extend the terminology of Definition 1.1(i) and say that $f : X \to \bar{\mathbb{R}}$ is convex at $\boldsymbol{A} \in X$ if Definition 1.1(i) holds with the space $\mathbb{M}^{m\times n}$ replaced by X; moreover we say that the integral $\int_Y f \circ b\, dm$ is well defined if $f \circ b$ is m measurable and either $\int_Y [f \circ b]_+\, dm < \infty$ or $\int_Y [f \circ b]_-\, dm < \infty$. Jensen's inequality (15) is proved in [7; Chapter X, Lemma 2.7] under the additional assumption that f is (globally) convex and lower semicontinuous.

Proof (i): Assertion (i) is similar to [15; Proposition 1]; the present form is stated without proof in [8] and the proof is given in [17; Proposition 16.1.4]. (ii): Inequality (15) is clear if $\int_Y f \circ b\, dm = \infty$; hence assume that $\int_Y f \circ b\, dm < \infty$. We can then change the map b on a set on m measure 0 in such a way that the resulting function, again denoted by b, satisfies $f \circ b < \infty$ for every point of Y without changing the values of the integrals in (15). For each $a \in \mathbb{R}$ define $f_a : X \to \bar{\mathbb{R}}$ by $f_a(z) = \max\{a, f(z)\}, z \in X$ and note that $\int_Y f_a \circ b\, dm \in \mathbb{R}$. We apply (i) to X replaced by $X \times \mathbb{R}$ and b replaced by the map $\beta_a := (b, f_a \circ b) : Y \to X \times \mathbb{R}$. The construction ensures that β_a is m integrable and thus by (i) there exist $\alpha_i \ge 0, t_i \in Y, i = 1, \dots, p$ such that

$$\int_Y b\, dm = \sum_{i=1}^p \alpha_i b(t_i), \quad \int_Y f_a \circ b\, dm = \sum_{i=1}^p \alpha_i f_a(b(t_i)), \quad \sum_{i=1}^p \alpha_i = 1.$$

The convexity of f at $\int_Y b\, dm$ then gives

$$f\Big(\int_Y b\, dm\Big) = f\Big(\sum_{i=1}^p \alpha_i b(t_i)\Big) \le \sum_{i=1}^p \alpha_i f(b(t_i)) \le \sum_{i=1}^p \alpha_i f_a(b(t_i)) = \int_Y f_a \circ b\, dm.$$

Noting that $\int_Y f_a \circ b\, dm \to \int_Y f \circ b\, dm$ as $a \to -\infty$ by the monotone convergence theorem then completes the proof. □

***Proof of Proposition* 1.6** (i): The implication f convex at $\boldsymbol{A} \Rightarrow f$ polyconvex at $\boldsymbol{A}$ is immediate. The implication f polyconvex at $\boldsymbol{A} \Rightarrow f$ rank 1 convex at $\boldsymbol{A}$ follows from the fact that if $\boldsymbol{A}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ are as in Definition 1.1(iv) then

$$\mathbf{M}(\boldsymbol{A}) = (1-a)\mathbf{M}(\boldsymbol{A} + a\boldsymbol{a} \otimes \boldsymbol{b}) + a\mathbf{M}(\boldsymbol{A} - (1-a)\boldsymbol{a} \otimes \boldsymbol{b})$$

since minors are rank 1 affine functions [14]. The implication f polyconvex at $\boldsymbol{A} \Rightarrow$ f quasiconvex at $\boldsymbol{A}$ will be proved if we show the second part of (i), i.e., that if f is polyconvex at $\boldsymbol{A}$ then (5) holds for every $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$ such that the integral in (5) is well defined. Thus let $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$ be such a function. We define the function $h : \mathbb{R}^{s(m,n)} \to \bar{\mathbb{R}}$ by (7) and note that f is polyconvex at $\boldsymbol{A}$ if and only if h is convex at $\mathbf{M}(\boldsymbol{A})$. We apply Lemma 2.1(ii) to $Y = E$, to m defined as the restriction of the Lebesgue measure to E divided by $|E|$, to $X = \mathbb{R}^{s(m,n)}$, to f replaced by h and to the map $b : E \to \mathbb{R}^{s(m,n)}$ given by $b(\boldsymbol{x}) = \mathbf{M}(\boldsymbol{A} + D\boldsymbol{u}(\boldsymbol{x}))$, $\boldsymbol{x} \in E$. Minors relations [14] give

$$\int_E \mathbf{M}(\boldsymbol{A} + D\boldsymbol{u})\, d\boldsymbol{x} = |E|\mathbf{M}(\boldsymbol{A})$$

and thus (15) reduces to $|E|h(\mathbf{M}(\boldsymbol{A})) \le \int_E h(\mathbf{M}(\boldsymbol{A} + D\boldsymbol{u}))\, d\boldsymbol{x} = \int_E f(\boldsymbol{A} + D\boldsymbol{u})\, d\boldsymbol{x}$ which is (5). (ii): This has been proved in the introduction. □

Proposition 2.2. *If* $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$ *then there exists a sequence of piecewise affine functions* $\boldsymbol{u}_j \in W_0^{1,\infty}(E, \mathbb{R}^m)$ *such that*

$$\operatorname{supp} \boldsymbol{u}_j \subset E, \tag{16}$$

$$\boldsymbol{u}_j \to \boldsymbol{u} \quad \text{uniformly on} \quad E, \tag{17}$$

$$D\boldsymbol{u}_j \to D\boldsymbol{u} \quad \text{a.e. on} \quad E, \tag{18}$$

$$\|D\boldsymbol{u}_j\|_{L^\infty(E)} \le c\|D\boldsymbol{u}\|_{L^\infty(E)}. \tag{19}$$

Here $c = b(m,n)$ *is a constant depending only on* m, n. *Also, there exists a sequence* $\boldsymbol{u}_j \in C_0^\infty(E, \mathbb{R}^m)$ *such that* (16)–(19) *hold.*

We include a proof for completeness.

Proof We extend $\boldsymbol{u}$ by $\mathbf{0}$ outside E and denote the extended function again by $\boldsymbol{u}$. We define a sequence $\boldsymbol{v}_j$ of functions by

$$\boldsymbol{v}_j(\boldsymbol{x}) = \varphi(j \operatorname{dist}(\boldsymbol{x}, \partial E))\boldsymbol{u}(\boldsymbol{x}),$$

$j \in \mathbb{N}$, $\boldsymbol{x} \in \mathbb{R}^n$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is given by

$$\varphi(t) = \begin{cases} 0 & \text{if } t \le 1/2, \\ 2(t - 1/2) & \text{if } 1/2 \le t \le 1, \\ 1 & \text{if } t \ge 1, \end{cases}$$

$t \in \mathbb{R}$. One finds that $\boldsymbol{v}_j$ are Lipschitz functions with

$$\operatorname{supp} \boldsymbol{v}_j \subset \{\boldsymbol{x} \in E : \operatorname{dist}(\boldsymbol{x}, \partial E)) \ge 1/j\},$$

$\boldsymbol{v}_j \to \boldsymbol{u}$ uniformly on $\mathbb{R}^n$, $D\boldsymbol{v}_j \to D\boldsymbol{u}$ pointwise on $\mathbb{R}^n$, and with the Lipschitz constant $\operatorname{Lip}(\boldsymbol{v}_j)$ satisfying $\operatorname{Lip}(\boldsymbol{v}_j) \le 3 \operatorname{Lip}(\boldsymbol{u})$. Thus if $\varepsilon > 0$ then taking j sufficiently large and putting $\boldsymbol{v} = \boldsymbol{v}_j$ we have constructed a function $\boldsymbol{v} \in W_0^{1,\infty}(E, \mathbb{R}^m)$ with

$$\operatorname{supp} \boldsymbol{v} \subset E, \tag{20}$$

$$\|\boldsymbol{v} - \boldsymbol{u}\|_{L^\infty(E)} < \varepsilon, \tag{21}$$

$$\|Dv\|_{L^\infty(E)} \le 3\|Du\|_{L^\infty(E)}, \tag{22}$$

$$\int_E |Dv - Du|\,dx < \varepsilon. \tag{23}$$

By passing to a sufficiently fine mollification of v we find a function, to be denoted by v again, such that $v \in C_0^\infty(E, \mathbb{R}^m)$ and (20)–(23) still hold. Finally we use a sufficiently fine triangulation of $\operatorname{supp} v$ to find a piecewise affine function $w \in W_0^{1,\infty}(E, \mathbb{R}^m)$ such that

$$\operatorname{supp} w \subset E,$$

$$\|w - u\|_{L^\infty(E)} < 3\varepsilon,$$

$$\|Dw\|_{L^\infty(E)} \le 3k\|Du\|_{L^\infty(E)},$$

$$\int_E |Dw - Du|\,dx < 3\varepsilon,$$

cf. [7; Chapter 10, Proposition 2.1] where $k = k(m,n)$ is a constant depending only on m and n. Thus taking $\varepsilon = 1/3j$ we find a sequence $u_j \in W_0^{1,\infty}(E, \mathbb{R}^m)$ satisfying (16), (17), and (19) with $c = 3k$ and

$$\int_E |Du_j - Du|\,dx < 1/j.$$

Thus for a subsequence of u_j, to be denoted by u_j again, we have (16)–(19). The proof also gives the sequence $u_j \in C_0^\infty(E, \mathbb{R}^m)$. □

***Proof of Proposition* 1.3** (i): This follows from Lebesgue's theorem and Proposition 2.2. (ii): This follows from (i) if one notes that any quasiconvex function $f : \mathbb{M}^{m\times n} \to \mathbb{R}$ is rank 1 convex by Proposition 1.6 and hence locally Lipschitz continuous by [13; Theorem 4.4.1]. □

***Proof of Proposition* 1.4** The function $f : \mathbb{M}^{m\times n} \to [-\infty, \infty)$ is rank 1 convex by Proposition 1.6. If $f(A) = -\infty$ for some $A \in \mathbb{M}^{m\times n}$ then the rank 1 convexity implies that $f \equiv -\infty$ on every rank 1 line through A, i.e., $f(A + a \otimes b) = -\infty$ for every $a \in \mathbb{R}^m, b \in \mathbb{R}^n$. The same argument then gives that $f \equiv -\infty$ on every rank 1 line through $A + a \otimes b$ and proceeding inductively one then obtains that $f \equiv -\infty$ on $\mathbb{M}^{m\times n}$. Thus either $f \equiv -\infty$ on $\mathbb{M}^{m\times n}$, in which case (5) trivially holds for each $u \in W_0^{1,\infty}(E, \mathbb{R}^m)$ or f is finite valued on $\mathbb{M}^{m\times n}$ in which case (5) holds for each $u \in W_0^{1,\infty}(E, \mathbb{R}^m)$ by Proposition 1.3(ii). □

***Proof of Theorem* 1.7** Thus let $g : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ be defined by

$$g(A) = \sup\{f(A) : f \in \mathcal{F}\}, \quad A \in \mathbb{M}^{m\times n},$$

where $\mathcal{F}$ is any set of quasiconvex functions $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$. Prove that g is quasiconvex at any $A \in \mathbb{M}^{m\times n}$. Let A be fixed. Let $E \subset \mathbb{R}^n$ be open bounded with $|\partial E| = 0$ and prove that

$$|E| g(A) \le \int_E g(A + Du)\,dx \tag{24}$$

for any $u \in Q(E, \mathbb{R}^m)$ such that the integral in (24) is well defined. This is clear if $\int_E g(A+Du)\,dx = \infty$; hence assume $\int_E g(A+Du)\,dx < \infty$, i.e., $g(A+Du) < \infty$ for a.e. point of E. Assume first that $g(A) < \infty$. Then for every $\varepsilon > 0$ there exists an $f \in \mathcal{F}$ such that $g(A) - \varepsilon \le f(A)$. Then $f(A + Du) < \infty$ for a.e. point of E and thus

$$|E|g(\boldsymbol{A}) - \varepsilon|E| \le |E|f(\boldsymbol{A}) \le \int_E f(\boldsymbol{A} + \boldsymbol{Du})\,d\boldsymbol{x} \le \int_E g(\boldsymbol{A} + \boldsymbol{Du})\,d\boldsymbol{x}. \tag{25}$$

Thus (24) holds. Next assume that $g(\boldsymbol{A}) = \infty$. For every $k \in \mathbb{R}$ there exists an $f \in \mathcal{F}$ such that $k < f(\boldsymbol{A})$ and then

$$k|E| \le |E|f(\boldsymbol{A}) \le \int_E f(\boldsymbol{A} + \boldsymbol{Du})\,d\boldsymbol{x} \le \int_E g(\boldsymbol{A} + \boldsymbol{Du})\,d\boldsymbol{x} < \infty$$

for all $k \in \mathbb{R}$, which is a contradiction. Thus there is no $\boldsymbol{u} \in Q(E, \mathbb{R}^m)$ with $g(\boldsymbol{A} + \boldsymbol{Du}) < \infty$ for a.e. point of E; hence the condition of quasiconvexity of f at $\boldsymbol{A}$ is vacuously satisfied. □

Remark 2.3. With the definition of quasiconvexity in [10–11] and [14] the above argument breaks down at (25): with a general $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$ there seems to be no guarantee that the function $f(\boldsymbol{A} + \boldsymbol{Du}(\cdot))$ is measurable on E.

***Proof of Theorem* 1.8** (i): Denote by g the function defined by the right-hand side of (12). Clearly $f^{qc} \le g$ and the proof will be complete if we show that g is quasiconvex. Thus let $\boldsymbol{A} \in \mathbb{M}^{m\times n}$, let $F \subset \mathbb{R}^n$ be open bounded with $|\partial F| = 0$ and prove that

$$|F|g(\boldsymbol{A}) \le \int_F g(\boldsymbol{A} + \boldsymbol{Du})\,d\boldsymbol{x} \tag{26}$$

for any $\boldsymbol{u} \in Q(F, \mathbb{R}^m)$ for which the integral in (26) is well defined. This is clear if $\int_F g(\boldsymbol{A}+\boldsymbol{Du})\,d\boldsymbol{x} = \infty$; hence assume that $\int_F g(\boldsymbol{A}+\boldsymbol{Du})\,d\boldsymbol{x} < \infty$, i.e., $f(\boldsymbol{A}+\boldsymbol{Du}) < \infty$ for a.e. point of F. Since $\boldsymbol{u}$ is Q-piecewise affine, there exists a finite number of disjoint open sets $G_\alpha \subset F$ such that

$$\left.\begin{aligned} &|F \sim \bigcup_\alpha G_\alpha| = 0 \\ &\boldsymbol{Du} = \boldsymbol{B}_\alpha = \text{const} \quad \text{on} \quad G_\alpha. \end{aligned}\right\} \tag{27}$$

Since $g(\boldsymbol{A} + \boldsymbol{B}_\alpha) < \infty$, for every $\varepsilon > 0$ and every α there exists a Q-piecewise affine function $\boldsymbol{v}_\alpha \in W_0^{1,\infty}(E, \mathbb{R}^m)$ such that

$$g(\boldsymbol{A} + \boldsymbol{B}_\alpha) + \varepsilon \ge |E|^{-1} \int_E f(\boldsymbol{A} + \boldsymbol{B}_\alpha + \boldsymbol{Dv}_\alpha)\,d\boldsymbol{x}. \tag{28}$$

We now invoke the construction in the proof of Proposition 1.2 to find $\boldsymbol{w}_\alpha \in Q(G_\alpha, \mathbb{R}^m)$ such that

$$|E|^{-1} \int_E f(\boldsymbol{A} + \boldsymbol{B}_\alpha + \boldsymbol{Dv}_\alpha) = |G_\alpha|^{-1} \int_{G_\alpha} f(\boldsymbol{A} + \boldsymbol{B}_\alpha + \boldsymbol{Dw}_\alpha)\ d\boldsymbol{x}. \tag{29}$$

Define $\boldsymbol{s} : \mathbb{R}^n \to \mathbb{R}^m$ by

$$\boldsymbol{s} = \begin{cases} \boldsymbol{u} + \boldsymbol{w}_\alpha & \text{on} \quad G_\alpha \quad \text{for any} \quad \alpha, \\ \boldsymbol{u} & \text{otherwise.} \end{cases} \tag{30}$$

Then $\boldsymbol{s} \in Q(F, \mathbb{R}^m)$. We invoke the construction in the proof of Proposition 1.2 again to find $\boldsymbol{r} \in Q(E, \mathbb{R}^m)$ such that

$$|F|^{-1} \int_F f(\boldsymbol{A} + \boldsymbol{Ds})\,d\boldsymbol{x} = |E|^{-1} \int_E f(\boldsymbol{A} + \boldsymbol{Dr})\,d\boldsymbol{x}. \tag{31}$$

Since $\boldsymbol{Du} = \boldsymbol{B}_\alpha$ on G_α, we have

$$Ds = \boldsymbol{B}_\alpha + D\boldsymbol{w}_\alpha \quad \text{on} \quad G_\alpha. \tag{32}$$

Then (27)–(32) give

$$\begin{aligned}\int_F g(\boldsymbol{A}+D\boldsymbol{u})\,d\boldsymbol{x} + \varepsilon|F| &= \sum_\alpha |G_\alpha|(g(\boldsymbol{A}+\boldsymbol{B}_\alpha)+\varepsilon)\\ &\geq \sum_\alpha \frac{|G_\alpha|}{|E|}\int_E f(\boldsymbol{A}+\boldsymbol{B}_\alpha+D\boldsymbol{v}_\alpha)\,d\boldsymbol{x}\\ &= \sum_\alpha \int_{G_\alpha} f(\boldsymbol{A}+\boldsymbol{B}_\alpha+D\boldsymbol{w}_\alpha)\,d\boldsymbol{x}\\ &= \int_F f(\boldsymbol{A}+D\boldsymbol{s})\,d\boldsymbol{x}\\ &= \frac{|F|}{|E|}\int_E f(\boldsymbol{A}+D\boldsymbol{r})\,d\boldsymbol{x}\\ &\geq |F|g(\boldsymbol{A}).\end{aligned}$$

(ii): Let $f < \infty$ on $\mathbb{M}^{m\times n}$; hence $f^{qc} < \infty$ on $\mathbb{M}^{m\times n}$. By Proposition 1.4 the either $f \equiv -\infty$ on $\mathbb{M}^{m\times n}$ or f^{qc} is finite valued and

$$|E|f^{qc}(\boldsymbol{A}) \leq \int_E f^{qc}(\boldsymbol{A}+D\boldsymbol{u})\,d\boldsymbol{x}$$

for any $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$. The inequality $f^{qc} \leq f$ implies $\int_E f^{qc}(\boldsymbol{A}+D\boldsymbol{u})\,d\boldsymbol{x} \leq \int_E f(\boldsymbol{A}+D\boldsymbol{u})\,d\boldsymbol{x}$ provided the last integral is well defined, which gives

$$f^{qc}(\boldsymbol{A}) \leq |E|^{-1}\int_E f(\boldsymbol{A}+D\boldsymbol{u})\,d\boldsymbol{x};$$

hence the infimum over the larger set specified in (ii) is the same as the infimum in (i). (iii): This follows from (ii) and the fact that the integral $\int_E f(\boldsymbol{A}+D\boldsymbol{u})\,d\boldsymbol{x}$ is well defined for any $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$. □

***Proof of Example* 1.5** Let F be the unit open ball in $\mathbb{R}^n$. BALL [2; Theorems 3.3 and 3.8] proves that if $m \geq n \geq 3$ or $n = 2$ and $m \geq 4$ then there exists a $\boldsymbol{B} \in \mathbb{M}^{m\times n}$ and a $\boldsymbol{w} \in C^1(\bar{F}, \mathbb{R}^m)$ with $\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{B}\boldsymbol{x}$ on ∂F such that the set $K := \operatorname{graph} D\boldsymbol{w} = \{D\boldsymbol{w}(\boldsymbol{x}) : \boldsymbol{x} \in \bar{F}\}$ contains no rank 1 connection (i.e., there is no pair $\boldsymbol{C}, \boldsymbol{D} \in K$ with $\operatorname{rank}(\boldsymbol{C}-\boldsymbol{D}) = 1$) and $\boldsymbol{B} \notin K$. If $f : \mathbb{M}^{m\times n} \to \{0, \infty\}$ is defined by

$$f(\boldsymbol{D}) = \begin{cases} 0 & \text{if} \quad \boldsymbol{D} \in K,\\ \infty & \text{otherwise,}\end{cases}$$

$\boldsymbol{D} \in \mathbb{M}^{m\times n}$, and if $\boldsymbol{u} \in W_0^{1,\infty}(E, \mathbb{R}^m)$ is defined by $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{w}(\boldsymbol{x}) - \boldsymbol{B}\boldsymbol{x}$, $\boldsymbol{x} \in F$, then

$$\infty = |F|f(\boldsymbol{B}) > \int_F f(\boldsymbol{B}+D\boldsymbol{v})\,d\boldsymbol{x} = \int_F f(D\boldsymbol{w})\,d\boldsymbol{x} = 0.$$

Prove that f is quasiconvex. Thus let E be a nonempty bounded open subset of $\mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{M}^{m\times n}$, $\boldsymbol{u} \in Q(E, \mathbb{R}^m)$ and prove that (5) holds. This is clear if $\int_E f(\boldsymbol{A}+D\boldsymbol{u})\,d\boldsymbol{x} = \infty$; hence assume that $\int_E f(\boldsymbol{A}+D\boldsymbol{u})\,d\boldsymbol{x} < \infty$, i.e., $f(\boldsymbol{A}+D\boldsymbol{u}) < \infty$ for a.e. point of E, which by the definition of f implies that $f(\boldsymbol{A}+D\boldsymbol{u}) = 0$ for a.e. point of E. By the definition of a Q-piecewise affine function there exist disjoint open subsets E_i, $i = 1, \ldots, p$ of E of positive measure and elements $\boldsymbol{A}_i$ such that

$$|E \sim \bigcup_{i=1}^{p} E_i| = 0$$

and

$$Du = A_i \qquad \text{for a.e. point of } E_i; \tag{33}$$

we furthermore assume that $A_i \neq A_j$ if $i \neq j$. From $f(A+Du) = 0$ for a.e. point of E we deduce that $A + A_i \in K$ for every i and from the fact that K has no rank 1 connection we deduce that $\operatorname{rank}[(A + A_i) - (A + A_j)] \geq 2$ if $i \neq j$. From (33) we deduce that u is affine on each connceted component C of E_i. Each such a component C can be decomposed into a finite number of simplexes in the sense that $C = N \cup \bigcup_{j=1}^{k} E \cap \Delta_j$ where Δ_j are nonempty simplexes and $|N| = 0$. Since each E_i has at most a countable number of connected components, we conclude that there is at most a countable number of disjoint simplexes $S_l \subset \mathbb{R}^n$, $l \in L$, such that

$$|E \sim \bigcup_{l \in L} E \cap S_l| = 0$$

and u is affine on each S_l. For any two simplexes S_j, S_m such that $\partial(E \cap S_l) \cap \partial(E \cap S_m)$ has a positive area we have that the values of Du on $E \cap S_l$ and $E \cap S_m$ are rank 1 connected. Since the range of Du contains no rank 1 connection, we deduce that necessarily Du is constant on E. Combining this with the boundary condition $u = 0$ on ∂E we finally obtain that $u \equiv \mathbf{0}$ on E. Thus the only $u \in Q(E, \mathbb{R}^m)$ with $\int_E f(A + Du)\, dx < \infty$ is $u = \mathbf{0}$ and hence

$$\int_E f(A + Du)\, dx = \int_E f(A)\, dx = |E| f(A)$$

which proves (5). Thus f is quasiconvex.

Finally, we give

Example 2.4. Let $m > 1$ or $n > 1$.

(i) There exists a non–Borelian convex (and hence polyconvex, quasiconvex, and rank 1 convex) function $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$;

(ii) there exists a family $\mathcal{F}$ of Borelian convex functions on $\mathbb{M}^{m\times n}$ such that the pointwise supremum of $\mathcal{F}$ is not Borelian.

Let B and S be the unit open ball and the unit sphere in $\mathbb{M}^{m\times n}$, and let $f : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ be a function such that $f(A) = 0$ if $A \in B$, $f(A) = \infty$ if $A \in \mathbb{M}^{m\times n} \sim (B \cup S)$, and $f|S : S \to \mathbb{R}$ an arbitrary non–Borelian nonnegative function. Such a function exists since the dimension of S is ≥ 1. The function f is convex. Consider further a family $\mathcal{F} := \{g_B, B \in S\}$, of functions $g_B : \mathbb{M}^{m\times n} \to \bar{\mathbb{R}}$ defined by

$$g_B(A) = \begin{cases} 0 & \text{if } A \in B \cup S, A \neq B, \\ f(B) & \text{if } A = B, \\ \infty & \text{if } A \in \mathbb{M}^{m\times n} \sim (B \cup S), \end{cases}$$

$A \in \mathbb{M}^{m\times n}$, $B \in S$. Then each g_B is a Borelian convex function. However, the pointwise supremum of $\mathcal{F}$ is the function f.

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