Preprint, Institute of Mathematics, AS CR, Prague. 2007-12-18

HIGHER-ORDER DISCRETE MAXIMUM PRINCIPLE FOR 1D DIFFUSION-REACTION PROBLEMS*

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Abstract. Sufficient conditions for the validity of the discrete maximum principle (DMP) for a 1D diffusion-reaction problem $-u'' + \kappa^2 u = f$ with the homogeneous Dirichlet boundary conditions discretized by the higher-order finite element method are presented. It is proved that the DMP is satisfied if the lengths h of all elements are shorter then one-third of the length of the entire domain and if $\kappa^2 h^2$ is small enough. The bounds for $\kappa^2 h^2$ are precisely specified in terms of the relative length of the elements. The obtained conditions are simple and easy to verify.

Key words. discrete maximum principle, discrete Green's function, diffusion-reaction problem, higher-order finite element method, *hp*-FEM, M-matrix

AMS subject classifications. 65N30, 65N50

1. Introduction. The standard (continuous) maximum principles for elliptic and parabolic problems, in particular, guarantee the nonnegativity of the solution provided that the data are nonnegative. This is especially important if naturally nonnegative quantities like temperature, concentration, density, etc. are modelled. There is a question if the discretization of these problems satisfies the (discrete) maximum principle (DMP) as well, or, equivalently, if the resulting discrete solution is guaranteed to be nonnegative provided the data are nonnegative.

Unfortunately, the standard methods, e.g., the finite element methods, do not satisfy the DMP in general. Therefore, additional conditions for the validity of the DMP are proposed and studied. Up to the author's knowledge the first paper about the DMP for elliptic problems [17] appeared in 1966. Since then many other papers about the DMP for various problems and various discretizations were published [4, 5, 7, 8, 12, 23].

Interestingly, the majority of the published works deal with the lowest-order approximations only. The results about the DMP for higher-order approximations are scarce, see [2, 11, 24] and the recent works of the author and his coauthors [18, 19, 22]. This paper extends the recent result [18] for the 1D Poisson problem to the 1D diffusion-reaction problem discretized by higher-order finite elements. In particular, this result is suitable for the *hp*-version of the finite element method (*hp*-FEM) because various polynomial degrees in different elements are allowed. More details about the *hp*-FEM can be found in books [6, 13, 14, 21, 20].

The generalization of the higher-order DMP to the diffusion-reaction problem is not straightforward. Many technical problems have to be overcome and new approaches introduced. For illustration let us mention that in contrast to the Poisson problem the bubble (interior) basis functions are not orthogonal to the vertex functions in the diffusion-reaction case, the reaction coefficient κ^2 complicates the analysis, the boundary layers appear for great values of κ^2, etc. Even for the lowest-order approximations, the DMPs for the diffusion-reaction problems were treated very recently [3, 10].

*This work was supported by the Czech Science Foundation, project No. 102/07/0496, and by the Czech Academy of Sciences, institutional research plan No. AV0Z10190503.

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Let us point out an interesting approach which can be used for 1D diffusion-reaction problem. The idea comes from [1]. It is possible to construct special (exponential) basis functions using the knowledge of the exact Green's function for the continuous problem. These special basis functions are in fact the exact minimum energy extensions, cf. Section 4, of the standard piecewise linear basis functions. The resulting approximation then naturally satisfies the DMP. This approach, however, cannot be generalized to the higher spatial dimension because the exact Green's function is known in 1D only. Nevertheless, our goal is to study the standard piecewise polynomial approximations with the hope that the understanding of the 1D behaviour gives us a hint how to treat the DMP in two and more dimensions.

The paper is organized as follows. Section 2 introduces the diffusion-reaction problem and briefly describes its discretization by the *hp*-FEM. In Section 3 the discrete maximum principle is defined and its relation to the discrete Green's function is explained. The usefull concept of discrete minimum energy extensions is introduced in Section 4 and it is used in Section 5 to define suitable basis functions for the higher-order finite element space. The splitting of the discrete Green's function to the vertex and bubble part is shown in Section 6 together with the proof of the nonnegativity of the vertex part. Section 7 is technically the most demanding. It analyzes the influence of the bubble part to the nonnegativity of the discrete Green's function in several steps. Section 8 summarizes the previous analysis and presents the main theorem of the paper. Certain technical assumptions of the main theorem have to be verified numerically. This is done in Section 9.

2. The problem and its discretization. Let us consider an open interval $\Omega \subset \mathbb{R}$, $\Omega = (a_\Omega, b_\Omega)$, and the 1D reaction-diffusion problem

$$-u'' + \kappa^2 u = f \quad \text{in } \Omega \tag{2.1}$$

with the homogeneous Dirichlet boundary conditions

$$u(a_\Omega) = u(b_\Omega) = 0,$$

where the reaction coefficient $\kappa \geq 0$ is assumed to be constant. The standard maximum principle for this problem is equivalent to the so-called conservation of nonnegativity

$$f \geq 0 \quad \Rightarrow \quad u \geq 0.$$

In what follows, we will study an analogue of this implication for the discrete solution obtained by the *hp*-FEM.

Let $a_\Omega = x_0 < x_1 < \cdots < x_{M+1} = b_\Omega$ be a division of the interval $\Omega = (a_\Omega, b_\Omega)$ into $M + 1 \geq 2$ finite elements $K_k = [x_{k-1}, x_k]$ with lengths $h_{K_k} = x_k - x_{k-1}$, $k = 1, 2, \ldots, M+1$. The set $\mathcal{T}_{hp} = \{K_k, k = 1, 2, \ldots, M+1\}$ is referred as the (finite element) mesh. Further, we consider an arbitrary distribution of polynomial degrees p_K assigned to the elements $K \in \mathcal{T}_{hp}$. The corresponding *hp*-FEM space V_{hp} is defined as follows

$$V_{hp} = \{v_{hp} \in H_0^1(\Omega) : v_{hp}|_K \in P^{p_K}(K),\ K \in \mathcal{T}_{hp}\}, \tag{2.2}$$

where $H_0^1(\Omega)$ is the standard Sobolev space of functions from $L^2(\Omega)$ with the generalized derivatives in $L^2(\Omega)$. The space $P^{p_K}(K)$ contains polynomials of degree at most p_K in the interval K. The *hp*-FEM solution $u_{hp} \in V_{hp}$ of problem (2.1) is defined by

$$a(u_{hp}, v_{hp}) = F(v_{hp}) \quad \forall v_{hp} \in V_{hp}, \tag{2.3}$$

where

$$a(u,v) = \int_\Omega u'v' \,\mathrm{d}x + \kappa^2 \int_\Omega uv \,\mathrm{d}x, \qquad F(v) = \int_\Omega fv \,\mathrm{d}x,$$

and f is assumed in $L^2(\Omega)$. Notice that there exists a unique solution $u_{hp} \in V_{hp}$ to problem (2.3).

3. Discrete maximum principle and the discrete Green's function.

DEFINITION 3.1. *Let V_{hp} given by (2.2) be the hp-FEM space based on the mesh $\mathcal{T}_{hp}$ and on the polynomial degrees p_K, $K \in \mathcal{T}_{hp}$. We say that approximate problem* (2.3) *satisfies the discrete maximum principle (DMP) if*

$$\max_{\overline{\Omega}} u_{hp} = \max_{\partial\Omega} u_{hp} = 0 \quad \textit{for all} \quad f \in L^2(\Omega),\ f \le 0 \ \textit{a.e. in}\ \Omega. \tag{3.1}$$

Notice that requirement (3.1) is equivalent to

$$u_{hp} \ge 0 \quad \text{for all} \quad f \in L^2(\Omega),\ f \ge 0 \text{ a.e. in } \Omega. \tag{3.2}$$

We remark that another possible definition of the DMP appears in the literature as follows. The discrete maximum principle is valid if $f \in L^2(\Omega)$, $f \le 0$ a.e. in Ω, and $\mathcal{T}_{hp}$ are such that the corresponding approximate solution $u_{hp} \in V_{hp}$ satisfies $\max_{\overline{\Omega}} u_{hp} = 0$.

If the DMP is valid according to this definition for certain $f \le 0$ and $\mathcal{T}_{hp}$ then there could exist another $f^* \le 0$ such that the maximum of the discrete solution u^*_{hp} obtained on the same mesh $\mathcal{T}_{hp}$ is attained in the interior of Ω. In this case the approximation based on the mesh $\mathcal{T}_{hp}$ does not satisfy the DMP according to Definition 3.1. In what follows, we exclusively use Definition 3.1.

The validity of the DMP according to Definition 3.1 is equivalent to the nonnegativity of the discrete Green's function G_{hp}, see Theorem 3.4 below.

DEFINITION 3.2. *Let $y \in \Omega$ and let $G_{hp,y} \in V_{hp}$ be the unique solution of the problem*

$$a(w_{hp}, G_{hp,y}) = \delta_y(w_{hp}) = w_{hp}(y) \quad \forall w_{hp} \in V_{hp}. \tag{3.3}$$

The function $G_{hp}(x,y) = G_{hp,y}(x)$, $(x,y) \in \Omega^2$, is called the discrete Green's function (DGF).

A combination of (2.3) and (3.3) yields the discrete Kirchhoff-Helmholtz representation formula

$$u_{hp}(y) = \int_\Omega G_{hp}(x,y) f(x) \,\mathrm{d}x, \quad y \in \Omega. \tag{3.4}$$

Interestingly, the DGF can be explicitely expressed in terms of a basis in V_{hp}.

THEOREM 3.3. *Let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be a basis in V_{hp} and let $\mathbb{A} \in R^{N\times N}$ be the stiffness matrix with entries $\mathbb{A}_{ij} = a(\varphi_i, \varphi_j)$, $i,j = 1,2,\ldots,N$. Then*

$$G_{hp}(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbb{A}^{-1}_{ij} \varphi_i(x) \varphi_j(y), \tag{3.5}$$

where $\mathbb{A}^{-1}_{ij}$ are the entries of the inverse matrix to $\mathbb{A}$.

Proof. See [18]. ∎

Notice that Theorem 3.3 and the symmetry of the bilinear form $a(\cdot,\cdot)$ imply $G_{hp}(x,y) = G_{hp}(y,x)$. Consequently, $G_{hp,x} = G_{hp}(x,\cdot) \in V_{hp}$.

THEOREM 3.4. *Problem* (2.3) *satisfies the DMP if and only if* $G_{hp}(x,y) \geq 0$ *for all* $(x,y) \in \Omega^2$.

Proof. Immediate consequence of (3.4). See [18] again. ∎

Thus, our goal is to prove the nonnegativity of G_{hp} in Ω^2. To this end, we will use (3.5). First, in Section 5, a suitable basis of V_{hp} will be constructed. For this purpose we will utilize the concept of the discrete minimum energy extensions which will be described in Section 4. The analysis of the nonnegativity of G_{hp} will be postponed to the subsequent sections.

4. Discrete minimum energy extensions. Let us consider a splitting of the space V_{hp} into a direct sum of two nontrivial subspaces $V_{hp} = V_{hp}^* \oplus V_{hp}^\#$. The discrete minimum energy extension $\psi^{\mathrm{me}} \in V_{hp}$ of a function $\psi^* \in V_{hp}^*$ with respect to $V_{hp}^\#$ is uniquely defined as

$$\psi^{\mathrm{me}} = \psi^* - \psi^\#,$$

where $\psi^\# \in V_{hp}^\#$ is the elliptic projection of ψ^* into $V_{hp}^\#$, i.e.,

$$0 = a(\psi^{\mathrm{me}}, v^\#) = a(\psi^* - \psi^\#, v^\#) \quad \text{for all } v^\# \in V_{hp}^\#. \tag{4.1}$$

Due to the symmetry of $a(\cdot,\cdot)$ and due to (4.1) we have

$$a(\psi^{\mathrm{me}}, \psi^{\mathrm{me}}) = a(\psi^{\mathrm{me}}, \psi^*) = a(\psi^*, \psi^*) - a(\psi^\#, \psi^*) = a(\psi^*, \psi^*) - a(\psi^\#, \psi^\#).$$

Hence, $\|\psi^{\mathrm{me}}\|^2 + \|\psi^\#\|^2 = \|\psi^*\|^2$, where $\|v\|^2 = a(v,v)$. Consequently,

$$\|\psi^{\mathrm{me}}\| \leq \|\psi^*\| \quad \text{and} \quad \|\psi^\#\| \leq \|\psi^*\|. \tag{4.2}$$

Now, let us compute the discrete minimum energy extensions of basis functions from V_{hp}^*. Let $\mathcal{B}^* = \{\varphi_1^*, \varphi_2^*, \ldots, \varphi_{N^*}^*\}$ be a basis in V_{hp}^* and let $\mathcal{B}^\# = \{\varphi_1^\#, \varphi_2^\#, \ldots, \varphi_{N^\#}^\#\}$ be a basis in $V_{hp}^\#$. The stiffness matrix corresponding to the basis $\mathcal{B}^* \cup \mathcal{B}^\#$ of V_{hp} has the following 2-by-2 block structure

$$\mathbb{A}^{*\#} = \begin{pmatrix} A^{**} & A^{*\#} \\ (A^{*\#})^T & A^{\#\#} \end{pmatrix},$$

where $A_{ij}^{**} = a(\varphi_i^*, \varphi_j^*)$, $i,j = 1,2,\ldots,N^*$, $A_{ij}^{*\#} = a(\varphi_i^*, \varphi_j^\#)$, $i = 1,2,\ldots,N^*$, $j = 1,2,\ldots,N^\#$, and $A_{ij}^{\#\#} = a(\varphi_i^\#, \varphi_j^\#)$, $i,j = 1,2,\ldots,N^\#$.

The discrete minimum energy extensions $\varphi_i^{\mathrm{me}} \in V_{hp}$ of $\varphi_i^* \in V_{hp}^*$ with respect to $V_{hp}^\#$ can be computed as

$$\varphi_i^{\mathrm{me}} = \varphi_i^* - \sum_{j=1}^{N^\#} C_{ij}^{*\#} \varphi_j^\#, \quad i = 1,2,\ldots,N^*. \tag{4.3}$$

The requirement (4.1) uniquely determines coefficients $C_{ij}^{*\#}$ as follows

$$0 = a(\varphi_i^*, \varphi_k^\#) - \sum_{j=1}^{N^\#} C_{ij}^{*\#} a(\varphi_j^\#, \varphi_k^\#) \quad \forall i = 1,2,\ldots,N^*,\ k = 1,2,\ldots,N^\#. \tag{4.4}$$

This can be written in a matrix form as $0 = A^{*\#} - C^{*\#}A^{\#\#}$, where the matrix $C^{*\#} \in \mathbb{R}^{N^* \times N^\#}$ consists of entries $C^{*\#}_{ij}$. Hence,

$$C^{*\#} = A^{*\#}(A^{\#\#})^{-1}. \tag{4.5}$$

The discrete minimum energy extensions $\varphi_i^{\mathrm{me}} \in V_{hp}$ can be used as an alternative basis $\mathcal{B}^{\mathrm{me}} = \{\varphi_1^{\mathrm{me}}, \varphi_2^{\mathrm{me}}, \dots, \varphi_{N^*}^{\mathrm{me}}\}$ in V_{hp}^*. It can be easily verified that the corresponding stiffness matrix $S^{**} \in \mathbb{R}^{N^* \times N^*}$ with entries $S^{**}_{ij} = a(\varphi_i^{\mathrm{me}}, \varphi_j^{\mathrm{me}})$, $i,j = 1,2,\dots,N^*$ is just the Schur complement

$$S^{**} = A^{**} - A^{*\#}(A^{\#\#})^{-1}(A^{*\#})^T. \tag{4.6}$$

Finally, the well known formula for the inversion of a 2-by-2 block matrix implies that the upper-left block of $(\mathbb{A}^{*\#})^{-1}$ is equal to the inverse of the Schur complement, i.e.,

$$(\mathbb{A}^{*\#})^{-1}_{ij} = (S^{**})^{-1}_{ij} \quad \forall i,j = 1,2,\dots,N^*. \tag{4.7}$$

5. Construction of the *hp*-FEM bases. The hp-FEM basis functions are constructed in the standard way as images of the shape functions defined on the reference element $K_{\mathrm{ref}} = [-1,1]$ under the reference maps

$$\chi_{K_k}(\xi) = \frac{h_{K_k}}{2}\xi + \frac{x_k + x_{k-1}}{2}, \quad \xi \in K_{\mathrm{ref}},\ k = 1,2,\dots,M+1. \tag{5.1}$$

For the shape functions we use the standard Lobatto polynomials, see, e.g., [14, 15, 21],

$$\begin{aligned} &l_0(\xi) = (1-\xi)/2, \qquad l_1(\xi) = (1+\xi)/2,\\ &l_j(\xi) = \sqrt{\frac{2j-1}{2}}\int_{-1}^{\xi} P_{j-1}(x)\,\mathrm{d}x, \quad j = 2,3,\dots, \end{aligned}$$

where $\xi \in K_{\mathrm{ref}}$ and $P_j(x) = \mathrm{d}^j/\mathrm{d}x^j (x^2-1)^j/(2^j j!)$ stands for the jth-degree Legendre polynomial. Thanks to the orthogonality of the Legendre polynomials the higher order Lobatto polynomials l_j, $j \ge 2$, vanish at ± 1 and, hence, we can factor out the root factors to obtain

$$l_j(\xi) = l_0(\xi) l_1(\xi) \lambda^{\mathrm{ker}}_{j-2}(\xi), \quad j \ge 2, \tag{5.2}$$

where the polynomial kernels $\lambda^{\mathrm{ker}}_{j-2}(\xi)$ of degree $j-2$ can be generated by the recurrence

$$\frac{j+4}{\sqrt{2j+7}}\,\lambda^{\mathrm{ker}}_{j+2}(\xi) = \sqrt{2j+5}\,\xi\,\lambda^{\mathrm{ker}}_{j+1}(\xi) - \frac{j+1}{\sqrt{2j+3}}\,\lambda^{\mathrm{ker}}_{j}(\xi), \quad j = 0,1,2,\dots.$$

For reference, we list the first nine kernels (see, e.g., in [15, Section 3.1] or in [21, Section 1.2]):

$$
\begin{aligned}
\lambda_0^{\ker}(\xi) &= -\sqrt{6},\\
\lambda_1^{\ker}(\xi) &= -\sqrt{10}\,\xi,\\
\lambda_2^{\ker}(\xi) &= -\frac{1}{4}\sqrt{14}(5\xi^2-1),\\
\lambda_3^{\ker}(\xi) &= -\frac{3}{4}\sqrt{2}(7\xi^2-3)\xi,\\
\lambda_4^{\ker}(\xi) &= -\frac{1}{8}\sqrt{22}(21\xi^4-14\xi^2+1),\\
\lambda_5^{\ker}(\xi) &= -\frac{1}{8}\sqrt{26}(33\xi^4-30\xi^2+5)\xi,\\
\lambda_6^{\ker}(\xi) &= -\frac{1}{64}\sqrt{30}(429\xi^6-495\xi^4+135\xi^2-5),\\
\lambda_7^{\ker}(\xi) &= -\frac{1}{64}\sqrt{34}(715\xi^6-1001\xi^4+385\xi^2-35)\xi,\\
\lambda_8^{\ker}(\xi) &= -\frac{1}{128}\sqrt{38}(2431\xi^8-4004\xi^6+2002\xi^4-308\xi^2+7).
\end{aligned}
$$

The shape functions l_0 and l_1 are called vertex shape functions and the l_j, $j \geq 2$, are referred as bubble (interior) shape functions. Similarly, we speak about the vertex and bubble basis functions defined element-by-element in Ω. The standard piecewise linear vertex functions φ_k are constructed for $k = 1, 2, \dots, M$ as follows

$$
\varphi_k(x) = \begin{cases} l_1\big(\chi_{K_k}^{-1}(x)\big), & \text{for } x \in K_k,\\ l_0\big(\chi_{K_{k+1}}^{-1}(x)\big), & \text{for } x \in K_{k+1},\\ 0 & \text{otherwise.} \end{cases}
$$

The $N - M$ bubble functions, where $N = -1 + \sum_{K\in\mathcal{T}_{hp}} p_K$ is the dimension of V_{hp}, are defined in a similar way. The $p_K - 1$ bubble functions $\varphi_2^{b,K}, \varphi_3^{b,K}, \dots, \varphi_{p_K}^{b,K}$ in an element K are obtained as

$$
\varphi_i^{b,K}(x) = \begin{cases} l_i(\chi_K^{-1}(x)), & \text{for } x \in K,\\ 0 & \text{otherwise,} \end{cases} \qquad i = 2, 3, \dots, p_K.
$$

As usual, we assemble the stiffness matrix $\mathbb{A} \in \mathbb{R}^{N\times N}$, $\mathbb{A}_{ij} = a(\varphi_i, \varphi_j)$, $i, j = 1, 2, \dots, N$, from the local stiffness matrices $\mathbb{A}^K \in \mathbb{R}^{(p_K+1)\times(p_K+1)}$, $\mathbb{A}_{ij}^K = a_K(\varphi_j, \varphi_i)$, $i, j = 1, 2, \dots, (p_K + 1)$, where p_K stands for the polynomial degree of the element $K \in \mathcal{T}_{hp}$ and

$$
a_K(\varphi_j, \varphi_i) = \int_K \varphi_j' \varphi_i' \,\mathrm{d}x + \kappa^2 \int_K \varphi_j \varphi_i \,\mathrm{d}x.
$$

Due to the existence of the vertex and bubble functions, the matrices $\mathbb{A}$ and $\mathbb{A}^K$ have a natural 2-by-2 block structure

$$
\mathbb{A} = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \quad \text{and} \quad \mathbb{A}^K = \begin{pmatrix} A^K & B^K \\ (B^K)^T & D^K \end{pmatrix},
$$

where $A \in \mathbb{R}^{M\times M}$, $B \in \mathbb{R}^{M\times(N-M)}$, and $D \in \mathbb{R}^{(N-M)\times(N-M)}$, $A^K \in \mathbb{R}^{2\times 2}$, $B^K \in \mathbb{R}^{2\times(p_K-1)}$, and $D^K \in \mathbb{R}^{(p_K-1)\times(p_K-1)}$. The entries of the element stiffness matrix

$\mathbb{A}^K$ can be easily computed, see, e.g. [14]. If the length of the element K is denoted by h then

$$hA^K = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{\kappa^2 h^2}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \tag{5.3}$$

$$hB^K = \kappa^2 h^2 \begin{pmatrix} -\sqrt{6}/12 & \sqrt{10}/60 & 0 & \cdots & 0 \\ -\sqrt{6}/12 & -\sqrt{10}/60 & 0 & \cdots & 0 \end{pmatrix}, \tag{5.4}$$

and the only nonzero elements of D^K are

$$hD^K_{ii} = 2 + \kappa^2 h^2 \frac{1}{(2i+3)(2i-1)}, \qquad i = 1, 2, \ldots, p_K - 1, \tag{5.5}$$

$$hD^K_{i,i+2} = hD^K_{i+2,i} = \kappa^2 h^2 \frac{1}{2(2i+3)\sqrt{(2i+1)(2i+5)}}, \quad i = 1, 2, \ldots, p_K - 3.$$

Obviously, if the element K is adjacent to the boundary then the matrix A^K reduces to a 1-by-1 matrix and B^K only contains one row.

It is convenient to multiply the formulas for A^K, B^K, and D^K by h because then the entries of matrices hA^K, hB^K, and hD^K are functions of a single parameter $\zeta = \kappa^2 h^2$. Further, we remark that the matrix D is block diagonal because each bubble function is supported in a single element. We denote $D = \text{blockdiag}\{D^K,\ K \in \mathcal{T}_{hp}\}$.

To prove the DMP it is convenient to introduce the discrete minimum energy extensions $\psi_1, \psi_2, \ldots, \psi_M$ of the vertex functions $\varphi_1, \varphi_2, \ldots, \varphi_M$ with respect to the space of all bubbles $V^b_{hp} = \text{span}\{\varphi^{b,K}_i,\ i = 2, 3, \ldots, p_K,\ K \in \mathcal{T}_{hp}\}$.

Notice that if the reaction coefficient κ vanishes then the standard piecewise linear vertex functions are orthogonal to the bubble functions and $\psi_i = \varphi_i$, $i = 1, 2, \ldots, M$, in this case, cf. (5.4).

In analogy with (4.4) we express

$$\psi_i = \varphi_i - \sum_{\substack{K \in \mathcal{T}_{hp} \\ K \subset \text{supp}\,\psi_i}} \sum_{j=1}^{p_K - 1} C^K_{\iota_K(i),j} \varphi^{b,K}_{j+1}, \quad i = 1, 2, \ldots, M. \tag{5.6}$$

where $\iota_K(i)$ is the standard connectivity mapping, see, e.g., [6, 14, 21]. In our case $\iota_K(i)$ equals to 1 or 2 depending whether ψ_i corresponds to the left or to the right endpoint of K. The matrix C^K of coefficients $C^K_{\iota_K(i),j}$ has one or two rows and $p_K - 1$ columns and it is given by (4.5) as $C^K = B^K (D^K)^{-1}$. We stress that the entries of C^K are functions of a single parameter $\zeta = \kappa^2 h_K^2$.

Further, by (4.1) the discrete minimum energy extensions $\psi_1, \psi_2, \ldots, \psi_M$ are orthogonal to all bubbles $\varphi^{b,K}_i$, $i = 2, 3, \ldots, p_K$, $K \in \mathcal{T}_{hp}$, where the orthogonality is understood in the energy inner product $a(\cdot, \cdot)$. Finally, by (4.6) the stiffness matrix $\mathbb{S} \in \mathbb{R}^{N \times N}$ formed from the discrete minimum energy extensions $\psi_1, \psi_2, \ldots, \psi_M$ and from the original bubbles $\varphi^{b,K}_i$, $i = 2, 3, \ldots, p_K$, $K \in \mathcal{T}_{hp}$, has the following structure

$$\mathbb{S} = \begin{pmatrix} S & 0 \\ 0 & D \end{pmatrix}, \tag{5.7}$$

where $S = A - BD^{-1}B^T$ stands for the Schur complement and D is block diagonal.

6. Nonnegativity of the discrete Green's function. The DGF corresponding to problem (2.3) can be expressed by (3.5) using the discrete minimum energy extensions ψ_i, $i = 1, 2, \ldots, M$, and the standard bubble functions $\varphi_i^{b,K}$, $i = 2, 3, \ldots, p_K$, $K \in \mathcal{T}_{hp}$. Thanks to the structure of the stiffness matrix $\mathbb{S}$, see (5.7), we can express the DGF as a sum of the vertex and bubble parts

$$G_{hp}(x,y) = G_{hp}^v(x,y) + G_{hp}^b(x,y), \quad (x,y) \in \Omega^2, \tag{6.1}$$

where

$$G_{hp}^v(x,y) = \sum_{i=1}^{M}\sum_{j=1}^{M} S_{ij}^{-1}\psi_i(x)\psi_j(y), \quad (x,y) \in \Omega^2, \tag{6.2}$$

$$G_{hp}^b(x,y) = \sum_{K\in\mathcal{T}_{hp}}\sum_{i=1}^{p_K-1}\sum_{j=1}^{p_K-1} (D^K)_{ij}^{-1}\varphi_{i+1}^{b,K}(x)\varphi_{j+1}^{b,K}(y), \quad (x,y) \in \Omega^2. \tag{6.3}$$

The following theorem introduces three sufficient conditions for the nonnegativity of the DGF G_{hp}.

THEOREM 6.1. *Let ψ_i, $i = 1, 2, \ldots, M$, $S \in \mathbb{R}^{M\times M}$, and G_{hp} be given by* (5.6), (5.7), *and* (6.1)–(6.3). *If*

(a) $\psi_i(x) \geq 0$ for all $i = 1, 2, \ldots, M$ and $x \in \Omega$,

(b) $S_{ij} \leq 0$ for all $i \neq j$, $i, j = 1, 2, \ldots, M$,

(c) $G_{hp}^v + G_{hp}^b \geq 0$ in K^2 for all $K \in \mathcal{T}_{hp}$,

then $G_{hp}(x,y) \geq 0$ for all $(x,y) \in \Omega^2$.

Proof. By the theory of M-matrices, see, e.g., [9, 16], if all offdiagonal entries of S are nonpositive and if S is symmetric and positive definite then S^{-1} consists of nonnegative entries, i.e, $S_{ij}^{-1} \geq 0$ for all $i, j = 1, 2, \ldots, M$. Hence, this fact together with (a) imply the nonnegativity of the vertex part G_{hp}^v in Ω^2, cf. (6.2). Since the support of any bubble function consists of a single element, we find that

$$G_{hp}^b(x,y) = 0 \quad \text{for } (x,y) \in K \times K^*,\ K, K^* \in \mathcal{T}_{hp},\ K \neq K^*.$$

This together with (c) proves the nonnegativity of $G_{hp} = G_{hp}^v + G_{hp}^b$ in the entire square Ω^2. ◻

6.1. Nonnegativity of the vertex DGF. We present two lemmas which show the validity of conditions (a) and (b) from Theorem 6.1 provided that the products $\kappa^2 h_K^2$ are bounded from above by values α^{p_K} and β^{p_K} for all elements $K \in \mathcal{T}_{hp}$. The bounds α^{p_K} and β^{p_K} can be computed for an arbitrary polynomial degree p_K as roots of certain polynomials.

LEMMA 6.2. *Let h_K and p_K stand for the length and polynomial degree of the element $K \in \mathcal{T}_{hp}$. Further, let ψ_i, $i = 1, 2, \ldots, M$, be given by* (5.6). *For each polynomial degree p, there exists $\alpha^p \in (0, \infty]$ with the following property. If*

$$\kappa^2 h_K^2 \leq \alpha^{p_K} \quad \textit{for all } K \in \mathcal{T}_{hp}$$

then $\psi_i(x) \geq 0$ for all $x \in \Omega$, i.e., condition (a) from Theorem 6.1 is satisfied.

Proof. Let $K = [x_{k-1}, x_k]$, $1 \leq k \leq M+1$, be an element in $\mathcal{T}_{hp}$ with the length $h_K = x_k - x_{k-1}$ and with the polynomial degree p_K. Further, let ψ_k be the vertex function corresponding to x_k (if $k \neq M+1$). We transform ψ_k from K to K_{ref} by (5.1) as follows

$$\psi_k|_K\big(\chi_K(\xi)\big) = l_1(\xi) - \sum_{m=1}^{p_K-1} C^K_{\iota_K(k),m} l_{m+1}(\xi)$$
$$= l_1(\xi)\left[1 - \sum_{m=1}^{p_K-1} C^K_{\iota_K(k),m} l_0(\xi)\lambda^{\mathrm{ker}}_{m-1}(\xi)\right] = l_1(\xi)\Psi_1^{p_K}(\kappa^2 h_K^2, \xi), \quad (6.4)$$

where we use (5.2). The connectivity mapping $\iota_K(k)$ was introduced above as well as the coefficients $C^K_{\iota_K(k),m}$. We recall that the coefficients $C^K_{\iota_K(k),m}$ are functions of the single parameter $\zeta = \kappa^2 h_K^2$. This justifies the definition of $\Psi_1^{p_K}(\kappa^2 h_K^2, \xi)$. The vertex function corresponding to x_{k-1} (if $k \neq 1$) can be transformed in a similar way

$$\psi_{k-1}|_K\big(\chi_K(\xi)\big) = l_0(\xi)\left[1 - \sum_{m=1}^{p_K-1} C^K_{\iota_K(k-1),m} l_1(\xi)\lambda^{\mathrm{ker}}_{m-1}(\xi)\right] = l_0(\xi)\Psi_0^{p_K}(\kappa^2 h_K^2, \xi).$$

Thus, ψ_{k-1} and ψ_k, are nonnegative in K_k if and only if $\Psi_i^{p_K}(\kappa^2 h_K^2, \xi)$, $i = 0, 1$, are nonnegative for all $\xi \in K_{\mathrm{ref}}$. Since $\Psi_i^{p_K}(0, \xi) = 1$ and $\Psi_i^{p_K}$ depends continuously on $\kappa^2 h_K^2$ then clearly $\alpha^{p_K} > 0$. ∎

We remark that $\Psi_i^p(\zeta, \xi)$, $i = 0, 1$, are polynomials of degree $p-1$ in ξ and rational functions in ζ. The nonnegativity of $\Psi_i^p(\zeta, \xi)$ for $\zeta \geq 0$ and $\xi \in [-1, 1]$ can be analyzed numerically to find the greatest possible value α^p for each polynomial degree p. The results for $p = 1, 2, \ldots, 10$ are presented in Table 9.1 below.

LEMMA 6.3. *Let h_K and p_K denote the length and the polynomial degree of the element $K \in \mathcal{T}_{hp}$. Further, let S be given by (5.7). For each polynomial degree p, there exists $\beta^p \in (0, \infty]$ with the following property. If*

$$\kappa^2 h_K^2 \leq \beta^{p_K} \quad \text{for all } K \in \mathcal{T}_{hp}$$

then $S_{ij} \leq 0$ for all $i \neq j$, $i, j = 1, 2, \ldots, M$, i.e., condition (b) from Theorem 6.1 is satisfied.

Proof. Clearly, the matrix S is tridiagonal, hence, the only nonzero off-diagonal entries are

$$a_K(\psi_{k-1}, \psi_k) = S_{k,k-1} = S_{k-1,k} = S^K_{12}, \quad k = 2, 3, \ldots, M,$$

where again ψ_{k-1} and ψ_k are the vertex functions corresponding to the endpoints of $K \in \mathcal{T}_{hp}$, S is given by (5.7), and $S^K = A^K - B^K(D^K)^{-1}(B^K)^T$ is the local Schur complement.

If we multiply S^K by $h = h_K$ (the length of K) we find that $hS^K = hS^{p_K}(\kappa^2 h^2)$ is a function of $\kappa^2 h^2$ only. Since $hS^{p_K}_{12}(0) = -1$ and since $hS^{p_K}(\cdot)$ is continuous we conclude that $S_{k,k-1} < 0$ for small enough $\kappa^2 h^2 > 0$. Consequently, the positive β^{p_K} exists. ∎

The function $hS^p_{12}(\cdot)$ is a rational function. However, $\tilde{q}_p(\zeta) = \det\big(hD^K(\zeta)\big) hS^p_{12}(\zeta)$ with $\zeta = \kappa^2 h^2$ is a polynomial of degree p in ζ. Since $\det\big(hD^K(\zeta)\big) > 0$ it suffices to investigate the nonpositivity of $\tilde{q}_p(\zeta)$. This is numerically an easy task and the largest possible bound $\beta^p = \min\{\zeta : \zeta \geq 0, \text{ and } \tilde{q}_p(\zeta) = 0\}$ can be computed for any degree p. The values of β^p for $p = 1, 2, \ldots, 10$ are presented in Table 9.1 below.

The conclusions from Theorem 6.1 and Lemmas 6.2 and 6.3 are summarized in the following corollary.

COROLLARY 6.4. *Let $\mathcal{T}_{hp}$ be a finite element mesh and let h_K and p_K denote the length and the polynomial degree of the element $K \in \mathcal{T}_{hp}$. If*

$$\kappa^2 h_K^2 \leq \min\{\alpha^{p_K}, \beta^{p_K}\} \quad \text{for all } K \in \mathcal{T}_{hp}$$

then $G_{hp}^v(x,y) \geq 0$ for all $(x,y) \in \Omega^2$.

7. The bubble part of the DGF. In this section we verify the validity of condition (c) from Theorem 6.1. This is the most difficult part because the bubble part G_{hp}^b defined by (6.3) is not nonnegative, in general. For $p = 3$, $p = 5$, and $p \geq 7$, there always are regions, where G_{hp}^b is negative. In these regions, the negative bubble part G_{hp}^b has to be compensated by the positive vertex part G_{hp}^v in order to obtain the nonnegativity of $G_{hp} = G_{hp}^v + G_{hp}^b$ and consequently the DMP. The good news is that condition (c) can be investigated for each element $K \in \mathcal{T}_{hp}$ independently.

Therefore, throughout this section, we consider an arbitrary but fixed element $K = [x_{k-1}, x_k]$ in $\mathcal{T}_{hp}$. The length and polynomial degree of this K are denoted by h and p.

For this K we will define two auxiliary DGFs $\widetilde{G}_{hp}$ and $\widehat{G}_{hp}$, see Figure 7.1 for an illustration. We will show that $\widehat{G}_{hp} \leq \widetilde{G}_{hp} \leq G_{hp}$ in K^2. The second auxiliary DGF $\widehat{G}_{hp}$ is simple enough to allow for the direct investigation of its nonnegativity. Below, in Section 7.4, we formulate conditions which guarantee $\widehat{G}_{hp} \geq 0$ and, hence, the condition (c) from Theorem 6.1. The DMP then follows from Theorems 6.1 and 3.4.

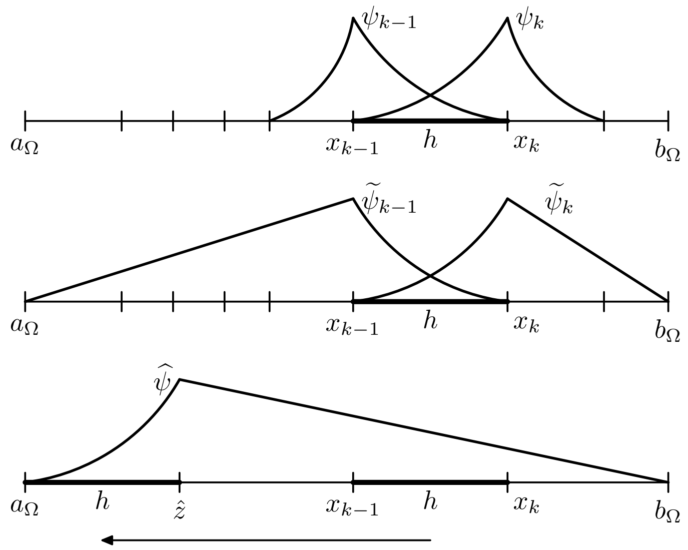

FIG. 7.1. *An illustration of the basis functions used for the construction of G_{hp}^v (top), $\widetilde{G}_{hp}^v$ (middle), and $\widehat{G}_{hp}^v$ (bottom) corresponding to the element $K = [x_{k-1}, x_k]$ of the length h.*

7.1. The first auxiliary DGF. An element $K \in \mathcal{T}_{hp}$ is called interior if it is not adjacent to the boundary of Ω, i.e., if $K \subset \Omega$. To define the first auxiliary DGF $\widetilde{G}_{hp}$, we assume for technical reasons that the element K is interior. Then, we consider a partition $a_\Omega < x_{k-1} < x_k < b_\Omega$ which defines a mesh $\widetilde{\mathcal{T}}_{hp}$ consisting of three elements. The polynomial degree assigned to the element $K = [x_{k-1}, x_k] \in \widetilde{\mathcal{T}}_{hp}$ is p while the degree of the other elements in $\widetilde{\mathcal{T}}_{hp}$ is set to 1. These polynomial degrees and the mesh $\widetilde{\mathcal{T}}_{hp}$ lead to an hp-FEM space $\widetilde{V}_{hp}$ defined in analogy with (2.2). In $\widetilde{V}_{hp}$ we consider the standard hp-FEM basis. It consists of two piecewise linear vertex functions $\widetilde{\varphi}_{k-1}$ and $\widetilde{\varphi}_k$ and of $p-1$ bubble functions $\varphi_2^{b,K}, \varphi_3^{b,K}, \dots, \varphi_p^{b,K}$. Notice that these bubble functions coincide with the bubbles defined on the original mesh $\mathcal{T}_{hp}$.

Further, we consider the discrete minimum energy extensions $\widetilde{\psi}_{k-1}$ and $\widetilde{\psi}_k$ of $\widetilde{\varphi}_{k-1}$ and $\widetilde{\varphi}_k$ with respect to the space $V_{hp}^{b,K} = \operatorname{span}\{\varphi_2^{b,K}, \varphi_3^{b,K}, \dots, \varphi_p^{b,K}\}$. Hence, $\widetilde{\psi}_{k-1}$ is linear in $[a_\Omega, x_{k-1}]$, $\widetilde{\psi}_{k-1} = \psi_{k-1}$ in $K = [x_{k-1}, x_k]$, and $\widetilde{\psi}_{k-1} = 0$ in $[x_k, b_\Omega]$. Similarly, $\widetilde{\psi}_k = 0$ in $[a_\Omega, x_{k-1}]$, $\widetilde{\psi}_k = \psi_k$ in $K = [x_{k-1}, x_k]$, and $\widetilde{\psi}_k$ is linear in $[x_k, b_\Omega]$. See the middle panel of Figure 7.1.

We construct a stiffness matrix $\widetilde{A} \in \mathbb{R}^{2\times 2}$ from $\widetilde{\varphi}_{k-1}$ and $\widetilde{\varphi}_k$ as follows

$$\widetilde{A}_{ij} = a(\widetilde{\psi}_{k-2+i}, \widetilde{\psi}_{k-2+j}), \quad i,j = 1,2. \tag{7.1}$$

In agreement with (6.1)–(6.3), we define the first auxiliary DGF

$$\widetilde{G}_{hp}(x,y) = \widetilde{G}_{hp}^v(x,y) + \widetilde{G}_{hp}^b(x,y), \quad (x,y) \in \Omega^2, \tag{7.2}$$

where

$$\widetilde{G}_{hp}^v(x,y) = \sum_{i=1}^{2}\sum_{j=1}^{2} (\widetilde{A})^{-1}_{ij} \widetilde{\psi}_{k-2+i}(x)\widetilde{\psi}_{k-2+j}(y), \quad (x,y) \in \Omega^2 \tag{7.3}$$

and $\widetilde{G}_{hp}^b(x,y) = G_{hp}^b(x,y)$, cf. (6.3).

The main result about $\widetilde{G}_{hp}(x,y)$ is formulated in the following lemma.

LEMMA 7.1. *Let condition (b) from Theorem 6.1 be satisfied. For an interior element $K \in \mathcal{T}_{hp}$, $K = [x_{k-1}, x_k] \subset \Omega$, $k = 2,3,\dots,M$, we consider the first auxiliary DGF $\widetilde{G}_{hp}(x,y)$ defined by* (7.2)–(7.3). *Then*

$$G_{hp}(x,y) \geq \widetilde{G}_{hp}(x,y) \quad \textit{for all } (x,y) \in K^2.$$

Proof. Clearly, it suffices to prove $G_{hp}^v(x,y) \geq \widetilde{G}_{hp}^v(x,y)$ for all $(x,y) \in K^2$. Let $K \in \mathcal{T}_{hp}$, $K = [x_{k-1}, x_k] \subset \Omega$, be an arbitrary but fixed interior element. First we consider the original vertex functions $\psi_1, \psi_2, \dots, \psi_M$. Let ψ_{k-1}^{me} and ψ_k^{me} be the discrete minimum energy extensions of ψ_{k-1} and ψ_k with respect to $V_{hp}^{v\#} = \operatorname{span}\{\psi_1, \dots, \psi_{k-2}, \psi_{k+1}, \dots, \psi_M\}$. Definition (4.3) yields $\psi_{k-1}^{\mathrm{me}}(x) = \psi_{k-1}(x)$ and $\psi_k^{\mathrm{me}}(x) = \psi_k(x)$ for all $x \in K$ because $\psi_j(x) = 0$ for all $x \in K$ and for all $\psi_j \in V_{hp}^{v\#}$. Using the definition of $\widetilde{\psi}_{k-1}$ and $\widetilde{\psi}_k$, we summarize

$$\psi_{k-1} = \widetilde{\psi}_{k-1} = \psi_{k-1}^{\mathrm{me}} \quad \text{and} \quad \psi_k = \widetilde{\psi}_k = \psi_k^{\mathrm{me}} \quad \text{in } K. \tag{7.4}$$

The stiffness matrix $S^{\mathrm{me}} \in \mathbb{R}^{2\times 2}$ corresponding to the basis functions ψ_{k-1}^{me} and ψ_k^{me} can be computed as a suitable Schur complement, cf. (4.7).

Now, let us concentrate on $\widetilde{\psi}_{k-1}$ and $\widetilde{\psi}_k$. We remark that the discrete minimum energy extensions $\widetilde{\psi}_{k-1}^{\mathrm{me}}$ and $\widetilde{\psi}_k^{\mathrm{me}}$ of $\widetilde{\psi}_{k-1}$ and $\widetilde{\psi}_k$ with respect to $V_{hp}^{v\#}$ are equal to the already defined discrete minimum energy extensions ψ_{k-1}^{me} and ψ_k^{me}, respectively. Indeed, cf. (4.1), if $0 = a(\psi_k^{\mathrm{me}}, v^\#) = a(\widetilde{\psi}_k^{\mathrm{me}}, v^\#)$ for all $v^\# \in V_{hp}^{v\#}$ then $0 = a(\psi_k^{\mathrm{me}} - \widetilde{\psi}_k^{\mathrm{me}}, v^\#)$ for all $v^\# \in V_{hp}^{v\#}$ and since $\psi_k^{\mathrm{me}} - \widetilde{\psi}_k^{\mathrm{me}} \in V_{hp}^{v\#}$ then $\psi_k^{\mathrm{me}} = \widetilde{\psi}_k^{\mathrm{me}}$. The same steps can be repeated to show that $\psi_{k-1}^{\mathrm{me}} = \widetilde{\psi}_{k-1}^{\mathrm{me}}$.

From (7.4) we conclude that

$$\widetilde{A}_{12} = a_K(\widetilde{\psi}_{k-1}, \widetilde{\psi}_k) = a_K(\psi_{k-1}^{\mathrm{me}}, \psi_k^{\mathrm{me}}) = S_{12}^{\mathrm{me}}.$$

Similarly, from (4.2) we infer the inequalities

$$\begin{aligned} \widetilde{A}_{11} &= a(\widetilde{\psi}_{k-1}, \widetilde{\psi}_{k-1}) \geq a(\widetilde{\psi}_{k-1}^{\mathrm{me}}, \widetilde{\psi}_{k-1}^{\mathrm{me}}) = a(\psi_{k-1}^{\mathrm{me}}, \psi_{k-1}^{\mathrm{me}}) = S_{22}^{\mathrm{me}}, \\ \widetilde{A}_{22} &= a(\widetilde{\psi}_k, \widetilde{\psi}_k) \geq a(\widetilde{\psi}_k^{\mathrm{me}}, \widetilde{\psi}_k^{\mathrm{me}}) = a(\psi_k^{\mathrm{me}}, \psi_k^{\mathrm{me}}) = S_{22}^{\mathrm{me}}. \end{aligned}$$

Hence, all entries of $\widetilde{A}$ are greater or equal to the corresponding entries of S^{me} and we write $\widetilde{A} \geq S^{\mathrm{me}}$. Condition (b) from Theorem 6.1 implies that both $\widetilde{A}$ and S^{me} are M-matrices. In particular, they have the nonnegative inverse and therefore

$$\widetilde{A}^{-1} \leq (S^{\mathrm{me}})^{-1}. \tag{7.5}$$

Using this fact, we obtain the following relation by (6.2), (4.7), (7.4), and by (7.5)

$$\begin{aligned} G_{hp}^v(x,y) = \sum_{i=1}^M \sum_{j=1}^M S_{ij}^{-1} \psi_i(x) \psi_j(y) &= \sum_{i=1}^2 \sum_{j=1}^2 (S^{\mathrm{me}})_{ij}^{-1} \psi_{k-2+i}^{\mathrm{me}}(x) \psi_{k-2+j}^{\mathrm{me}}(y) \\ &\geq \sum_{i=1}^2 \sum_{j=1}^2 (\widetilde{A})_{ij}^{-1} \widetilde{\psi}_{k-2+i}(x) \widetilde{\psi}_{k-2+j}(y) = \widetilde{G}_{hp}^v(x,y) \end{aligned}$$

for all $(x,y) \in K^2$. ∎

7.2. DGF in the elements adjacent to the boundary. Let us analyze the auxiliary DGF $\widetilde{G}_{hp}$ in more detail. For the arbitrarily chosen element $K \in \mathcal{T}_{hp}$, $K = [x_{k-1}, x_k]$, with the polynomial degree p and with the length h we introduce a parameter $t \in [0,1]$ such that

$$\begin{aligned} x_{k-1} &= (1-t)a_\Omega + t(b_\Omega - h), \\ x_k &= (1-t)(a_\Omega + h) + t b_\Omega. \end{aligned} \tag{7.6}$$

Clearly, the parameter t determines the position of K in $\Omega = (a_\Omega, b_\Omega)$. For example, if $t = 0$ then K is adjacent to the left endpoint a_Ω of Ω, if $t = 1$ then K is adjacent to b_Ω, and if $t \leq 1/2$ then the midpoint of K lies in the left half of Ω. Moreover, we define the relative length $H_{\mathrm{rel}} = H_{\mathrm{rel}}^K$ of an element $K \in \mathcal{T}_{hp}$ and an auxiliary parameter θ as

$$H_{\mathrm{rel}} = \frac{h}{b_\Omega - a_\Omega} \quad \text{and} \quad \theta = \frac{H_{\mathrm{rel}}}{1 - H_{\mathrm{rel}}} = \frac{h}{b_\Omega - a_\Omega - h}. \tag{7.7}$$

Notice that $H_{\mathrm{rel}} \in (0,1]$ and $\theta \in (0, \infty]$.

Here, we restrict ourselves to the interior elements only, i.e., we assume $t \in (0,1)$. To express the the stiffness matrix $\widetilde{A} \in \mathbb{R}^{2\times 2}$ assembled from $\widetilde{\psi}_{k-1}$ and $\widetilde{\psi}_k$ we introduce two auxiliary functions

$$r(\zeta) = (hS_{11}^K)(\zeta) = (hS_{22}^K)(\zeta) \quad \text{and} \quad q(\zeta) = (hS_{12}^K)(\zeta), \tag{7.8}$$

where $\zeta = \kappa^2 h^2$ and $S^K = A^K - B^K (D^K)^{-1} (B^K)^T$, cf. (5.7), with matrices A^K, B^K, D^K given by (5.3)–(5.5). We stress that $hS^K_{11} = hS^K_{22}$ and hS^K_{12} are rational functions of the paremeter $\zeta = \kappa^2 h^2$. Further notice that $r(\zeta) = r(\kappa^2 h^2) = a_K(\psi_{k-1}, \psi_{k-1}) = a_K(\psi_k, \psi_k) > 0$ for $h > 0$ and that $q(\zeta) = q(\kappa^2 h^2) \le 0$ for $\zeta = \kappa^2 h^2 \le \beta^p$, see Lemma 6.3. For illustration, if $p = 3$ then

$$r(\zeta) = \frac{6300 + 2880\zeta + 135\zeta^2 + \zeta^3}{15(10+\zeta)(42+\zeta)} \quad \text{and} \quad q(\zeta) = \frac{-25200 + 1080\zeta - 30\zeta^2 + \zeta^3}{60(10+\zeta)(42+\zeta)}.$$

Using (7.8) and the parameters t and θ, we can express $\widetilde{A}$ as

$$h\widetilde{A} = \begin{pmatrix} r(\kappa^2 h^2) + \dfrac{\theta}{t} + \dfrac{\kappa^2 h^2}{3} \dfrac{t}{\theta} & q(\kappa^2 h^2) \\ q(\kappa^2 h^2) & r(\kappa^2 h^2) + \dfrac{\theta}{1-t} + \dfrac{\kappa^2 h^2}{3} \dfrac{1-t}{\theta} \end{pmatrix}.$$

Our goal is to study the limit $t \to 0$. Since we analyze the DGF, we need to compute the inverse $(h\widetilde{A})^{-1}$. The entry $(h\widetilde{A})^{-1}_{11} \to 0$ for $t \to 0$ and, therefore, we concentrate on

$$s(t,\theta,\zeta) = (h\widetilde{A})^{-1}_{22} = \left(r(\zeta) + \frac{\theta}{1-t} + \frac{\zeta}{3}\frac{1-t}{\theta} - \frac{q^2(\zeta)}{r(\zeta) + \dfrac{\theta}{t} + \dfrac{\zeta}{3}\dfrac{t}{\theta}} \right)^{-1}, \tag{7.9}$$

which is well defined for $t \in (0,1)$, $\theta \in (0,\infty)$, and $\zeta = \kappa^2 h^2 \in [0,\infty)$. For $t = 0$ we define $s(t,\theta,\zeta)$ by the following limit

$$s(0,\theta,\zeta) = \lim_{t \to 0_+} s(t,\theta,\zeta) = \left(r(\zeta) + \theta + \frac{\zeta}{3\theta} \right)^{-1}. \tag{7.10}$$

For convenience, we also define $s(0,0,\zeta) = 0$ for all $\zeta \in [0,\infty)$ by a limit.

LEMMA 7.2. *If* $s(t,\theta,\zeta)$ *is defined by* (7.9) *and* (7.10) *then*

$$s(t,\theta,\zeta) \ge s(0,\theta,\zeta) \quad \text{for all } \theta \in (0,1/2],\ t \in [0,1/2],\ \zeta \in [0,\infty). \tag{7.11}$$

Proof. The inequality (7.11) is equivalent to

$$s^*(t,\theta,\zeta) = \frac{\theta}{1-t} - \frac{\zeta}{3\theta} - \frac{q^2(\zeta)}{r(\zeta)t + \theta + \dfrac{\zeta}{3}\dfrac{t^2}{\theta}} \le 0. \tag{7.12}$$

Clearly, since $r(\zeta) > 0$, the function $s^*(t,\theta,\zeta)$ is increasing in the variable t. Hence,

$$s^*(t,\theta,\zeta) \le s^*(1/2,\theta,\zeta) = 2\theta - \frac{\zeta}{3\theta} - \frac{q^2(\zeta)}{\dfrac{1}{2}r(\zeta) + \theta + \dfrac{\zeta}{12\theta}}. \tag{7.13}$$

Differentiating $s^*(1/2,\theta,\zeta)$ with respect to θ and using the fact that $\det(hA^K) = r^2(\zeta) - q^2(\zeta) \ge 0$ we find out that $s^*(1/2,\theta,\zeta)$ is increasing in θ. Thus,

$$s^*(1/2,\theta,\zeta) \le s^*(1/2,1/2,\zeta) = 1 - \frac{2}{3}\zeta - \frac{2q^2(\zeta)}{r(\zeta) + 1 + \dfrac{\zeta}{3}}. \tag{7.14}$$

Similarly, it can be verified that $s^*(1/2,1/2,\zeta)$ is decreasing in ζ and, therefore,

$$s^*(1/2,1/2,\zeta) \le s^*(1/2,1/2,0) = 0 \tag{7.15}$$

because $r(0) = 1$ and $q(0) = -1$. The combination of (7.12)–(7.15) finishes the proof. ☐

7.3. The second auxiliary DGF. The second auxiliary DGF $\widehat{G}_{hp}$ is the limit case of the first auxiliary DGF $\widetilde{G}_{hp}$ if $t \to 0_+$ or if $t \to 1_-$. Without the loss of generality, we restrict ourselves to the former case and consider an arbitrary but fixed element $K = [x_{x-1}, x_k] \in \mathcal{T}_{hp}$ not adjacent to the right endpoint b_Ω of the interval Ω. If h and p stand for the length and polynomial degree of this K then we set $\hat{z} = a_\Omega + h$ and consider two elements $\widehat{K} = [a_\Omega, \hat{z}]$ and $[\hat{z}, b_\Omega]$ with polynomial degrees p and 1.

The standard *hp*-FEM basis comprises one piecewise linear vertex function $\widehat{\varphi}$ and $p-1$ bubble functions $\widehat{\varphi}_2^{b,\widehat{K}}, \widehat{\varphi}_3^{b,\widehat{K}}, \dots, \widehat{\varphi}_p^{b,\widehat{K}}$ supported in $\widehat{K}$. The values of the vertex function are $\widehat{\varphi}(\hat{z}) = 1$ and $\widehat{\varphi}(a_\Omega) = \widehat{\varphi}(b_\Omega) = 0$.

As before, we define $\widehat{\psi}$ as the discrete minimum energy extension of the vertex function $\widehat{\varphi}$ with respect to the space of the bubbles $V_{hp}^{b,\widehat{K}} = \operatorname{span}\{\widehat{\varphi}_2^{b,\widehat{K}}, \widehat{\varphi}_3^{b,\widehat{K}}, \dots, \widehat{\varphi}_p^{b,\widehat{K}}\}$, cf. the bottom panel of Figure 7.1. Notice that $\widehat{\psi}$ is a linear function in $[\hat{z}, b_\Omega]$ and that $\widehat{\psi}$ restricted to $\widehat{K}$ is just the shifted function $\widetilde{\psi}_k = \psi_k$ restricted to K, i.e.,

$$\widehat{\psi}(x + a_\Omega - x_{k-1}) = \widetilde{\psi}_k(x) = \psi_k(x) \quad \text{for all } x \in K. \tag{7.16}$$

In agreement with (6.1)–(6.3) we define

$$\widehat{G}_{hp}(x,y) = \widehat{G}^v_{hp}(x,y) + \widehat{G}^b_{hp}(x,y), \quad (x,y) \in \Omega^2, \tag{7.17}$$

where

$$\widehat{G}^v_{hp}(x,y) = \frac{1}{a\big(\widehat{\psi},\widehat{\psi}\big)} \widehat{\psi}(x)\widehat{\psi}(y), \quad (x,y) \in \Omega^2, \tag{7.18}$$

$$\widehat{G}^b_{hp}(x,y) = \sum_{i=1}^{p-1}\sum_{j=1}^{p-1} (D^K)^{-1} \widehat{\varphi}_{i+1}^{b,\widehat{K}}(x) \widehat{\varphi}_{j+1}^{b,\widehat{K}}(y), \quad (x,y) \in \Omega^2. \tag{7.19}$$

We recall that D^K is given by (5.5). The equality

$$\frac{1}{a\big(\widehat{\psi},\widehat{\psi}\big)} = hs(0,\theta,\kappa^2 h^2) \tag{7.20}$$

is not surprising and can be easily verified.

In Section 7.1 we did not define the first auxiliary DGF $\widetilde{G}_{hp}$ for the elements adjacent to the boundary of Ω. For completeness, if $K \in \mathcal{T}_{hp}$ is adjacent to the left endpoint of Ω then we set

$$\widetilde{G}_{hp}(x,y) = \widehat{G}_{hp}(x,y) \quad \text{for all } (x,y) \in \Omega^2. \tag{7.21}$$

If the element $K \in \mathcal{T}_{hp}$ is adjacent to the right endpoint of Ω then $\widehat{G}_{hp}(x,y) = \widetilde{G}_{hp}(x,y)$ are defined in a symmetric way.

The following lemma shows the relation between the first and the second auxiliary DGF.

LEMMA 7.3. *Let conditions (a) and (b) from Theorem 6.1 be satisfied. Further, let $K \in \mathcal{T}_{hp}$ be such that $t \le 1/2$, cf. (7.6), and $H_{\rm rel} \le 1/3$, cf. (7.7). If $\widehat{G}_{hp}(x,y)$ and $\widetilde{G}_{hp}(x,y)$ are given by (7.17) and (7.2)–(7.3) with (7.21) then*

$$\widehat{G}_{hp}(\hat{x},\hat{y}) \le \widetilde{G}_{hp}(x,y) \quad \textit{for all } (x,y) \in K^2,$$

where $\hat{x} = x - x_{k-1} + a_\Omega$ and $\hat{y} = y - x_{k-1} + a_\Omega$.

Proof. First, if K is adjacent to the left endpoint then there is nothing to prove due to (7.21). The element K cannot be adjacent to the right endpoint because of the assumptions $t \le 1/2$ and $H_{\rm rel} \le 1/3$. Thus, it remains to consider the interior elements $K \in \mathcal{T}_{hp}$.

The bubble functions $\widehat{\varphi}_2^{b,\widehat{K}}, \widehat{\varphi}_3^{b,\widehat{K}}, \ldots, \widehat{\varphi}_p^{b,\widehat{K}}$ in $\widehat{K}$ are just shifted bubble functions $\varphi_2^{b,K}, \varphi_3^{b,K}, \ldots \varphi_p^{b,K}$ from K, cf. (7.16). Therefore,

$$\widehat{G}^b_{hp}(\hat{x},\hat{y}) = \widetilde{G}^b_{hp}(x,y) \quad \text{for all } (x,y) \in K^2,$$

where $\hat{x} = x - x_{k-1} + a_\Omega$ and $\hat{y} = y - x_{k-1} + a_\Omega$.

By (7.16)–(7.19), Lemma 7.2, the facts that $\widetilde{A}^{-1} \ge 0$, cf. (7.1), $\widetilde{\psi}_{k-1} \ge 0$ and $\widetilde{\psi}_k \ge 0$ for all $x \in K$, and by (7.3) we obtain

$$\begin{aligned}\widehat{G}^v_{hp}(\hat{x},\hat{y}) = hs(0,\theta,\kappa^2h^2)\widehat{\psi}(\hat{x})\widehat{\psi}(\hat{y}) &\le hs(t,\theta,\kappa^2h^2)\widetilde{\psi}_k(\hat{x})\widetilde{\psi}_k(\hat{y}) \\ &\le \sum_{i=1}^{2}\sum_{j=1}^{2} (\widetilde{A})^{-1}_{ij}\widetilde{\psi}_{k-2+i}(x)\widetilde{\psi}_{k-2+j}(y) = \widetilde{G}^v_{hp}(x,y)\end{aligned}$$

for all $(x,y) \in K^2$ with $\hat{x} = x - x_{k-1} + a_\Omega$ and $\hat{y} = y - x_{k-1} + a_\Omega$. We remark that Lemma 7.2 can be used because the inequality $H_{\rm rel} \le 1/3$ is equivalent to $\theta \le 1/2$. ∎

COROLLARY 7.4. *Let G_{hp} be given by (6.1)–(6.3). Further, let $K \in \mathcal{T}_{hp}$ be an element and let $\widehat{G}_{hp}$ be the corresponding second auxiliary DGF defined by (7.17)–(7.19). Let the relative length $H_{\rm rel}$ of K, cf. (7.7), be limited by $H_{\rm rel} \le 1/3$. If*

$$\widehat{G}_{hp}(\hat{x},\hat{y}) \ge 0 \quad \textit{for all } (\hat{x},\hat{y}) \in \hat{K}^2 \tag{7.22}$$

then

$$G_{hp}(x,y) \ge 0 \quad \textit{for all } (x,y) \in K^2,$$

i.e., the condition (c) from Theorem 6.1 is satisfied.

Proof. Let $K \in \mathcal{T}_{hp}$ be arbitrary. If the parameter t defined by (7.6) is less or equal to $1/2$ then assumption (7.22) and Lemmas 7.3 and 7.1 imply

$$0 \le \widehat{G}_{hp}(\hat{x},\hat{y}) \le \widetilde{G}_{hp}(x,y) \le G_{hp}(x,y) \quad \forall (x,y) \in K^2,$$

where $\hat{x} = x - x_{k-1} + a_\Omega$ and $\hat{y} = y - x_{k-1} + a_\Omega$. The same conclusion is valid also for $t > 1/2$ due to the symmetry. ∎

7.4. Nonnegativity of the second auxiliary DGF. The last step is to verify the nonnegativity of $\widehat{G}_{hp}(\hat{x},\hat{y})$ for $(\hat{x},\hat{y}) \in \widehat{K}^2$. Again, without the loss of generality we assume that the element K is not adjacent to the right endpoint of Ω.

It is convenient to transform $\widehat{G}_{hp}$ from $\widehat{K}^2 = [a_\Omega, \hat{z}]$ to $K^2_{\mathrm{ref}} = [-1,1]^2$ using $\hat{x} = \chi_{\widehat{K}}(\xi)$ and $\hat{y} = \chi_{\widehat{K}}(\eta)$, where the reference map $\chi_{\widehat{K}}$ is given by (5.1). As the first step, we transform $\widehat{\psi}(\hat{x})$, $\hat{x} \in \widehat{K}$, as follows, cf. (6.4),

$$\widehat{\psi}(\chi_{\widehat{K}}(\xi)) = l_1(\xi) - \sum_{m=1}^{p-1} C^K_{\iota_K(k),m} l_{m+1}(\xi) = l_1(\xi)\left[1 - l_0(\xi)\sum_{m=1}^{p-1} C^K_{\iota_K(k),m}\lambda^{\mathrm{ker}}_{m+1}(\xi)\right] = l_1(\xi)\Psi^p_1(\kappa^2h^2,\xi),$$

where we recall that $C^K = B^K(D^K)^{-1}$ is a function of κ^2h^2 only, cf. (4.5) and (5.4)–(5.5). We stress that $\Psi^p_1(\kappa^2h^2,\xi)$ is the same as in (6.4).

With the help of (7.17)–(7.19), (5.2), and (7.9), the transformed DGF $\widehat{G}_{hp}$ can be expressed as follows

$$G^{\mathrm{ref}}_{hp}(\xi,\eta) = \widehat{G}_{hp}(\chi_{\widehat{K}}(\xi),\chi_{\widehat{K}}(\eta)) = hs(0,\theta,\kappa^2h^2)\widehat{\psi}(\chi_{\widehat{K}}(\xi))\widehat{\psi}(\chi_{\widehat{K}}(\eta)) + \sum_{i=1}^{p-1}\sum_{j=1}^{p-1}(D^K)^{-1}_{ij} l_{i+1}(\xi) l_{j+1}(\eta) = hl_1(\xi)l_1(\eta)\omega^p(\theta,\kappa^2h^2,\xi,\eta),$$

where

$$\omega^p(\theta,\zeta,\xi,\eta) = s(0,\theta,\zeta)\Psi^p_1(\zeta,\xi)\Psi^p_1(\zeta,\eta) + l_0(\xi)l_0(\eta)\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}(hD^K)^{-1}_{ij}\lambda^{\mathrm{ker}}_{i-1}(\xi)\lambda^{\mathrm{ker}}_{j-1}(\eta)$$

with the short-hand notation $\zeta = \kappa^2h^2$. An important fact is that the entries of $(hD^K)^{-1}$ depend on $\zeta = \kappa^2h^2$ only, see (5.5). Finally, we define

$$\widehat{\omega}^p(\theta,\zeta) = \min_{(\xi,\eta)\in K^2_{\mathrm{ref}}} \omega^p(\theta,\zeta,\xi,\eta). \tag{7.23}$$

The following lemma presents conditions which guarantee the nonnegativity of $\widehat{G}_{hp}$ in $\widehat{K}^2$.

LEMMA 7.5. *Consider an element $K \in \mathcal{T}_{hp}$ with the length h and the polynomial degree p. Let $\hat{z} = a_\Omega + h$, and $\widehat{K} = [a_\Omega, \hat{z}]$. Further, let $\widehat{G}_{hp}$ be given by* (7.17) *and $\widehat{\omega}^p(\theta,\zeta)$ by* (7.23)*. If*

$$\widehat{\omega}^p(\theta,0) > 0 \quad \text{for all } \theta \in (0,1/2] \tag{7.24}$$

then for each $\theta \in (0,1/2]$ there exists $\sigma^p(\theta) > 0$ such that

$$\kappa^2h^2 \le \sigma^p(\theta) \quad \Rightarrow \quad \widehat{G}_{hp}(\hat{x},\hat{y}) \ge 0 \quad \forall(\hat{x},\hat{y}) \in \widehat{K}^2. \tag{7.25}$$

Moreover, if $\theta = 0$ then there exists $\sigma^p(0) \ge 0$ with property (7.25)*.*

Proof. The nonnegativity of $\widehat{G}_{hp}$ in $\widehat{K}^2$ is equivalent to the nonnegativity of G^{ref}_{hp} in K^2_{ref} which is further equivalent to the inequality $\widehat{\omega}^p(\theta,\zeta) \ge 0$ with $\zeta = \kappa^2h^2$. Let $\theta \in (0,1/2]$ be fixed. Since $\widehat{\omega}^p(\theta,\zeta)$ depends continuously on $\zeta \ge 0$, we conclude by (7.24) that $\widehat{\omega}^p(\theta,\zeta) > 0$ even for small enough $\zeta > 0$. Thus, $\sigma^p(\theta) > 0$ with property (7.25) exists.

Moreover, $\widehat{\omega}^p(0,0) \ge 0$ by (7.24) and by continuity in θ, therefore, $\sigma^p(\theta) \ge 0$. ∎

The importance of Lemma 7.5 lies in the fact that the largest possible $\sigma^p(\theta)$ with property (7.25) can be found numerically. This is done below in Section 9, cf. Figure 9.1.

We mention that the assumption (7.24) was verified in [18] for polynomial degrees $p = 1, 2, \dots, 100$. Let us comment more on this. If $\zeta = \kappa^2 h^2 = 0$ then $s(0, \theta, 0) = (1 + \theta)^{-1} = 1 - H_{\mathrm{rel}}$ by (7.10) and (7.7). Further, we have $\Psi_1^p(0, \xi) = 1$ because $C^K = B^K (D^K)^{-1} = 0$ for $\zeta = \kappa^2 h^2 = 0$, see (5.4)–(5.5). In [18] we find that

$$H_{\mathrm{rel}}^* = 1 + \min_{(\xi,\eta)\in K_{\mathrm{ref}}^2} \left[l_0(\xi) l_0(\eta) \sum_{i=1}^{p-1} \sum_{j=1}^{p-1} (hD^K)_{ij}^{-1} \lambda_{i-1}^{\mathrm{ker}}(\xi) \right] \ge \frac{9}{10}$$

for $p = 1, 2, \dots, 100$ and $\kappa = 0$. Interestingly, the smallest value $9/10$ was attained for $p = 3$. If $\theta \in [0, 1/2]$ then $H_{\mathrm{rel}} \in [0, 1/3]$ and we conclude that $\widehat{\omega}^p(\theta, 0) = H_{\mathrm{rel}}^* - H_{\mathrm{rel}} \ge 9/10 - 1/3 = 17/30 > 0$.

8. The main result. In this section we formulate simple conditions which yield the DMP. The conditions can be easily verified element-by-element. They limit the relative length H_{rel} of all elements from two sides. The crucial condition in Theorem 8.2 below is formulated in terms of the following two parameters

$$\gamma^p = 2\big(\sigma^p(1/2) - \sigma^p(0)\big) \quad \text{and} \quad \delta^p = \sigma^p(0). \tag{8.1}$$

Before we present the main theorem, we introduce one more auxiliary lemma.

LEMMA 8.1. *Consider the function $\sigma^p(\theta)$ defined in Lemma 7.5, where we assume $\zeta \le \alpha^p$ in addition. If γ^p and δ^p are given by* (8.1) *then $\gamma^p \ge 0$ and $\delta^p \ge 0$.*

Proof. The fact that $\delta^p \ge 0$ follows immediately from Lemma 7.5. Definition (7.10) implies $s(0, 0, \zeta) = 0$ and $s(0, 1/2, \zeta) \ge 0$. Further, $\Psi_1^p(\zeta, \xi) \ge 0$ for all $\xi \in K_{\mathrm{ref}} = [-1, 1]$ and for all $\zeta \le \alpha^p$, see Lemma 6.2. These facts yield $\omega^p(0, \zeta, \xi, \eta) \le \omega^p(1/2, \zeta, \xi, \eta)$. Consequently, if $\widehat{\omega}^p(0, \zeta) \ge 0$ then $\widehat{\omega}^p(1/2, \zeta) \ge 0$. Thus, $\sigma^p(0) \le \sigma^p(1/2)$ which finishes the proof. ☐

THEOREM 8.2. *Let us consider the hp-FEM problem* (2.3) *discretized on a mesh $\mathcal{T}_{hp}$. Denote by h_K and p_K the lengths and the polynomial degrees of elements $K \in \mathcal{T}_{hp}$. Further, consider the relative lengths of elements $H_{\mathrm{rel}} = H_{\mathrm{rel}}^K$ given by* (7.7) *and constants α^p, β^p, γ^p, and δ^p introduced in Lemma 6.2, Lemma 6.3, and* (8.1)*. Let us assume* (7.24) *and let the function $\sigma^p(\theta)$ defined by Lemma 7.5 be concave downward. If the following conditions*

$$H_{\mathrm{rel}}^K \le 1/3, \tag{8.2}$$

$$\kappa^2 h_K^2 \le \min\left\{ \alpha^{p_K}, \beta^{p_K}, \gamma^{p_K} \frac{H_{\mathrm{rel}}^K}{1 - H_{\mathrm{rel}}^K} + \delta^{p_K} \right\} \tag{8.3}$$

are satisfied for all elements $K \in \mathcal{T}_{hp}$ then the approximate problem (2.3) *satisfies the DMP.*

Proof. If $\sigma^{p_K}(\theta)$ is concave downward then

$$\gamma^{p_K} \theta + \delta^{p_K} \le \sigma^{p_K}(\theta) \quad \text{for all } \theta \in [0, 1/2]. \tag{8.4}$$

This and (8.3) imply

$$\kappa^2 h_K^2 \le \gamma^{p_K} \theta + \delta^{p_K} \le \sigma^{p_K}(\theta) \quad \text{for all } K \in \mathcal{T}_{hp} \tag{8.5}$$

because $\theta = \theta^K = H_{\mathrm{rel}}^K/(1 - H_{\mathrm{rel}}^K)$ by (7.7). The DMP now follows from (8.5), (8.2), Lemma 7.5, Corollary 7.4, Lemmas 6.2 and 6.3, and Theorems 6.1 and 3.4. ∎

The values of parameters α^p, β^p, γ^p, and δ^p are listed in Table 9.1 below for $p = 1, 2, \ldots, 10$. Furthermore, there are two technical assumptions in Theorem 8.2. The validity of the first technical assumption (7.24) follows from the analysis of the Poisson problem which was done in [18]. This issue was already discussed at the end of Section 7.4. The second technical assumption requires $\sigma^p(\theta)$ being concave downward. This is verified in Section 9.

The fundamental conditions for the validity of the DMP are (8.2) and (8.3). Notice that (8.2) and (8.3) limit the relative length H_{rel}^K of elements from both sides. Indeed, the inequality

$$\kappa^2 h_K^2 \le \gamma^{p_K} \frac{H_{\mathrm{rel}}^K}{1 - H_{\mathrm{rel}}^K} + \delta^{p_K}, \tag{8.6}$$

cf. (8.3), is equivalent to

$$\kappa^2 h_K^2 - \delta^{p_K} \le (\kappa^2 h_K^2 - \delta^{p_K} + \gamma^{p_K}) H_{\mathrm{rel}}^K. \tag{8.7}$$

Now, we split the analysis into two cases (i) $\kappa^2 h_K^2 \le \delta^{p_K}$ and (ii) $\kappa^2 h_K^2 > \delta^{p_K}$. If (i) $\kappa^2 h_K^2 \le \delta^{p_K}$ then, clearly, (8.6) holds because $\gamma^{p_K} \ge 0$ by Lemma 8.1. Hence, there is no lower bound for H_{rel} in this case. However, this case is rare. Condition (i) can be nontrivially satisfied for $p = 1, 2, 4$, and 6 only, cf. Table 9.1.

On the other hand, if (ii) $\kappa^2 h_K^2 > \delta^{p_K}$ then $\kappa^2 h_K^2 - \delta^{p_K} + \gamma^{p_K} > 0$ again by Lemma 8.1 and we obtain from (8.7) the following lower bound for H_{rel}^K

$$\frac{\kappa^2 h_K^2 - \delta^{p_K}}{\kappa^2 h_K^2 - \delta^{p_K} + \gamma^{p_K}} \le H_{\mathrm{rel}}^K.$$

This lower bound, however, allows for the lengths of the elements to be arbitrarily small. Indeed, if h_K is already small enough to satisfy (8.6) then an element of the length $h_K/2$ with the same polynomial degree satisfies (8.6) as well. This follows immediately from

$$\kappa^2 (h_K/2)^2 = \frac{1}{4}\kappa^2 h_K^2 \le \frac{1}{4}\gamma^{p_K} \frac{H_{\mathrm{rel}}^K}{1 - H_{\mathrm{rel}}^K} + \frac{1}{4}\delta^{p_K} \le \gamma^{p_K} \frac{H_{\mathrm{rel}}^K/2}{1 - H_{\mathrm{rel}}^K/2} + \delta^{p_K},$$

where the last inequality holds if $H_{\mathrm{rel}}^K \le 2/3$ but this is guaranteed by (8.2). Thus, we have shown that if a mesh $\mathcal{T}_{hp}$ with a distribution of polynomial degrees p_K, $K \in \mathcal{T}_{hp}$, satisfies the conditions of Theorem 8.2 then its uniform refinement satisfies the same conditions and consequently the DMP.

The final part of this section treats the special case of the linear and quadratic elements. The results are summarized in the following theorem.

Theorem 8.3. *Let the finite element mesh $\mathcal{T}_{hp}$ consist of liner and quadratic elements only. If*

$$\kappa^2 h_K^2 \le 6 \quad \textit{for all } K \in \mathcal{T}_{hp} \textit{ with the polynomial degree } p_K = 1 \tag{8.8}$$

and if

$$\kappa^2 h_K^2 \le 20/3 \quad \textit{for all } K \in \mathcal{T}_{hp} \textit{ with the polynomial degree } p_K = 2 \tag{8.9}$$

Table 9.1
The critical values α^p, β^p, γ^p, and δ^p.

p	α^p	β^p	γ^p	δ^p
1	∞	6	0	∞
2	20/3	∞	0	∞
3	38.61	25.89	5.608	0
4	18.91	∞	2.936	3.614
5	49.44	59.82	7.799	0
6	37.56	∞	7.247	0.887
7	72.82	107.81	9.791	0
8	62.62	∞	9.709	0
9	104.09	169.85	11.510	0
10	94.10	∞	10.644	0

then approximate problem (2.3) *satisfies the DMP.*

Proof. The statement follows from Theorem 6.1. Indeed, $G^b_{hp} = 0$ for $p = 1$ and $G^b_{hp} \geq 0$ for $p = 2$. Hence, the most complicated condition (c) from Theorem 6.1 is trivially satisfied. For linear and quadratic elements, the validity of conditions (a) and (b) can be easily verified. It turns out that condition (a) always holds for $p = 1$ and it is equivalent to (8.9) for $p = 2$. Condition (b) is equivalent to (8.8) for $p = 1$ and it is always satisfied for $p = 2$, cf. Table 9.1. ∎

Notice, that the requirement (8.9) for quadratic elements is weaker than the requirement (8.8) for linear elements. Hence, interestingly, the quadratic elements behave slightly better with respect to the DMP than the linear elements.

9. Computation of the critical parameters. In this section, we verify the assumptions of Theorem 8.2 numerically for $p = 1, 2, \ldots, 10$ because these polynomial degrees are most often used in practice. The computed values of $\sigma^p(\theta)$ for $p = 3, 4, \ldots, 10$ are shown in Figure 9.1. We see that $\sigma^p(\theta)$ is concave downward and that it can be estimated from below by the linear function (8.4).

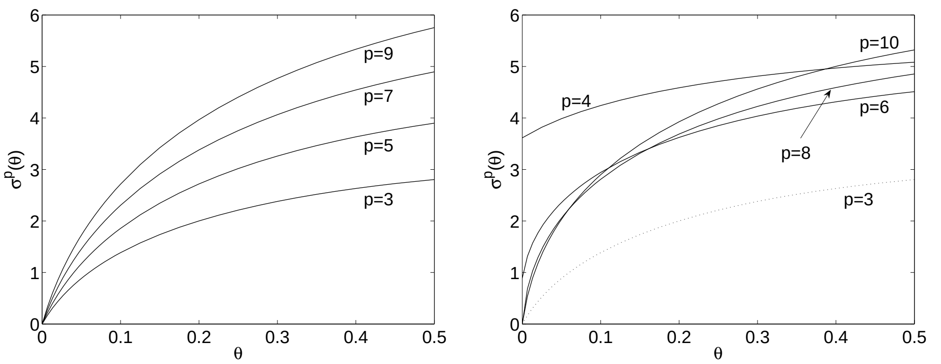

Fig. 9.1. *The graphs of $\sigma^p(\theta)$ for $p = 3, 5, 7, 9$ (left) and for $p = 4, 6, 8, 10$ (right). The dotted line in the right panel shows $\sigma^3(\theta)$ to indicate that $\sigma^3(\theta) \leq \sigma^p(\theta)$ for all $p = 3, 4, \ldots, 10$, $\theta \in [0, 1/2]$.*

In Section 6.1 we described how to compute the critical parameters α^p and β^p. The parameters γ^p and δ^p are defined by (8.1). The values of these parameters for $p = 1, 2, \ldots, 10$ are summarized in Table 9.1. Practically, it does not make much sense to consider the function $\sigma^p(\theta)$ for $p = 1$ and for $p = 2$. However, for consistency, we define $\gamma^1 = \gamma^2 = 0$ and $\delta^1 = \delta^2 = \infty$.

The results observed in Table 9.1 allow to simplify assumption (8.3) in Theorem 8.2. Requirement (8.2) is equivalent to the inequality $H_{\mathrm{rel}}^K/(1-H_{\mathrm{rel}}^K) = \theta^K \leq 1/2$, cf. 7.7. From Table 9.1 we see that

$$\gamma^{p_K}\theta^K + \delta^{p_K} \leq \gamma^{p_K}\frac{1}{2} + \delta^{p_K} \leq \min\{\alpha^{p_K}, \beta^{p_K}\} \quad \text{for } p_K = 3,4,\ldots,10.$$

Thus, the crucial assumption (8.3) in Theorem 8.2 can be simplified as follows

$$\kappa^2 h_K^2 \leq \gamma^{p_K}\frac{H_{\mathrm{rel}}^K}{1-H_{\mathrm{rel}}^K} + \delta^{p_K} \quad \text{for all } K \in \mathcal{T}_{hp},\ p_K = 3,4,\ldots,10. \tag{9.1}$$

Another observation from the numerical results is that $\sigma^3(\theta) \leq \sigma^p(\theta)$ for all $p = 3,4,\ldots,10$, see Figure 9.1. This implies that condition (8.3) in Theorem 8.2 or its simplified version (9.1) is the most strict for $p = 3$. In other words, if the DMP is valid on a mesh with cubic elements only then it is also valid on the same mesh with the arbitrary distribution of polynomial degrees (up to the degree 10). This observation (that the cubic elements are the "worst") is in agreement with the previous results for the Poisson problem, cf. [18].

The growing trend of values $\sigma^p(\theta)$ for increasing p observed in Fig. 9.1 allows us to conclude this paper the following Conjecture.

CONJECTURE 9.1. *Let $\mathcal{T}_{hp}$ be a finite element mesh in an interval $\Omega = (a_\Omega, b_\Omega)$. Let us consider an arbitrary distribution of polynomial degrees in $\mathcal{T}_{hp}$. Denote by h_K and $H_{\mathrm{rel}}^K = h_K/(b_\Omega - a_\Omega)$ the length and the relative length of the element $K \in \mathcal{T}_{hp}$, respectively. If*

$$\frac{\kappa^2 h_K^2}{\kappa^2 h_K^2 + \gamma^3} \leq H_{\mathrm{rel}}^K \leq 1/3 \quad \textit{for all } K \in \mathcal{T}_{hp},$$

where $\gamma^3 \approx 5.608797$, then the approximate problem (2.3) satisfies the DMP.

Acknowledgments. The author thanks Pavel Kůs for carefull reading of the manuscript and for valuable suggestions.

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