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ON EFFICIENCY OF THE CRITERION FOR THE EQUIVALENCE OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract.

In this paper we study the effectiveness of the criterion for equivalence in the category of linear differential equations.

I. Motivation

Certain procedures, operations, constructions in mathematics are called effective. E.g. in geometry we say that a construction is effective if it can be accomplished by using a ruler and a compass finite times. In analysis a procedure is considered as effective if it requires a finite number of elementary algebraic operations, compositions of functions and quadratures, i.e. finding primitive functions.

Here we present an examples of an effective operation having an origin in a problem of equivalence of objects in an Ehresmann groupoid. We study the effectiveness of the criterion for equivalence in the category of linear differential equations.

II. Category and equivalence

A class is called a *category* if to each pair of its elements P, Q, *objects,* a set $Hom(P,Q)$ of *morphisms* is assigned such that the following axioms are satisfied:

1. The sets $Hom(P,Q)$ are disjoint for different pairs (P,Q).

2. A composition $\alpha\beta \in Hom(P,T)$ is defined for each $\alpha \in Hom(P,Q)$ and $\beta \in Hom(Q,T)$ such that

a) the associativity $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ holds for each $\gamma \in Hom(T,U)$,

b) there exists an identity ι_P for each object P, ι_Q for Q:

$$\iota_P\alpha = \alpha, \quad \alpha\iota_Q = \alpha \text{ for each } \alpha \in Hom(P,Q).$$

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A category is an *Ehresmann* groupoid if each morphism α has an inverse $\alpha^{-1} \in Hom(Q,P)$:

$$\alpha\alpha^{-1} = \iota_P, \qquad \alpha^{-1}\alpha = \iota_Q.$$

Moreover, an Ehresmann groupoid is a *Brandt groupoid* if $Hom(P,Q)$ is not empty for any pair (P,Q) of its objects P,Q.

An Ehresmann groupoid is a collection of connected components, Brandt groupoids, also called *classes of equivalent objects.* The set $Hom(P,P)$ is a group, a *stationary group* of the object P.

Consider two categories $\mathbf{A}_1$ and $\mathbf{A}_2$ with their sets of morphisms denoted by Hom_1 and Hom_2, respectively. A mapping Φ defined on objects and morphisms of $\mathbf{A}_1$ is called a covariant functor of the category $\mathbf{A}_1$ into the category $\mathbf{A}_2$ if

for each object $P_1 \in \mathbf{A}_1$, $\Phi(P_1) \in \mathbf{A}_2$, and

for each morphism $\alpha \in Hom_1(P_1, Q_1)$, $\Phi(\alpha) \in Hom_2(\Phi(P_1), \Phi(Q_1))$, and

$$\Phi(\iota_{P_1}) = \iota_{\Phi(P_1)}, \qquad \Phi(\alpha\beta) = \Phi(\alpha)\Phi(\beta)$$

whenever $\alpha\beta$ is defined. For details see e.g. [**3**].

Historically one may observe that the following problems were studied when Ehresmann groupoids occurred:

Criterion of equivalence: sufficient and necessary conditions whether two given objects are equivalent or not, i.e. when they are in the same Brandt groupoid.

Canonical forms and their *stationary groups* in each Brandt groupoid of the Ehresmann groupoid under our consideration. This enables us to describe a structure of all morphisms of our category.

Invariants in each Brandt groupoid are of our interest in connection with a criterion of equivalence.

III. Category of linear differential equations

Transformations of linear differential equations were considered by many authors. For the second order equations it was E. E. Kummer [**7**] in 1834, for higher order equations e.g. E. Laguerre [**8**], P. Stäckel [**13**], and others. linear differential equations of the form (1). O. Borůvka [**1**] required *global* transformations for the second order equations, i.e. transformations of solutions of the corresponding equations on their whole intervals of definition. His approach was extended to linear differential equations of an arbitrary order, see [**10**]. Summarizing the ideas we give the following definition of the *global transformation* .

Consider a linear differential equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_0(x)y = 0 \qquad \text{on} \qquad I, \qquad (P_n)$$

I being an open interval of the reals, p_i are real continuous functions defined on I for $i = 0, 1, \ldots, n-1$, i.e. $p_i \in C^0(I), p_i : I \to \mathbb{R}, n \geq 2$. Denote also

$$z^{(n)} + q_{n-1}(t)z^{(n-1)} + \cdots + q_0(t)z = 0 \qquad \text{on} \qquad J, \tag{Q_n}$$

another equation of this type.

Definition. We say that equation (P_n) is globally transformable into equation (Q_n) if there exist two functions,

$$f \in C^n(J), f(t) \neq 0 \quad \text{for each} \quad t \in J, \quad \text{and}$$

$$h \in C^n(J), h'(t) \neq 0 \quad \text{for each} \quad t \in J, \quad \text{and} \quad h(J) = I,$$

such that whenever $y : I \to \mathbb{R}$ is a solution of (P_n) then

$$z : J \to \mathbb{R}, \quad z(t) := f(t) \cdot y(h(t)), \quad t \in J, \tag{f,h}$$

is a solution of (Q_n). This transformation is global in the sense that solutions are transformed on their whole intervals of definition: $h(J) = I$.

Equations (P_n) and their global transformations (f, h) can be viewed as objects and morphisms, respectively of the category of linear differential equations $\mathbb{A}$.

Now consider linear differential equations (P_n) with vanishing coefficients by the $(n-1)$st derivative

$$y^{(n)} + p_{n-2}(x)y^{(n-2)} + \cdots + p_0(x)y = 0 \quad \text{on} \qquad I. \tag{P_n^0}$$

Remark 1. The coefficient of p_{n-1} can always be eliminated if it is sufficiently smooth, i.e. of the class $C^{(n-1)}$, see e.g. [**5, 10, 14**].

If f in (f, h) is taken as $|h'|^{\frac{n-1}{2}}$ for $h'(t) \neq 0, h \in C^{n+1}(J)$, then the function z defined by the transformation $(f, h) =: \tau_n(h)$

$$z : J \to \mathbb{R}, \quad z(t) := |h'(t)|^{\frac{n-1}{2}} \cdot y(h(t)), \quad t \in J, \tag{$\tau_n(h)$}$$

converts solutions y of equation (P_n^0) into solutions z of a differential equation of the same form

$$z^{(n)} + q_{n-2}(t)z^{(n-2)} + \cdots + q_0(t)z = 0 \qquad \text{on} \qquad J. \tag{Q_n^0}$$

Notation 1. We will write $\tau_n(h)(P_n^0) = (Q_n^0)$. In this way we identify equations (P_n^0) and (Q_n^0) with their solution spaces represented also by their n-tuples of linearly independent solutions.

Remark 2. Equations of the form (P_n^0) as objects and transformations $\tau_n(h)$ as morphisms form a subcategory $\mathbb{A}^0$ of $\mathbb{A}$.

IV. Criterion of global equivalence

Solutions of the second order linear differential equation in the Jacobi form

$$y'' + p_0(x)y = 0, \quad x \in I, \qquad (P_2^0)$$

$(P_2^0) \in \mathbb{A}_2^0$, have either only a finite or only an infinite number of zeros on the interval I. According to O. Borůvka [**1**], if the equation (P_2^0) is nonoscillatory, it is called of *finite type* m, m being the maximal number of zeros of all its nontrivial solutions. It means that the equation (P_2^0) has solutions with m zeros but no of its solutions has $m+1$ zeros on I. An equation (P_2^0) of the finite type m is either of a *general character* if it has two linearly independent solutions with $m-1$ zeros on I, otherwise it is of a *special character.* If the equation (P_2^0) is oscillatory, then it is of the *type* ∞ and its character is either *one-side* (right- or left-side) oscillatory or *both-side* oscillatory.

Borůvka's criterion for $\mathbb{A}_2^0$. *Equations* (P_2^0) *and* (Q_2^0), always considered with their intervals of definition, *are globally equivalent if and only if they are of the same type and at the same time of the same character.*

Remark 3. It can be shown that the same criterion holds also for the second order linear differential equations in general form (P_2) from $\mathbb{A}_2$, not only for the Jacobi type equations (P_2^0) from $\mathbb{A}_2^0$, see [**10**].

Remark 4. Borůvka's criterion is **not effective** , because we do not have a formula giving the number of zeros of solutions of equations (P_2^0) for general p_0. However for the third and higher order equations we have in general an effective criterion. Before we formulate it, we need to introduce a special differential equations, called *iterative.*

Iterative differential equations.

Consider again the second order linear differential equations in the Jacobi form

$$y'' + p_0(x)y = 0 \quad \text{on} \quad I, \qquad (P_2^0)$$

$p_0 \in C^{n-2}(I)$, and global transformations

$$z(t) = |h'(t)|^{-1/2}.y(h(t)), \qquad (\tau_2(h))$$

where $h(J) = I, h \in C^{n+1}(J), h'(t) \neq 0$ on J.

For z we get again the second order equation

$$z'' + q_0(t)z = 0, \quad \text{on} \quad J\ , \quad q_0 \in C^{n-2}(J). \qquad (Q_2^0)$$

Open problem. The above conditions on smoothness of functions are required because of the following constructions. It would be an interesting question whether under weaker conditions. we could obtain similar results.

Consider a pointwise mapping Φ of solutions of the second order linear differential equations into solutions of n-th order linear differential equations satisfying the following requirements.

For two linearly independent solutions $y_1(x), y_2(x)$ of equation (P_2^0) let Φ map $\mathbb{R}^2 \setminus \{(0,0)\}$ into $\mathbb{R}^n$, $\Phi(y_1(x), y_2(x))$ being the n-tuple of linearly independent solutions of an equation (P_n^0). Let this equation (P_n^0) depend only on the equation (P_2^0) and not on a particular choice of its solutions y_1, y_2. We will also write shortly $\Phi(P_2^0) = (P_n^0)$. In fact, mappings under considerations convert complete solution set of one equation into again complete solution space of another equation. This justifies our notation that uses the same symbol for mapping solutions and equations.

Moreover, we will require φ to be a covariant functor of $\mathbb{A}_2^0$ into $\mathbb{A}_n^0$:

$$\Phi\tau_2 = \tau_n\Phi, \tag{2}$$

i.e.

$$\begin{array}{ccc} P_2^0 & \xrightarrow{\Phi} & P_n^0 \\ \tau_2(h)\downarrow & & \tau_n(h)\downarrow \\ Q_2^0 & \xrightarrow{\Phi} & Q_n^0 \end{array}$$

In addition, let Φ keep smoothness, i.e., let it map linear second order differential equations with real analytic coefficients into n-th order equations again with real analytic coefficients.

Proposition. *Under the above conditions on Φ, Φ is uniquely determined. For y_1, y_2 two linearly independent solutions of $(P_2^0) \in \mathbb{A}_2^0$*

$$\Phi(P_2^0)) = (P_n^0), \quad \Phi(y_1, y_2) = (y_1^{n-1}, y_1^{n-2}y_2, \cdots, y_2^{n-1}),$$

this being an n-tuple of linearly independent solutions of equation $(P_n^0) \in \mathbb{A}_n^0$.

Proof of this statement can be found in was given in [**9, 10**]. It is based on the form of the general solution of the functional equation for homogeneous functions, see e.g. [**6**]. Such a functional equation is induced by the commutativity condition (1). □

The explicit form of equation $\Phi(P_2^0)$ is

$$y^{(n)} + \binom{n+1}{3} p_0(x) y^{(n-2)} + 2\binom{n+1}{4} p_0'(x) y^{(n-3)} +$$

$$+\binom{n+1}{5}\left(3p_0'(x)+\frac{5n+7}{3}\big(p_0(x)\big)^2\right)y^{(n-4)}+\cdots=0.$$

Equations $\Phi(P_2^0)$ have already occurred in the mathematical literature under the name *iterative equations,* [**2, 4, 11, 12, 14**]. What is new it in our presentation is the categorical approach using covariant functors. And there is more important its application in deriving an

Effective criterion for the third and higher order equations from $\mathbb{A}_n^0$.

Hence, consider equations (P_n^0) and (Q_n^0) from $\mathbb{A}_n^0$. Write the differential operators on the left sides of the equations such that:

first we put the operators corresponding the n-th order iterative equations whose first two terms are identical with the first two terms of given equations (P_n^0) and (Q_n^0);

then the rests R and S are operators that start with terms for derivatives of lower orders than $(n-2)$nd:

$$y^{(n)}+p_{n-2}(x)y^{(n-2)}+p_{n-3}(x)y^{(n-3)}+\cdots+p_0(x)y=$$

$$=\text{left side of }\Phi\left(\frac{1}{\binom{n+1}{3}}p_{n-2}\right)+R,$$

and similarly

$$z^{(n)}+q_{n-2}(t)z^{(n-2)}+q_{n-3}(t)z^{(n-3)}+\cdots+q_0(t)z=$$

$$=\text{left side of }\Phi\left(\frac{1}{\binom{n+1}{3}}q_{n-2}\right)+S.$$

Let R begin with $r(x)y^{(n-k)}$ and S with $s(t)z^{(n-l)}$. If $k\neq l$, then (P_n^0) and (Q_n^0) are not globally equivalent. If $k=l$ and (P_n^0) is equivalent to (Q_n^0), then h in $\tau_n(h)$ should satisfy

$$s(t)=r(h(t)).(h'(t))^k. \tag{3}$$

On intervals where r and s do not vanish, (3) gets h by quadratures. Then we must verify whether this h also transforms the whole (P_n^0) into (Q_n^0) by means of the transformation $\tau_n(h)$. For those intervals where either r or s vanish (but not both), relation (3) shows that the equations are not equivalent. Or it does not determine h, because both r and s vanish. In this last case we need to go to a similar relation for higher k, and so forth. If such a k does not exist, then both R and S are vanishing. Hence our equations are iterative, and their equivalence or non-equivalence follows from the equivalence or non-equivalence of the corresponding second order equations, see again [**1**] or [**10**].

V. Final conclusion

Preceding considerations and constructions can be summarized in the following

Theorem.

For $n-$th order linear differential equations, $n \geq 3$, there exists an **effective** *criterion determining in general whether two given equations are globally equivalent or not. Only for one type of equations it is not possible, namely for iterative equations. For those equations the equivalence depends on equivalence of the corresponding second order equations for which Borůvka's criterion is not effective.*

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