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Yang-Mills bar connections over compact Kähler manifolds

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Abstract

In this note we introduce a Yang-Mills bar equation on complex vector bundles over compact Hermitian manifolds as the Euler-Lagrange equation for a Yang-Mills bar functional. We show the existence of a non-trivial solution to this equation over compact Kähler manifolds as well as a short time existence of the negative Yang-Mills bar gradient flow. We also show a rigidity of holomorphic connections among a class of Yang-Mills bar connections over compact Kähler manifolds of positive Ricci curvature.

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1 Introduction

Let M^{2n} be a compact Hermitian manifold of dimension $2n$ and E be a complex vector bundle over M^{2n}. The following Koszul-Malgrange criterion [6], see also ([2], 2.1.53, 2.1.54) establishes the equivalence between the existence of a holomorphic structure on E and a partial flatness of E.

Koszul-Malgrange criterion. *A complex vector bundle E over a complex manifold M^{2n} carries a holomorphic structure, if and only if there is a* *connection A on E such that the $(0,2)$-component $F_A^{0,2}$ of the curvature F_A of A vanishes.*

Thus we shall call a connection A satisfying the Koszul-Malgrange criterion a holomorphic connection. It is well-known (see e.g. [2]) that we can replace the connection A in the Kozsul-Malgrange criterion by a unitary connection A for any given choice of a compatible (Hermitian) metric h on E.

We introduce in section 2 (see (2.5.1) and (2.5.2)) a Yang-Mills bar equation as the Euler-Lagrange equation for the Yang-Mills bar functional which is the square of the L_2-norm of the $(0,2)$-component $F_A^{0,2}$ of a unitary connection A on (E,h). Solutions of a Yang-Mills bar equation are called Yang-Mills bar connections. The Yang-Mills bar equation has an advantage over the equation for a holomorphic connection, because the later one is overdetermined if the complex dimension of the bundle is greater or equal to 2, and the first one is elliptic modulo a degeneracy which is formally generated by an action of the complex gauge group of the complex vector bundle E (the degeneracy is formal generated since the action of this group on the "small" space does not preserve the Yang-Mills bar functional, see 2.7.b and Remark 5.13). Thus we hope that by using this equation we shall be able to find useful sufficient conditions under which a complex vector bundle carries a holomorphic structure. Appropriate sufficient conditions for the existence of a holomorphic structure on complex vector bundles over projective algebraic manifolds could be a key step in solving the Hodge conjecture, if the conjecture is correct. A particular result in this direction is our Theorem 4.25 which states that an almost holomorphic connection over a compact Kähler manifold of positive Ricci curvature is holomorphic, in particular any Yang-Mills bar connection on a 4-dimensional compact Kähler manifold of positive Ricci curvature is holomorphic.

In section 2 after introducing the Yang-Mills equation we also discuss the symmetry of this equation in 2.7. In section 3 we give a proof of the Hodge-Kähler identities for general unitary connections over Kähler manifolds and show the existence of non-trivial Yang-Mills bar connections. In section 4 we derive a Bochner-Weitzenböck type identity on compact Kähler manifolds and prove Theorem 4.25. In section 5 we introduce the notion of affine integrability condition, a negative Yang-Mills bar gradient flow and find an affine integrability condition for this flow (Theorem 5.9). Unlike previously known cases for weakly parabolic equations (Ricci flow, Yang-Mills flow), our affine integrability is not derived from an action of a group, which preserves the Lagrangian on the space where our flow is considered (see 2.7.b and Remark 5.13.i). The automorphism group of the Yang-Mill bar equation gives us only "half" of the integrability condition. In the last section 6 we prove the short time existence, uniqueness and smoothness of a solution of an evolution equation with affine integrability condition, slightly extending a Hamilton's result.

2 Yang-Mills bar equation

Let $(V, \langle , \rangle)$ be a Euclidean space. Denote by $V_{\mathbb{C}}$ its complexification. Then $\langle , \rangle$ extends uniquely to a complex bilinear form $\langle , \rangle_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$. Denote by $(v, w) := \langle v, \bar{w} \rangle_{\mathbb{C}}$ the associated Hermitian form on $V_{\mathbb{C}}$ and by $\langle v, w \rangle = Re(v, w)$ the Euclidean metric on the space $(V_{\mathbb{C}}) \otimes \mathbb{R}$. We note that the restriction of this metric to V coincides with the original metric $\langle , \rangle$. Conversely any Hermitian metric (J-invariant Euclidean metric) on a complex space (V, J) considered as a complexification of a real vector space V_0 is obtained in this way.

In this note we shall define by the same $(,)$ (and resp. $\langle , \rangle$) the Hermitian form (resp. the Euclidean metric) extended in the above way from any vector bundle $(E, \langle , \rangle)$ provided with a fiber-wise Euclidean metric $\langle , \rangle$ to its complexification $V_{\mathbb{C}}$ (resp. considered as a real space). If A is a connection on $(E, \langle , \rangle)$ then A can be extended to a unitary connection also denoted by A on the complexification $V_{\mathbb{C}}$ with that extended metric by setting $d_A(\sqrt{-1}\phi) := \sqrt{-1} d_A(\phi)$.

Now let A be a connection on a complex vector bundle (E, J) over a Hermitian manifold M^{2n}. Denote by $\Omega^{p,q}(E)$ the space of E-valued (p, q)-forms on M^{2n}: $\Omega^{p,q}(E) = \Omega^{p,q}(M) \otimes_{\mathbb{C}} E$. We have the decomposition

$$d_A = \partial_A \oplus \bar{\partial}_A : \Omega(E) \to \Omega^{1,0}(E) \oplus \Omega^{0,1}(E).$$

In general we have the inclusion

$$d_A(\Omega^{p,q}(E)) \subset \Omega^{p+1,q}(E) \oplus \Omega^{p,q+1}(E),$$

since for $g \in \Omega^0(E)$ and $\phi \in \Omega^{p,q}(M^{2n})$ we have

$$d_A(g \otimes \phi) = d_A(g) \otimes \phi + g \otimes d\phi \in \Omega^{p+1,q}(E) \oplus \Omega^{p,q+1}(E).$$

(The operator d_A is well defined on $\Omega^{p,q}(E)$, since $d_A(Jg) = Jd_A(g)$.) For $\phi \in \Omega^{p,q}(E)$ we shall denote by $\partial_A(\phi)$ the projection of $d_A(\phi)$ on the first factor and by $\bar{\partial}_A(\phi)$ the projection on the second factor w.r.t the above decomposition.

We note that the curvature $F_A \in \Omega^2(End_J E)$ of A can be consider as an element in $\Omega^2_{\mathbb{C}}(End_J(E))$.

Let (E, h) be a Hermitian vector bundle, i.e. a complex vector bundle (E, J) provided with a Hermitian metric h but E need not to be holomorphic. There is a natural (Killing) metric on the space u_E, defined by $\langle A, B \rangle = -Re\, Tr(AB)$. We can also write $End_J E = u_E \oplus \sqrt{-1} u_E$. Thus the metric h extends to a positive definite bilinear form on $End_J E$ (defined by $\langle A, B\rangle = Re\, Tr(AB^*))$. Here B^* is the conjugate transpose of B, the adjoint of B w.r.t the unitary metric h. We note that this metric is invariant under the original complex structure on $End_J(E)$ induced by J which we denoted above by multiplication with $\sqrt{-1}$. Hence by the remark at the beginning of the section, this metric extends to a metric on the space $\Omega^k_{\mathbb{C}}(End_J E)$ by combining with the Hermitian metric on M^{2n}. The decomposition $\Omega^k_{\mathbb{C}}(End_J E) = \sum_{p+q=k} \Omega^{p,q}(End_J E)$ is an orthogonal decomposition w.r.t this metric.

If A is a unitary connection on (E, h), then $F_A \in \Omega^2(u_E) \subset \Omega^2(End_J E)$. We also note that in the decomposition for the curvature of unitary connection A:

$$F_A = (F_A)^{2,0} + (F_A)^{1,1} + (F_A)^{0,2}$$

we have $(F_A)^{0,2} = -((F_A)^{2,0})^*$. The Kozsul-Malgrange criterion suggests us to consider the following Yang-Mills bar functional on the space of all unitary connections A on (E, h) over M^{2n}

$$\mathcal{YM}^b(A) = (1/2) \int_{M^{2n}} ||(F_A)^{0,2}||.$$

It is easy to see that the functional $\mathcal{YM}^b$ is invariant under the gauge transformation of the Hermitian vector bundle (E, h). We shall derive the first variation formula for the Yang-Mills bar equation. First we shall extend the usual Hodge operator $* : \Omega^p(M^{2n}) \to \Omega^{2n-p}(M^{2n})$ to $\bar{*} : \Omega^p(End_J E) \to \Omega^{2n-p}(End_J E)$ defined as follows . First we extend $*$ to $\Omega^p_{\mathbb{C}}(M^{2n})$ so that for all $\alpha \in \Omega^p_{\mathbb{C}}(M^{2n})$ and $\beta \in \Omega^p_{\mathbb{C}}(M^{2n})$ we have (see [5], chapter III, §2, or [3], chapter I, §2)

$$\langle \alpha(x), \beta(x)\rangle = \langle vol_x M^{2n}, \alpha(x) \wedge (\bar{*}\beta(x))\rangle. \tag{2.1}$$

Then we extend $\bar{*} : \Omega^p(End_J E) \to \Omega^{2n-p}(End_J E)$ so that for each $\alpha \in \Omega^p_{\mathbb{C}}(End_J E)$ and $\beta \in \Omega^p_{\mathbb{C}}(End_J E)$ we have

$$\langle \alpha(x), \beta(x)\rangle = \langle vol_x M^{2n}, \alpha(x) \wedge^{(,)} (\bar{*}\beta(x))\rangle. \tag{2.2}$$

Here $\wedge^{(,)}$ means that we compose the wedge product with the contraction of the coefficients in $End_J E$ of $\alpha = \sum u^i_A \otimes \theta^i_A$ and $\bar{*}\beta = \bar{*}(\sum u^j_B \otimes \theta^j_B)$ via the natural Hermitian form $(,)$ on $End_J E$. Now we define the operator $\bar{\partial}^*_A : \Omega^{p,q}(End_J E) \to \Omega^{p,q-1}(End_J E)$ as follows (see also [5], chapter III, (2.19), or [3], chapter 1, §2, for the case E is absent)

$$(\bar{\partial}^*_A)\beta^{p,q} := (-1)\bar{*}\bar{\partial}_A\bar{*}\beta^{p,q}. \tag{2.3}$$

Using the following identity for the formal adjoint d_A^* of d_A on an even dimensional manifold M^{2n} (see e.g. [B-L1982, (2.27)] for the real case, the complex case can be proved by the same way by using the Stocks formula locally):

$$(d_A^*)\beta = (-1)\bar{*}d_A\bar{*}\beta$$

and taking into account (2.3) which implies that $\bar{\partial}_A^*$ is the component with correct bi-degree of d_A^*, we conclude that $\bar{\partial}_A^*$ is the formal adjoint of $\bar{\partial}_A$. Now using the formula $(F_{A+ta})^{0,2} = (F_A)^{0,2} + t\bar{\partial}_A a^{0,1} + t^2 a^{0,1}\wedge a^{0,1}$ and taking into account (2.2) we get immediately

2.4. Lemma. *Let M^{2n} be a compact Hermitian manifold with (possibly empty) boundary. The first variation of the Yang-Mills bar functional is given by the formula*

$$\frac{d}{dt}_{|t=0}\mathcal{YM}^b(A+ta) = \int_{M^{2n}}\langle(\bar{\partial}_A)^*F_A^{0,2}, a\rangle + \int_{\partial M^{2n}}\langle vol_x, a\wedge^{(,)}\bar{*}F_A^{0,2}\rangle.$$

We shall call a smooth unitary connection A a Yang-Mills bar connection, if it satisfies the following two conditions

$$(\bar{\partial}_A)^*F_A^{0,2} = 0, \tag{2.5.1}$$

$$(\bar{*}F_A^{0,2})_{|\partial M^{2n}} = 0. \tag{2.5.2}$$

Let $\triangle_A^{\bar{\partial}} := \bar{\partial}_A(\bar{\partial}_A)^* + (\bar{\partial}_A)^*\bar{\partial}_A$. Using the Bianchi identity $\bar{\partial}_A F_A^{0,2} = 0$ (which follows from the usual Bianchi identity) and using the equality $\langle vol_x, a\wedge^{(,)}\bar{*}b\rangle = \langle vol_x, b\wedge^{(,)}\bar{*}a\rangle$ we conclude that we can replace (2.5.1) in the system of two equations (2.5.1) and (2.5.2) by the following condition

$$\triangle_A^{\bar{\partial}}(F_A)^{0,2} = 0, \tag{2.6.1}$$

to get an equivalent system of equations.

2.7. Symmetries of the Yang-Mills bar equation. a) We can vary the Yang-Mills bar functional among all compatible Hermitian metrics h' on (E, J) in order to get an invariant of the complex vector bundle E. Let A_t be a family of unitary connections w.r.t. a compatible metric h_t. We note that we can write $h_t = g_t(h)$, where g_t is a (complex) gauge transformation of (E, J). Clearly $(g_t)^{-1}A_t$ is a unitary connection w.r.t. h (i.e. $d_{(g_t)^{-1}A_t}h = 0$). Now we have $F_{A_t}^{0,2} = Ad_{g_t}F_{(g_t)^{-1}A_t}^{0,2}$. Moreover

$$||F_{A_t}^{0,2}||_{h_t} = ||Ad_{g_t}^{-1}F_{A_t}^{0,2}||_h = ||F_{(g_t)^{-1}(A_t)}^{0,2}||_h. \tag{2.7.1}$$

(We can get (2.7.1) easily by noticing that our inner products on $End_J E$ induced by h and $g(h)$ satisfy the following relation

$$\langle A, B\rangle_{g(h)} = \sum_i \langle A(g(e_i)), B(g(e_i))\rangle_{g(h)} = \sum_i \langle Ad_{g^{-1}}A(e_i), Ad_{g^{-1}}B(e_i)\rangle_h$$

where e_i is an orthonormal basis in E w.r.t. h.)

Hence the infimum of the Yang-Mills bar functional is a constant which does not depend on the unitary metric h.

b) The linearization of the Yang-Mills bar equation is not elliptic, because the equation is invariant under the gauge group $\mathcal{G}(E,h)$ of (E,h) (see (2.7.1). The complexification of this group is the gauge group $\mathcal{G}(E)$. This complexified group acts also on the space of all unitary connections w.r.t. a fixed compatible metric h [D-K1990, (6.1.4)]. For $g \in \mathcal{G}(E)$ and a unitary connection A we denote by $\hat{g}$ the new (non-canonical) action of g defined as follows

$$\bar{\partial}_{\hat{g}(A)} = g\bar{\partial}_A g^{-1} = \bar{\partial}_A - (\bar{\partial}_A g)g^{-1},$$
$$\partial_{\hat{g}(A)} = \partial_A + [(\bar{\partial}_A g)g^{-1}]^*.$$

Then $\hat{g}(A)$ is a unitary connection. Thought this action of $\hat{\mathcal{G}}(E)$ does not preserve the Yang-Mills bar functional, infinitesimally it fails to do it at a connection A only by a quadratic term in $F_A^{0,2}$ (see (5.3)).

In the rest of this note we shall consider only compact Kähler manifolds without boundary. Clearly any unitary holomorphic connection is a Yang Mills bar connection. We shall call a unitary connection A almost holomorphic, if $\partial_A F_A^{0,2} = 0$. It follows from the Kähler identity (3.1) that an almost holomorphic connection is a Yang-Mills bar connection. Since $\partial_{g(A)} F^{0,2}_{g(A)} = Ad_g \partial_A F_A^{0,2}$, the notion of almost holomorphic connections is also invariant under the (canonical) gauge transformations of the complex vector bundle E. So we shall say that a complex vector bundle is almost holomorphic if it admits an almost holomorphic connection.

We shall see in the next section that there are almost holomorphic connections on a complex vector bundle which carries no holomorphic structure.

3 Yang-Mills bar connections over compact Kähler manifolds

Suppose that A is a unitary connection on a Hermitian vector bundle E over a Kähler manifold M^{2n} with a Kähler form ω. As before denote by $\bar{\partial}_A^*$ the formal adjoint of $\bar{\partial}_A : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ defined by (2.3), and by ∂_A^* the formal adjoint of $\partial_A : \Omega^{p,q}(E) \to \Omega^{p+1,q}(E)$ defined in the same way.

Denote by $\Lambda : \Omega^{p,q}(E) \to \Omega^{p-1,q-1}(E)$ the adjoint of the wedge multiplication by ω, an algebraic operator. The following Hodge-Kähler identities

$$\bar{\partial}_A^* = \sqrt{-1}[\partial_A, \Lambda], \tag{3.1}$$

$$\partial_A^* = -\sqrt{-1}[\bar{\partial}_A, \Lambda], \tag{3.2}$$

are well-known for the case of a holomorphic bundle E and A being the metric connection compatible with the holomorphic structure ([5], chapter III, (2.39)), or ([3], chapter 0, §7, chapter 1, §2), where they are called the Hodge identities. These identities have been called Kähler identities in ([2], §6.1). For the sake of completeness we shall give a proof of the general case here.

Note that it suffices to prove these identities locally (i.e. its suffices to consider their actions on forms with small support), so we can assume that the bundle is $U(n)$-trivial and $\partial_A = \partial + A^{1,0}$, where $A^{1,0} = \sum_{i=1}^n A_i dz_i$, $A_i \in End_J(E)$. Similarly $\bar{\partial}_A = \bar{\partial} + A^{0,1}$ with $A^{1,0} = \sum_{i=1}^n -(A_i)^* d\bar{z}_i$.

Here we define $\bar{\partial}$ and ∂ to be the $(1,0)$ and $(0,1)$ components of the unique unitary connection which is compatible with the trivial holomorphic structure. Assuming the validity of the Hodge-Kähler identity for $A = 0$, we reduce the Hodge-Kähler identities (3.1) and (3.2) to following equations

$$[A^{0,1}]^* = \sqrt{-1}[A^{1,0}, \Lambda], \tag{3.3}$$

$$[A^{1,0}]^* = -\sqrt{-1}[A^{0,1}, \Lambda]. \tag{3.4}$$

In view of the Hermitian linearity of LHS of (3.3) and (3.4):

$$(\lambda A + \gamma B)^* = \bar{\lambda} A^* + \bar{\gamma} B^*$$

for $\lambda, \gamma \in \mathbb{C}$, and taking into account the unitary of A which implies $A^{1,0} = -(A^{0,1})^*$, it suffices to prove these identities for a $\mathbb{C}$-basic $\{A^{1,0} = e_{ij} dz_k, | \, 1 \le i, j \le \dim_{\mathbb{C}} E,\, 1 \le k \le \dim_{\mathbb{C}} M^{2n} = n\}$ of $(0,1)$-forms in $\Omega^{0,1}(End_J E)$. Here e_{ij} is an elementary matrix in $End_J(E)$. We also assume that the Kähler metric at a given point x is $\sum_i dz_i d\bar{z}_i$. Denote by i_k and $\bar{i}_k$ the adjoint of the multiplication operators $dz_k \wedge$ and $d\bar{z}_k \wedge$ correspondingly. Then we have

$$[A^{1,0}]^* = (e_{ji} i_k),\ [A^{0,1}]^* = -(e_{ij} \bar{i}_k)$$

$$\Lambda = -\frac{\sqrt{-1}}{2} \sum_{k=1}^n \bar{i}_k i_k.$$

Substituting these identities in LHS of (3.3) and (3.4) we conclude that (3.3) and (3.4) are equivalent to the following identities for all i, j, k

$$-(e_{ij}\bar{i}_k) = \sqrt{-1}[e_{ij}dz_k, -\frac{\sqrt{-1}}{2}\sum_{k=1}^{n}\bar{i}_k i_k], \tag{3.5}$$

$$(e_{ji}i_k) = -\sqrt{-1}[-e_{ji}d\bar{z}_k, -\frac{\sqrt{-1}}{2}\sum_{k=1}^{n}\bar{i}_k i_k], \tag{3.6}$$

In their turn (3.5) and (3.6) are immediate consequences of the following identities

$$-\bar{i}_k = \frac{1}{2}[dz_k\wedge, \sum_{k=1}^{n}\bar{i}_k i_k]. \tag{3.7}$$

$$-i_k = \frac{1}{2}[d\bar{z}_k\wedge, \sum_{k=1}^{n}\bar{i}_k i_k], \tag{3.8}$$

To prove (3.7) (and (3.8) resp.) we shall compare the action of LHS of (3.7) (and of (3.8) resp.) and the action of RHS of (3.7) (and of (3.8) resp) on $\phi = dz_J \wedge d\bar{z}_K$. We use the following formulas proved in p.112-113 of [3]

$$i_k(dz_J \wedge d\bar{z}_K) = 0, \text{ if } k \notin J, \tag{3.9}$$

$$i_k(dz_k \wedge dz_J \wedge d\bar{z}_K) = 2dz_J \wedge d\bar{z}_K, \tag{3.10}$$

$$\bar{i}_k(dz_J \wedge d\bar{z}_K) = 0, \text{ if } k \notin K, \tag{3.11}$$

$$\bar{i}_k(d\bar{z}_k \wedge dz_J \wedge d\bar{z}_K) = 2dz_J \wedge d\bar{z}_K. \tag{3.12}$$

Withe help of (3.9) -(3.12) we compute the action of RHS of (3.7) on $dz_J \wedge d\bar{z}_K$. For a multi-index $J = (j_0, \cdots, j_l, \cdots j_p)$ we shall use the following abbreviations

$$(-1)^{\#(J)} := p+1, \qquad (-1)^{(j_l)_J} dz_{J\setminus\{j_l\}} := (-1)^l dz_{j_0} \wedge \cdots_{\hat{dz}_{j_l}} \cdots \wedge dz_{j_p}.$$

Now we have

$$\frac{1}{2}[dz_k \wedge \sum_l \bar{i}_l i_l(dz_J \wedge d\bar{z}_K) - \sum_l \bar{i}_l i_l(dz_k \wedge dz_J \wedge d\bar{z}_K)] =$$

if $k \notin \bar{K}$,

$$= 2dz_k \wedge \sum_{l \in J \cap K} (-1)^{\#(J)}(-1)^{l_J} dz_{J\setminus\{l\}} \wedge (-1)^{l_K} d\bar{z}_{K\setminus\{l\}}$$

$$-2 \sum_{l \in J \cap K} (-1)^{\#(J)+1)}(-1)^{l_J+1} dz_k \wedge dz_{J\setminus\{l\}} \wedge (-1)^{l_K} d\bar{z}_{K\setminus\{l\}} = 0; \tag{3.13}$$

and if $k \in \bar{K}$,

$$= -2(-1)^{\#(J)}(-1)^{k_K} dz_J \wedge d\bar{z}_{K\setminus\{k\}}. \tag{3.14}$$

Comparing (3.13) and (3.14) with (3.11) and (3.12) we get (3.7) immediately. It is easy to see that (3.8) can be obtained from (3.7) by changing the complex orientation. □

Set $\triangle^{\partial}_A := \partial_A \partial^*_A + \partial^*_A \partial_A$.

3.15. Corollaries. *For $\phi, \psi \in \Omega^{0,p}(E)$ we have the following simple expressions*

$$\bar{\partial}^*_A \phi = -\sqrt{-1}\Lambda \partial_A(\phi), \tag{3.15.1}$$

(3.15.2)
$$\int_{M^{2n}} \langle \sqrt{-1}\Lambda F^{1,1}_A \phi, \psi\rangle = \int_{M^{2n}} -\langle \bar{\partial}^*_A \phi, \bar{\partial}^*_A \psi\rangle + \langle \partial_A \phi, \partial_A \psi\rangle - \langle \bar{\partial}_A \phi, \bar{\partial}_A \psi\rangle.$$

More generally, for all $\phi \in \Omega^{p,q}(End_J E)$ we have

$$(\triangle^{\partial}_A - \triangle^{\bar{\partial}}_A)\phi = -\sqrt{-1}[F^{1,1}_A \wedge, \Lambda]\phi \tag{3.15.3}$$

$$\triangle^{\bar{\partial}}_A \phi = \frac{1}{2}(\triangle^d_A + \sqrt{-1}[-F^{0,2}_A + F^{2,0}_A + F^{1,1}_A, \Lambda])\phi \tag{3.15.4}$$

Proof. 1) The first statement follows immediately from the Hodge-Kähler identity (3.1).

2) Substituting $F^{1,1}_A = \bar{\partial}_A \partial_A + \partial_A \bar{\partial}_A$ we get

$$\int_{M^{2n}} \langle \sqrt{-1}\Lambda F^{1,1}_A \phi, \psi\rangle = \int_{M^{2n}} \langle \sqrt{-1}\Lambda(\bar{\partial}_A \partial_A + \partial_A \bar{\partial}_A)\phi, \psi\rangle.$$

Now applying the Hodge-Kähler identities to this equation we get

$$\int_{M^{2n}}\langle\sqrt{-1}\Lambda F_A^{1,1}\phi,\psi\rangle=\int_{M^{2n}}\langle\sqrt{-1}(\bar\partial_A\Lambda\partial_A-\sqrt{-1}\partial_A^*\partial_A)\phi,\psi\rangle-\int_{M^{2n}}\langle\bar\partial_A^*\bar\partial_A\phi,\psi\rangle$$

$$(3.16)\qquad =\int_{M^{2n}}\langle\sqrt{-1}\Lambda\partial_A\phi,\bar\partial_A^*\psi\rangle+\int_{M^{2n}}\langle\partial_A\phi,\partial_A\psi\rangle-\int_{M^{2n}}\langle\bar\partial_A\phi,\bar\partial_A\psi\rangle.$$

Using (3.15.1) we get Corollary 3.15.2 immediately from (3.16).

3) Using the Hodge-Kähler identities (3.1) and (3.2) we get

$$-\sqrt{-1}\triangle_A^{\partial}=\partial_A(\Lambda\bar\partial_A-\bar\partial_A\Lambda)+(\Lambda\bar\partial_A-\bar\partial_A\Lambda)\partial_A$$

$$(3.17)\qquad =\partial_A\Lambda\bar\partial_A-\partial_A\bar\partial_A\Lambda+\Lambda\bar\partial_A\partial_A-\bar\partial_A\Lambda\partial_A$$

In the same way we have

$$\sqrt{-1}\triangle_A^{\bar\partial}=\bar\partial_A(\Lambda\partial_A-\partial_A\Lambda)+(\Lambda\partial_A-\partial_A\Lambda)\bar\partial_A$$

$$(3.18)\qquad =\bar\partial_A\Lambda\partial_A-\bar\partial_A\partial_A\Lambda+\Lambda\partial_A\bar\partial_A-\partial_A\Lambda\bar\partial_A.$$

Using the identities

$$-(\partial_A\bar\partial_A+\bar\partial_A\partial_A)=-F_A^{1,1}\wedge$$

we get from (3.17) and (3.18)

$$-\sqrt{-1}(\triangle_A^{\partial}-\triangle_A^{\bar\partial})=-[F_A^{1,1}\wedge,\Lambda].$$

which yields (3.15.3) immediately.

4) We have

$$\triangle_A^d=(\partial_A+\bar\partial_A)(\partial_A^*+\bar\partial_A^*)+(\partial_A^*+\bar\partial_A^*)(\partial_A+\bar\partial_A)$$

$$(3.19)\qquad =\triangle_A^{\partial}+\triangle_A^{\bar\partial}+(\partial_A\bar\partial_A^*+\bar\partial_A\partial_A^*+\partial_A^*\bar\partial_A+\bar\partial_A^*\partial_A)$$

Using the Hodge-Kähler identity (3.1) and replacing $\partial_A\partial_A$ by $F_A^{2,0}\wedge$ we get
(3.20)

$$(\partial_A\bar\partial_A^*+\bar\partial_A^*\partial_A)=-\sqrt{-1}\partial_A(\Lambda\partial_A-\partial_A\Lambda)-\sqrt{-1}(\Lambda\partial_A-\partial_A\Lambda)\partial_A=\sqrt{-1}[F_A^{2,0}\wedge,\Lambda].$$

Similarly
(3.21)

$$(\bar\partial_A\partial_A^*+\partial_A^*\bar\partial_A)=-\sqrt{-1}\bar\partial_A(\bar\partial_A\Lambda-\Lambda\bar\partial_A)-\sqrt{-1}(\bar\partial_A\Lambda-\Lambda\bar\partial_A)\bar\partial_A=-\sqrt{-1}[F_A^{0,2}\wedge,\Lambda].$$

Using corollary 3.15.3 we get from (3.19), (3.20), (3.21)

$$\triangle_A^d = 2\triangle_A^{\bar\partial} - \sqrt{-1}[F_A^{1,1}\wedge, \Lambda] + \sqrt{-1}[F_A^{0,2}\wedge, \Lambda] - \sqrt{-1}[F_A^{0,2}\wedge, \Lambda]$$

which yields (3.15.4) immediately.

□

3.22. Remark. Clearly (3.15.2) follows directly from (3.15.3).

Using Corollary 3.15.1 we observe that a connection A over a compact Kähler manifold is Yang-Mills bar, iff $\Lambda\partial_A F_A^{0,2} = 0$. We shall call a connection A almost holomorphic, if $\partial_A F_A^{0,2} = 0$. Using the Bianchi identity $\bar\partial_A F_A^{0,2} = 0$ we get that $\partial_A F^{0,2} = 0$, iff $d_A F_A^{0,2} = 0$. Since $F_A^{2,0} = -(F_A^{0,2})^*$ we observe that $d_A F_A^{0,2} = 0$, iff $d_A F^{2,0} = 0$. Using the Bianchi identity $d_A F_A = 0$ we observe that A is almost holomorphic, iff $dF_A^{1,1} = 0$. If $F_A^{1,1} = 0$ we shall call A almost flat holomorphic connection.

If dimension of M equals 4 it is easy to check that

$$\Lambda\partial_A F_A^{0,2} = 0 \Longleftrightarrow \partial_A F_A^{0,2} = 0.$$

Thus any Yang-Mills bar connection over M^4 is an almost holomorphic connection.

3.23. Existence of almost holomorphic connections. Let T^4 be a 2-dimensional complex torus with coordinates $z_1 = x_1 + \sqrt{-1}y_1$, $z_2 = x_2 + \sqrt{-1}y_2$. Let L be a complex line bundle whose Chern class is represented by the cohomology class c_1 of $dz_1 \wedge dz_2 + d\bar z_1 \wedge d\bar z_2$. Let A be a unitary connection of L. Then $F_A = \sqrt{-1}(dz_1 \wedge dz_2 + d\bar z_1 \wedge d\bar z_2) + \sqrt{-1}d\alpha$, where $\alpha \in \Omega^1(T^4)$. The new connection $A' = A - \alpha$ has the curvature $\sqrt{-1}(dz_1 \wedge dz_2 + d\bar z_1 \wedge d\bar z_2)$ whose component $F_{A'}^{1,1}$ vanishes. Thus A' is an almost flat holomorphic connection. We observe that by the Hodge theorem L carries no holomorphic structure.

To get an almost holomorphic connection in vector bundles of higher dimension we can take the sum of line bundles or a tensor product of a complex line bundle with a holomorphic vector bundles.

In the next section we shall show that if M^{2n} is a Kähler manifold of positive Ricci curvature, then any almost holomorphic connection is a holomorphic connection (Theorem 4.25), in particular any almost flat holomorphic connection is a flat connection.

In general the Hodge theory implies that on any Hermitian complex line bundle over a Kähler manifold there is a Yang-Mills bar connection which realizes the infimum of the Yang-Mills bar functional.

4 Yang-Mills bar equation over compact Kähler manifolds of positive Ricci curvature

Suppose that A is a unitary connection on a Hermitian vector bundle E over a Kähler manifold M^{2n}. Let D be the Levi-Civita connection on T^*M^{2n}:

$$D : \Omega^1(M^{2n}) \to \Omega^1(M^{2n}) \otimes T^*M^{2n}.$$

The connection D extends $\mathbb{C}$-linearly to a connection also denoted by $D : \Omega^1_{\mathbb{C}}(M^{2n}) \to \Omega^1_{\mathbb{C}}(M^{2n}) \otimes_{\mathbb{C}} T^*_{\mathbb{C}}M^{2n} =_{\mathbb{R}} \Omega^1_{\mathbb{C}}(M^{2n}) \otimes_{\mathbb{R}} T^*M^{2n}$. Since M^{2n} is Kähler we have $D_v(\phi \pm \sqrt{-1}J\phi) = D_v(\phi) \pm \sqrt{-1}JD_v(\phi)$ for all $v \in T^*_{\mathbb{C}}M^{2n}$ and for all $\phi \in \Omega^{0,1}(M^{2n})$. It follows that $D(\Omega^{0,1}(M^{2n})) \subset \Omega^{0,1}(M^{2n}) \otimes_{\mathbb{C}} T^*_{\mathbb{C}}M^{2n}$, and iterating we have $D(\Omega^{0,p}(M^{2n})) \subset \Omega^{0,p}(M^{2n}) \otimes_{\mathbb{C}} T^*_{\mathbb{C}}M^{2n}$ for all p. Now we denote by $\bar{D}$ the projection $\pi^{0,1} \circ D : \Omega^{0,p}(M^{2n}) \to \Omega^{0,p}(M^{2n}) \otimes_{\mathbb{C}} T^{0,1}M^{2n}$. Clearly for all $\phi \in \Omega^{0,p}(M)$ the following formula holds

$$D_{v^{0,1}}(\phi) = \bar{D}_{v^{0,1}}(\phi), \tag{4.1}$$

where $v^{0,1}$ denotes the $(0,1)$-component of v: $v^{0,1} = (1/2)(v + \sqrt{-1}Jv)$.

Combining $\bar{D}$ with $\bar{\partial}_A : \Omega(E) \to \Omega^{0,1}(E)$ we define the following partial connection

$$\bar{\nabla}_A : \Omega^{0,p}(E) \to \Gamma(E \otimes_{\mathbb{C}} \Lambda^{0,p}T^*_{\mathbb{C}}M^{2n} \otimes_{\mathbb{C}} T^{0,1}M^{2n}).$$

In view of (4.1) we have

$$\bar{\nabla}_A = \pi^{0,1} \circ \nabla A_{|\Omega^{0,p}(E)},$$

where ∇_A is the tensor product of d_A and D which preserves the natural induced metric on the bundle $E \otimes_{\mathbb{C}} \Lambda^p T^*_{\mathbb{C}}M^{2n}$:

$$\nabla_A : \Omega^p_{\mathbb{C}}(E) \to \Gamma(E \otimes_{\mathbb{C}} \Lambda^p T^*_{\mathbb{C}}M^{2n} \otimes_{\mathbb{C}} T^*_{\mathbb{C}}M^{2n}).$$

In view of (4.1) we also have $\nabla_A(\Omega^{0,p}(E)) \subset \Omega^{0,p}(E) \otimes_{\mathbb{C}} T^*_{\mathbb{C}}M^{2n}$.

We shall use the following notation. For any element $\phi \in \Omega^p(E)$ the expression $\phi_{v_1,\cdots,v_p}$ denotes the value of ϕ at $(v_1, \cdots, v_p) \in \Lambda^p(T_*M^{2n})$.

Now we define a basic zero order operator $\mathcal{R}^A : \Omega^1_{\mathbb{C}}(End_J E) \to \Omega^1_{\mathbb{C}}(End_J E)$ by setting

$$\mathcal{R}^A(\phi)_X = \sum_{j=1}^{2n} [(F_A)_{e_j,X}, \phi_{e_j}] \in End_J E, \tag{4.2.0}$$

where $(e_1, \cdots, e_{n+k} = Je_k, \cdots, e_{2n})$ is a unitary basis of the tangent space T_xM^{2n} at the point x in question. We also consider (as before) F_A as an element in $\Omega^2_{\mathbb{C}}(End_J(E))$.

Recall that the Ricci transformation $Ric : (T_xM^{2n})_{\mathbb{C}} \to (T_xM^{2n})_{\mathbb{C}}$ is defined by

$$Ric(X) = \sum_{j=1}^{2n} R_{X,e_j} e_j$$

where R denotes the curvature tensor of the Levi-Civita connection on the tangent space TM^{2n} whose action extends $\mathbb{C}$-linearly on $T_{\mathbb{C}}M^{2n}$. We modify this transformation as follows

$$Ric^-(X) = \sum_{j=1}^{2n} R_{X,e_j} e_j^{0,1}.$$

Since $J \circ R = R \circ J$ we have $Ric^-(X) = \pi^{0,1} \circ Ric(X)$. Here $\pi^{0,1}$ denotes the projection on the $(0,1)$-component.

Now we define a $\mathbb{C}$-linear transformation $Ric : \Omega^{0,1}(End_J E) \to \Omega^{0,1}(End_J E)$ as follows. For any $(0,1)$-vector X let

$$(\phi \circ Ric)_X := \phi_{Ric(X)}, \qquad (\phi \circ Ric^-)_X := \phi_{Ric^-(X)}.$$

If $\phi \in \Omega^{0,1}(E)$, then we have $\phi \circ Ric^- = \phi \circ Ric$.

4.2. Lemma. *Suppose that (E, h) is a Hermitian vector bundle provided with a unitary connection A. We have the following simple formulas for any $\phi \in \Omega^{0,p}(E)$ and $(0,1)$-vectors X_i*

$$(\bar\partial_A \phi)_{X_0,\cdots,X_p} = \sum_{k=0}^{p} (-1)^k ((\nabla_A)_{X_k} \phi)_{X_0,\cdots,\hat{X}_k,\cdots X_p}, \tag{4.3}$$

$$(\bar\partial_A^* \phi)_{X_1,\cdots,X_{p-1}} = -\sum_{j=1}^{2n} ((\nabla_A)_{e_j^{1,0}} \phi)_{e_j^{0,1}, X_1,\cdots,X_{p-1}}, \tag{4.4}$$

where $(e_1, \cdots, e_{n+k} = Je_k, \cdots, e_{2n})$ is an unitary frame at a given point, and $e_j^{1,0} = \frac{1}{2}(e_j - \sqrt{-1} Je_j)$ is the $(1,0)$-component of e_j.

Proof. First we extend a well-known formula in real case (see e.g. [B-L1981, (2.12), (2.13)]) to complex forms $\phi \in \Omega^k_{\mathbb{C}}(E)$ and $X_i \in T^*_{\mathbb{C}}M^{2n}$.

$$(d_A \phi)_{X_0,\cdots,X_p} = \sum_{k=0}^{p} (-1)^k ((\nabla_A)_{X^k} \phi))_{X_0,\cdots,\hat{X}_k,\cdots X_p}, \tag{4.5}$$

Formula (4.5) holds, since it holds for all real forms $\phi \in \Omega^k(E) \subset \Omega^k_{\mathbb{C}}(E)$ and for all $X_i \in T_* M^{2n}$, and because both LHS and RHS of (4.5) are $\mathbb{C}$-linear w.r.t. to variables ϕ and X_k.

Next we observe that, by definition the LHS of (4.3) equals the LHS of (4.5) and clearly the RHS of (4.3) equals the RHS of (4.5). Hence we get (4.3).

In the same way, for $\phi \in \Omega^{0,p}(E)$ and for a $(1,0)$-vector X_0 and $(0,1)$-vectors X_i, $1 \le i \le p$, using (4.5), we have

$$(\partial_A \phi)_{X_0, X_1 \cdots, X_p} = \sum_{k=0}^{p} (-1)^k ((\nabla_A)_{X_k} \phi)_{X_0, \cdots, \hat{X}_k, \cdots X_p}, \tag{4.6}$$

since LHS of (4.6) coincides with the value $(d_A\phi)_{X_0, X_1 \cdots, X_p}$. Since $\phi \in \Omega^{0,p}(E)$ we get

$$\sum_{k=0}^{p} (-1)^k ((\nabla_A)_{X_k} \phi)_{X_0, \cdots, \hat{X}_k, \cdots, X_p} = ((\nabla_A)_{X_0} \phi)(X_1, \cdots, X_p). \tag{4.7}$$

Thus we get

$$(\partial_A \phi) = \sum_{i=1}^{n} dz_i \wedge (\nabla_A)_{e_i^{1,0}} \phi. \tag{4.8}$$

Now using the Kähler identity $\bar\partial^*_A = -\sqrt{-1}\Lambda \partial_A$ we get from (4.8)

$$(\bar\partial^*_A \phi)_{X_1, \cdots, X_{p-1}} = \frac{-1}{2} \sum_{k=1}^{n} \sum_{j=1}^{n} [\bar{i}_k i_k dz_j \wedge ((\nabla_A)_{e_j^{1,0}} \phi)]_{X_1, \cdots, X_{p-1}}$$

$$= - \sum_{j=1}^{n} [\bar{i}_j ((\nabla_A)_{e_j^{1,0}} \phi)]_{X_1, \cdots, X_p}. \tag{4.9}$$

Clearly the last term of (4.9) equals the RHS of (4.4). $\square$

The following Proposition is a complex analogue of Theorem 3.2 in [1].

4.10. Proposition. *We have for* $\phi \in \Omega^{0,1}(End_J E)$

$$\triangle^{\bar\partial}_A \phi = \bar\nabla^*_A \bar\nabla_A(\phi) + \phi \circ Ric + \mathcal{R}^A(\phi). \tag{4.10.1}$$

Proof. Let $X \in T^{0,1}_x(M^{2n})$. We extend X locally on M^{2n} so that $DX(x) = 0$. We also extend the unitary frame $\{e_1, \cdots, e_{n+k} := Je_k, \cdots, e_{2n}\}$ locally so that $De_i(x) = 0$. Now using (4.3) and (4.4), and since $(Je_j)^{0,1} = -\sqrt{-1}e_j^{0,1}$ and $(Je_j)^{1,0} = \sqrt{-1}e_j^{1,0}$, we get at the point x

$$(\bar{\partial}_A \bar{\partial}_A^* \phi)_X = (\nabla_A)_X\{\bar{\partial}_A^* \phi\} = -(\nabla_A)_X\{\sum_{j=1}^{2n} [(\nabla_A)_{e_j}\phi]_{e_j^{0,1}}\}$$

$$= -\sum_{j=1}^{2n} [(\nabla_A)_X(\nabla_A)_{e_j}\phi]_{e_j^{0,1}}. \tag{4.11}$$

$$(\bar{\partial}_A^* \bar{\partial}_A \phi)_X = -\sum_{j=1}^{2n} \{(\nabla_A)_{e_j}(\bar{\partial}_A\phi)\}_{e_j^{0,1},X}$$

$$= -\sum_{j=1}^{2n} (\nabla_A)_{e_j}\{[(\nabla_A)_{e_j^{0,1}}\phi]_X - [(\nabla_A)_X\phi]_{e_j^{0,1}}\}$$

$$= -\sum_{j=1}^{2n} \{[(\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi]_X - [(\nabla_A)_{e_j}(\nabla_A)_X\phi]_{e_j^{0,1}}\}. \tag{4.12}$$

Summing (4.11) and (4.12) we get

$$(\triangle_A^{\bar{\partial}}\phi)_X = -\sum_{j=1}^{2n} \{[(\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi]_X + \sum_{j=1}^{2n} (R^A_{e_j,X}\phi)_{e_j^{0,1}}\}. \tag{4.13}$$

Here we denote by R^A the curvature of the tensor product connection on the bundle $T^*_{\mathbb{C}}M^{2n} \otimes_{\mathbb{C}} End_J E = (T^*M^{2n} \otimes_{\mathbb{R}} End_J E)_{\mathbb{C}}$. This curvature coincides with the one on $T^*M^{2n} \otimes_{\mathbb{R}} End_J E$, if we consider $\Omega^2(T^*M \otimes_{\mathbb{R}} End_J E)$ as a subspace in $\Omega^2_{\mathbb{C}}(T^*_{\mathbb{C}}M^{2n} \otimes_{\mathbb{C}} End_J E)$. Now we observe that for $\psi \in \Omega^{0,1}(End_J E)$ we get

$$\langle -\sum_{j=1}^{2n} [(\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi], \psi\rangle =$$

$$-\sum_{j=1}^{2n} [(\nabla_A)_{e_j}\langle(\nabla_A)_{e_j^{0,1}}\phi, \psi\rangle - \langle(\nabla_A)_{e_j^{0,1}}\phi, (\nabla_A)_{e_j}\psi\rangle]. \tag{4.14}$$

Denote by θ the real valued 1-form on M

$$\theta(X) := \langle(\nabla_A)_{X^{0,1}}\phi, \psi\rangle.$$

Then

$$-\sum_{j=1}^{2n}(\nabla_A)_{e_j}\langle(\nabla_A)_{e_j^{0,1}}\phi,\psi\rangle(x)=(-d^*\theta)(x), \tag{4.15}$$

and

$$\langle(\nabla_A)_{e_j^{0,1}}\phi,(\nabla_A)_{e_j}\psi\rangle=\langle\bar{\nabla}_A\phi,\bar{\nabla}_A\psi\rangle+\sum_{i=1}^{2n}\langle(\nabla_A)_{e_j^{0,1}}\phi,(\nabla_A)_{e_j^{1,0}}\psi\rangle.$$

Since $(Je_j)^{0,1}=-\sqrt{-1}e_j^{0,1}$ and $(Je_j)^{1,0}=\sqrt{-1}e_j^{1,0}$, we get

$$\langle(\nabla_A)_{e_i^{0,1}}\phi,(\nabla_A)_{e_j^{1,0}}\psi\rangle+\langle(\nabla_A)_{(Je_i)^{0,1}}\phi,(\nabla_A)_{(Je_j)^{1,0}}\psi\rangle=0$$

$$\Longrightarrow\sum_{i=1}^{2n}\langle(\nabla_A)_{e_j^{0,1}}\phi,(\nabla_A)_{e_j^{1,0}}\psi\rangle=0$$

$$\Longrightarrow\langle(\nabla_A)_{e_j^{0,1}}\phi,(\nabla_A)_{e_j}\psi\rangle=\langle\bar{\nabla}_A\phi,\bar{\nabla}_A\psi\rangle. \tag{4.16}$$

From (4.14), (4.15), (4.16) we get

$$\int_{M^{2n}}-\langle(\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi,\psi\rangle=\int_{M^{2n}}-d^*\theta+\int_{M^{2n}}\langle\bar{\nabla}_A\phi,\bar{\nabla}_A\psi\rangle. \tag{4.17}$$

Next we have

$$(R^A_{e_j,X}\phi)_{e_j^{0,1}}=(F_A)_{e_j,X}\phi_{e_j^{0,1}}-\phi(R_{e_j,X}e_j^{0,1}). \tag{4.18}$$

Clearly Proposition 4.10 follows from (4.13), (4.17) and (4.18).

□

Denote by $\mathcal{R}^A$ the following linear operator : $\Omega^{0,2}(End_JE)\to\Omega^{0,2}(End_JE)$ such that for all $(0,1)$-vectors X,Y we have

$$(\mathcal{R}^A(\phi))_{X,Y}=\sum_{j=1}^{2n}\{[(F_A)_{e_j,X},\phi_{e_j,Y}]-[(F_A)_{e_j,Y},\phi_{e_j,X}]\}. \tag{4.19.0}$$

We also associate to each $\phi\in\Omega^{0,2}(End_JE)$ a new $(0,2)$-form $\phi\circ(Ric\wedge I)\in\Omega^{0,2}(End_JE)$

$$(\phi\circ(Ric\wedge I))_{X,Y}:=\phi(Ric(X),Y)-\phi(Ric(Y),X),$$

4.19. Proposition. We have for $\phi \in \Omega^{0,2}(End_J E)$

$$\triangle_A^{\bar\partial}\phi = \bar\nabla_A^*\bar\nabla_A\phi + \phi\circ(Ric\wedge I) + \mathcal{R}^A(\phi). \tag{4.19.1}$$

Proof. (Cf. Theorem 3.10 in [1].) We use the notations $X, Y, e_1, \cdots, e_n$ as in the proof of Proposition 4.10. Then at the point x and for $(0,1)$-vectors X and Y we have

$$(\bar\partial_A\bar\partial_A^*\phi)_{X,Y} = ((\bar\partial_A)_X\bar\partial_A^*\phi))_Y - ((\bar\partial_A)_Y\bar\partial_A^*\phi)_X$$

$$= -(\nabla_A)_X\{\sum_{j=1}^{2n}((\nabla_A)_{e_j}\phi)_{e_j^{0,1},Y}\} + (\nabla_A)_Y\{\sum_{j=1}^{2n}((\nabla_A)_{e_j}\phi)_{e_j^{0,1},X}\}$$

$$= -\sum_{j=1}^{2n}\{[(\nabla_A)_X(\nabla_A)_{e_j}\phi]_{e_j^{0,1},Y} - [(\nabla_A)_Y(\nabla_A)_{e_j}\phi]_{e_j^{0,1},X}\}. \tag{4.20}$$

We also have

$$(\bar\partial_A^*\bar\partial_A\phi)_{X,Y} = -\sum_{j=1}^{2n}((\nabla_A)_{e_j}\bar\partial_A\phi)_{e_j^{0,1},X,Y}$$

$$= -\sum_{j=1}^{2n}(\nabla_A)_{e_j}\{((\nabla_A)_{e_j^{0,1}}\phi)_{X,Y} + ((\nabla_A)_Y\phi)_{e_j^{0,1},X} + ((\nabla_A)_X\phi)_{Y,e_j^{0,1}}\}$$

(4.21)

$$= -\sum_{j=1}^{2n}\{[(\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi]_{X,Y} + [(\nabla_A)_{e_j}(\nabla_A)_Y\phi]_{e_j^{0,1},X} - [(\nabla_A)_{e_j}(\nabla_A)_X\phi]_{e_j^{0,1},Y}\}.$$

Summing (4.20) and (4.21) we get

(4.22)

$$(\triangle_A^{\bar\partial}\phi)_{X,Y} = -\sum_{j=1}^{2n}[(\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi]_{X,Y} + \sum_{j=1}^{2n}\{[R^A_{e_j,X}\phi]_{e_j^{0,1},Y} - [R^A_{e_j,Y}\phi]_{e_j^{0,1},X}]\}.$$

As in the proof of Proposition 4.10 (see (4.14)) we have for $\psi \in \Omega^{0,2}(End_J E)$

$$\int_{M^{2n}}\langle -\sum_{j=1}^{2n}(\nabla_A)_{e_j}(\nabla_A)_{e_j^{0,1}}\phi, \psi\rangle = \int_{M^{2n}}\langle\bar\nabla_A\phi, \bar\nabla_A\psi\rangle. \tag{4.23}$$

We use the following identity

$$(R^A_{X,Y}\phi)_{Z,W} = [(F_A)_{X,Y}, \phi_{Z,W}] - \phi(R_{X,Y}Z, W) - \phi(Z, R_{X,Y}W)$$

and combining with (4.23) to rewrite (4.22) as follows

$$(\triangle_A^{\bar\partial}\phi)_{X,Y} = (\bar\nabla_A^*\bar\nabla_A\phi)_{X,Y} + \mathcal{R}^A(\phi)_{X,Y} + \phi(Ric^-(X),Y)$$

$$-\sum_{j=1}^{2n}\phi_{e_j^{0,1},R_{e_j,X}Y} - \phi(Ric^-(Y),X) + \sum_{j=1}^{2n}\phi_{e_j^{0,1},R_{e_j,Y}X}. \tag{4.24}$$

Using the Bianchi identity

$$-R_{e_j,X}Y - R_{Y,e_j}X = R_{X,Y}e_j$$

and taking into account that the following quantity vanishes for all $\phi \in \Omega^{0,2}(End_J E)$ and for all $X, Y \in T^{0,1}M^{2n}$

$$(\phi\circ R)_{X,Y} := \sum_{j=1}^{2n}\phi(e_j, R_{X,Y}e_j),$$

because $(Je_j)^{0,1} = -\sqrt{-1}e_j^{0,1}$ and $(Je_j)^{1,0} = \sqrt{-1}e_j^{0,1}$, we get Proposition 4.19 immediately from (4.24). $\square$

4.25. Theorem. *Let M be a compact Kähler manifold with positive Ricci curvature. If A is an almost holomorphic connection, then A is holomorphic.*

Proof. First let us prove the following formula for $\phi \in \Omega^{0,2}(End_J E)$.

$$\mathcal{R}^A(\phi) = -\sqrt{-1}\{\Lambda F_A^{1,1}\wedge\phi - (\Lambda F_A^{1,1})\phi\} \overset{(4.25.0)}{=} \bar R(A)\phi. \tag{4.26}$$

We also take convention on $e_i, d\bar z_i$ as before for computing the value $\mathcal{R}^A(\phi)(x)$. Let us rewrite the expression in (4.19.0) as follows

$$\mathcal{R}^A(\phi) = \sum_{1\le k<l\le n}\sum_{j=1}^{2n}\{[(F_A)_{e_j,e_k^{0,1}}, \phi_{e_j,e_l^{0,1}}] - [(F_A)_{e_j,e_l^{0,1}}, \phi_{e_j,e_k^{0,1}}]\}d\bar z_k d\bar z_l. \tag{4.27}$$

We shall use the following abbreviation. For any $\phi \in \Omega^{k,p}(End_J E)$ denote by

$$\phi_{i_1\cdots i_k,\bar j_1\cdots\bar j_p} := \phi(e_{i_1}^{1,0}, \cdots, e_{i_k}^{1,0}, e_{j_1}^{0,1}, \cdots, e_{j_p}^{0,1}).$$

Since $\phi \in \Omega^{0,2}(End_J E)$ we get from (4.27)

(4.28)

$$\mathcal{R}^A(\phi) = \sum_{1\le k<l\le n}\sum_{j=1}^{2n}\{[(F_A)_{j\bar k}, \phi_{\bar j\bar l}] - [(F_A)_{j\bar l}, \phi_{\bar j\bar k}] + [(F_A)_{\bar j\bar k}, \phi_{\bar j\bar l}] - [(F_A)_{\bar j\bar l}, \phi_{\bar j\bar k}]\}d\bar z_k d\bar z_l.$$

Since $(J(e_j))^{0,1} = -\sqrt{-1}e_j^{0,1}$ and $(J(e_j))^{1,0} = \sqrt{-1}e_j^{1,0}$ we get from (4.28)

$$\mathcal{R}^A(\phi) = 2 \sum_{1\le k<l\le n} \sum_{j=1}^{n} \{[(F_A)_{j\bar{k}}, \phi_{\bar{j}\bar{l}}] - [(F_A)_{j\bar{l}}, \phi_{\bar{j}\bar{k}}]\} \, d\bar{z}_k d\bar{z}_l. \tag{4.29}$$

Now expanding the expression in local coordinates

$$\sqrt{-1}\Lambda F^{1,1} \wedge \phi = \frac{1}{2}\sum_{p=1}^{n} \bar{i}_p i_p \{\sum_{i,j}\sum_{k<l} [(F_A)_{i\bar{j}}, \phi_{\bar{k}\bar{l}}] \, dz_i d\bar{z}_j d\bar{z}_k d\bar{z}_l\}$$

$$= \sqrt{-1}(\Lambda F_A^{1,1})\phi - 2\sum_{1\le i\le n}\sum_{1\le j,l\le n} [(F_A)_{i\bar{j}}, \phi_{\bar{i}\bar{l}}] d\bar{z}_j d\bar{z}_l.$$

and comparing it with the RHS of (4.29) we get (4.26) immediately.

Now let A be a Yang-Mills bar connection. Applying (4.19.1) to $F_A^{0,2}$ and using (4.26) we get

(4.30)

$$0 = \int_{M^{2n}} \langle \bar{\nabla}_A F_A^{0,2}, \bar{\nabla}_A F_A^{0,2}\rangle + \langle F_A^{0,2} \circ (Ric \wedge I), F_A^{0,2}\rangle + \int_M \langle \bar{R}(A) F_A^{0,2}, F_A^{0,2}\rangle.$$

Since A is a Yang-Mills bar connection, differentiating (2.7.1) we get

$$\langle (\Lambda F^{1,1}) F_A^{0,2}, F_A^{0,2}\rangle = 0. \tag{4.31}$$

Now let A be an almost holomorphic connection. Using (4.30), (4.31), (3.15.2) we get immediately that $F_A^{0,2} = 0$. □

4.32. Remark Theorem 4.25 implies that any Yang-Mills bar connection on a compact 4-dimensional Kähler manifold of positive Ricci curvature is holomorphic. It is easy to extend this theorem for a larger class of Yang-Mills bar connections, but we shall consider this extension only in a relation with a topology of the underlying complex vector bundle in a subsequent note.

5 Short time existence of a Yang-Mills bar gradient flow over a compact Kähler manifold

5.1. Affine integrability condition. The following identity holds for any $\theta \in \Omega(End_J E)$ and unitary connection A

$$\int_{M^{2n}} \langle [\theta, F_A^{0,2}], F_A^{0,2}\rangle = -\int_{M^{2n}} \langle [F_A^{0,2}, \theta], F_A^{0,2}\rangle. \tag{5.2}$$

We shall prove that at any point $x \in M^{2n}$ we have

$$\langle [\theta, F_A^{0,2}], F_A^{0,2}\rangle = -2\langle \theta, \Lambda\Lambda F_A^{0,2} \wedge F_A^{2,0}\rangle. \tag{5.3}$$

We write $\theta = \theta^+ + \sqrt{-1}\theta^-$ where $\theta^+, \theta^- \in u_E$. In the same way at a fixed point $x \in M^{2n}$ we can take coordinates such that the Kähler metric g has the form $g(x) = \sum dz_i \otimes d\bar{z}_i$. We shall write

$$F_A^{0,2} = \sum_{1\le i<j\le n} (F_{ij}^+ + \sqrt{-1}F_{ij}^-)d\bar{z}_i d\bar{z}_j,$$

where $F_{ij}^\pm \in u_E$. Then $F_A^{2,0} = \sum_{ij}(F_{ij}^+ - \sqrt{-1}F_{ij}^-)dz_i dz_j$. Recall that $||d\bar{z}_i d\bar{z}_j||^2 = 4$. A direct computation at a point x shows

$$\langle [\theta, F_A^{0,2}], F_A^{0,2}\rangle = \sum_{1\le i<j\le n} \langle [\theta^-, F_{ij}^+]d\bar{z}_i d\bar{z}_j, F_{ij}^- d\bar{z}_i d\bar{z}_j\rangle +$$

$$\sum_{1\le i<j\le n} \langle -[\theta^-, F_{ij}^-]d\bar{z}_i d\bar{z}_j, F_{ij}^+ d\bar{z}_i d\bar{z}_j\rangle = 8\langle \theta^-, \sum_{1\le i<j\le n} [F_{ij}^+, F_{ij}^-]\rangle. \tag{5.4}$$

Now we compute

$$\langle \theta, \Lambda\Lambda F_A^{0,2} \wedge F_A^{2,0}\rangle = -2 \sum_{1\le i<j\le n} \langle \theta^-, \Lambda\Lambda [F_{ij}^+, F_{ij}^-] dz_i dz_j d\bar{z}_i d\bar{z}_j\rangle$$

(5.5)
$$= -4\sqrt{-1} \sum_{1\le i<j\le n} \langle \theta^-, \Lambda [F_{ij}^+, F_{ij}^-](dz_j d\bar{z}_j + dz_i d\bar{z}_i)\rangle = -16\langle \theta^-, \sum_{1<i<j\le n} [F_{ij}^+, F_{ij}^-]\rangle.$$

Clearly (5.3) follows from (5.4) and (5.5).

Now substituting $[F_A^{0,2}, \theta] = \bar\partial_A \bar\partial_A \theta$ in the RHS of (5.2) and taking into account (5.3) we get

$$-\int_{M^{2n}} \langle \theta, 2\Lambda\Lambda F_A^{0,2} \wedge F_A^{2,0}\rangle = \int_{M^{2n}} \langle \theta, \bar\partial_A^* \bar\partial_A^* F_A^{0,2}\rangle. \tag{5.6}$$

Thus we get the following identity

$$\bar\partial_A^* \bar\partial_A^* F_A^{0,2} + 2\Lambda\Lambda F_A^{0,2} \wedge F_A^{2,0} = 0. \tag{5.7}$$

Define the following operator $P_A : \Omega^{0,1}(End_J E) \times \Omega^{0,1}(End_J E) \to \Omega(End_J E)$

$$P_A(a)\phi := \bar\partial_{A+a}^* \phi + 2\Lambda\Lambda F_{A+a}^{0,2} \wedge F_{A+a}^{2,0}. \tag{5.8}$$

Clearly $P_A(a)\phi$ is a differential operator of order 1 in a and order 1 in ϕ. Moreover $P_A(a)\phi$ is an affine differential operator w.r.t. ϕ, i.e. $P_A(a)\phi = L_A(a)\phi + C_A(a)$, where $L_A(a)\phi$ is a linear differential operator w.r.t. ϕ. By (5.6) we have $P_A(a)\bar{\partial}^*_{A+a}F^{0,2}_{A+a} = 0$. Thus we shall call $P_A(a)$ an affine integrability condition for the differential operator $\bar{\partial}^*_{A+a}F^{0,2}_{A+a} : \Omega^{0,1}(End_J E) \to \Omega^{0,1}(End_J E)$.

5.9. Proposition. *Let $\xi \in T^*_x M^{2n} \setminus \{0\}$. All the eigenvalues of the eigenspace of the symbol $\sigma_\xi D(-1)\partial^*_{A+a}F^{0,2}_{A+a} : \Omega^{0,1}(End_J E) \to \Omega^{0,1}(End_J E)$ in Null $\sigma_\xi P_A(a)$ are positive. Hence the evolution equation*

$$\frac{da}{dt} = -\bar{\partial}^*_{A+a}F^{0,2}_{A+a}, \tag{5.9.1}$$

has a unique smooth solution for a short time which may depend on a.

Proof. Since $F^{0,2}_{A+a+th} = F^{0,2}_A + t\bar{\partial}_{A+a} \wedge h + t^2 h \wedge h$ for $h \in \Omega^{0,1}(End_J E)$, we have the following expression for the linearization of $\bar{\partial}^*_{A+a}F^{0,2}_{A+a}$ at point $a \in \Omega^{0,1}(End_J E)$

$$D_a(\bar{\partial}^*_{A+a}F^{0,2}_{A+a})(h) = \bar{\partial}^*_{A+a}\bar{\partial}_{A+a}h + \{ \text{ terms of lower order } \}. \tag{5.10}$$

We may assume that $\xi = dx_1$. Then a direct computation using the Hodge-Kähler identity $\bar{\partial}^*_{A+a} = -\sqrt{-1}\Lambda\partial_{A+a}$ and (5.10) shows

$$-\sigma_\xi D_a(\partial^*_{A+a}F^{0,2}_{A+a})(\alpha_1 d\bar{z}_1, \cdots, \alpha_n d\bar{z}_n) = (0, \alpha_2 d\bar{z}_2, \cdots, \alpha_n d\bar{z}_n). \tag{5.11}$$

Clearly the linearization $D_\phi P_A(a)\phi$ with respect to the variable ϕ is

$$[D_\phi P_A(a)\phi]h = \frac{d}{dt}_{|t=0}\bar{\partial}^*_{A+a}(\phi + th) + 2\Lambda\Lambda F^{0,2}_{A+a} \wedge F^{2,0}_{A+a} = \bar{\partial}^*_{A+a}(h).$$

We note that this linearization does not depend on ϕ. A short computation shows

$$\sigma_\xi D_\phi P_A(a)(\alpha_1 d\bar{z}_1, \cdots, \alpha_n d\bar{z}_n) = \sqrt{-1}\alpha_1. \tag{5.12}$$

Now (5.11) and (5.12) imply the first statement of Proposition 5.9. The second statement follows from Hamilton's theory for evolution equation with integrability condition [4], Theorem 5.1, actually from its slightly extended version in Theorem 6.6 below. (We note that though in the statement of his Theorem [4], Theorem 5.1, Hamilton did not require the linearity w.r.t. h of the integrability condition $L(f)h$, (in our case $L(f)h = P_A(a)\phi), a = f, h = \phi$, but in his proof, it is important (and we shall see that it is sufficient) to have the linearization $D_h L(f)(h)$ w.r.t. h independent on h. Our operator $P_A(a)\phi$ satisfies this condition, see the next section for more details.) □

5.13. Remarks. 1. By taking derivative of (2.7.1) in the time t we also get (5.2) and hence (5.7). In the same way we can get (5.2) (and hence (5.7)) as an infinitesimal consequence of the non-canonical action of the complex gauge group on the space of unitary connections w.r.t. a fixed Hermitian metric on the bundle.

2. Using (2.3) it is easy to prove that $\partial_A^* F_A^{0,2}$ also satisfies an affine integrability condition analogous to (5.8), if the ground manifold M^{2n} is Hermitian but not necessary Kähler.

6 Evolution equations with affine integrability condition

In his work [4] Hamilton introduced the notion of an evolution equation with integrability condition. Let us rapidly recall the Hamilton concept from section 5 of that paper.

We shall consider an evolution equation

$$\frac{df}{dt} = E(t),$$

where $E(f)$ is a non-linear differential operator of degree 2 in f. We suppose f belong to an open set U in a vector bundle F over a compact manifold X, and $E(f)$ takes its values in F also. Then E is a smooth map

$$E : C^\infty(X, U) \subset C^\infty(X, F) \to C^\infty(X, F)$$

of an open set in a Fréchet space to itself.

We shall consider problems where some of the eigenvalues of the symbol $\sigma DE(f)\xi$ are zero. This happens when $E(f)$ satisfies an integrability condition.

6.1. Definition. [4] Let $g = L(f)h : C^\infty(X, U) \times C^\infty(F) \to C^\infty(G)$ be a differential operator of degree 1 on sections $f \in U \subset F$, $h \in F$, and G another vector bundle over X. We call $L(f)h$ the integrability condition for $E(f)$, if the operator $Q(f) = L(f)E(f)$ only has degree at most one in f.

Suppose that $L(f)h$ is an integrability condition for $E(f)$. Taking a variation in $\tilde{f}$ we see that

$$L(f)DE(f)\tilde{f} + DL(f)\{E(f), \tilde{f}\} = DQ(f)\tilde{f}. \tag{6.2}$$

Since $DQ(f)\tilde{f}$ as well as $L(f)DE(f)\tilde{f}$ only have degree 1 in f the operator $L(f)DE(f)\tilde{f}$ also have degree 1. hence $\sigma L(f)(\xi)\sigma DE(f)(\xi) = 0$. Therefore we get

$$Im\,\sigma DE(f)(\xi) \subset Null\,\sigma L(f)(\xi). \tag{6.3}$$

6.4. Theorem ([4], Theorem 5.1). *Let $df/dt = E(f)$ be an evolution equation with integrability condition $L(f)$. Suppose that all the eigenvalues of the eigenspaces of $\sigma DE(f)(\xi)$ in $Null\,\sigma L(f)(\xi)$ is positive. Then the initial value problem $f = f_0$ at $t = 0$ has a unique smooth solution for a short time $0 \le t \le \varepsilon$ where ε may depend on f_0.*

6.5. Remark. Hamilton's notation in (6.2) indicates that $L(f)h$ is a linear w.r.t. h. (In fact, in section 4 of that paper Hamilton stressed that $L(f)h$ is linear w.r.t. h.) A closer look at Hamilton's proof (see also our proof of Theorem 6.6 below) shows that, the linearity of $L(f)h$ w.r.t. h is important. We shall call such integrability condition $L(f)h$ linear in the argument (and f shall be considered as parameter). Now we shall call an integrability condition $L(f)h$ an affine integrability condition, if $L(f)h = L_0(f)h + A(f)$, where $L_0(f)h$ is linear w.r.t. h. The linearization $(D_\phi L(f)h)\tilde{h} = L_0(f)\tilde{h}$ does not depend on h.

6.6. Theorem. *Let $df/dt = E(f)$ be an evolution equation with integrability condition $L(f)$ which is affine in the argument: $L(f)h = L_0(f)h + A(f)$. Suppose that all the eigenvalues of the eigenspaces of $\sigma DE(f)(\xi)$ in $Null\,\sigma L_0(f)(\xi)$ is positive. Then the initial value problem $f = f_0$ at $t = 0$ has a unique smooth solution for a short time $0 \le t \le \varepsilon$ where ε may depend on f_0.*

Proof of Theorem 6.6. We follow Hamilton's argument, replacing $L(f)h$ in his proof by $L_0(f)h$ in some places, and re-arranging parameters which do not depend on h. To keep our notations as close as possible with those of Hamilton, we denote by DL the derivative of $L(f)h$ w.r.t the parameter f. We divide the proof in 3 steps.

STEP 1. *Reduction of Theorem 6.6 to a version of the Nash-Moser inverse function theorem.*

In this step we reduce Theorem 6.6 to the following

6.7. Lemma. *Suppose that $\bar{f}$ is a solution of the perturbed evolution equation by a term $\bar{h}(t, x)$*

$$\frac{d\bar{f}(t,x)}{dt} = E(\bar{f}(t,x)) + \bar{h}(t,x),$$

$$\bar{f}(0,x) = \bar{f}_0(x)$$

over the interval $0 \le t \le 1$. *Then for any* f_0 *near* $\bar{f}_0$ *and* h *near* $\bar{h}$ *there exists a unique solution of the perturbed equation*

$$\frac{df(t,x)}{dt} = E(f(t,x)) + h(t,x),$$

$$f(0,x) = f_0(x)$$

over the interval $0 \le t \le 1$.

Now we explain how to get Theorem 6.6 from Lemma 6.7. Let $\bar{f}(t,x)$ be any function satisfying

$$\frac{d\bar{f}(t,x)}{dt}_{|t=0} = E(f(0,x)),$$

$$\bar{f}(0,x) = f_0(x).$$

Set

$$\bar{h}(t,x) := \frac{d\bar{f}(t,x)}{dt} - E(\bar{f}(t,x)).$$

Then $\bar{h}(0,x) = 0$.

Since X is compact, for any $\delta \rangle 0$ there exist a number $\varepsilon > 0$ and a function $h(t,x)$ such that $H(t,x)$ is δ-close to $\bar{h}(t,x)$ and moreover $h(t,x) = 0$ for a short time $0 \le t \le \varepsilon$. Applying Lemma 6.7 to the pair $(\bar{h}, h)$ we conclude that the equation

$$\frac{df(t,x)}{dt} = E(f(t,x)) + h(t,x),$$

$$f(0,x) = f_0(x)$$

has solution up to time ε. This solution in the interval $(0,\varepsilon)$ is a solution of our original equation in that time interval. This completes the first step.

STEP 2. *Reduction of Lemma 6.7 to a case of a weakly parabolic linear system of (6.14.1) and (6.14.2).* We can apply the Nash-Moser inverse function theorem to the operator

$$\mathcal{E} : C^\infty(X \times [0,1], F) \to C^\infty(X \times [0,1], F) \times C^\infty(X, F),$$

$$\mathcal{E}(f) = (df/dt - E(f),\ f|\{t=0\}).$$

Its derivative is the operator

$$D\mathcal{E}(f)\tilde{f} = (\frac{d\tilde{f}}{dt} - DE(f)\tilde{f}, \tilde{f}|\{t=0\}).$$

We must show that the linearized equation

$$d\tilde{f}/dt - DE(f)\tilde{f} = \tilde{h} \tag{6.8}$$

has a unique solution for the initial value problem $\tilde{f} = \tilde{f}_0$ at $t = 0$, and verify that the solution $\tilde{f}$ is a smooth tame function of $\tilde{h}$ and $\tilde{f}_0$.

We make the substitution $\tilde{g} = L(f)\tilde{f}$. Then $\tilde{g}$ satisfies the evolution equation

$$\frac{d\tilde{g}}{dt} = L_0(f)\frac{d\tilde{f}}{dt} + DL(f)\{\tilde{f}, \frac{df}{dt}\}. \tag{6.9}$$

Now differentiating the integrability condition $L(f)E(f) = Q(f)$ we get

$$L_0(f)DE(f)\tilde{f} = -DL(f)\{E(f), \tilde{f}\} + DQ(f)\tilde{f}. \tag{6.10}$$

Substituting $d\tilde{f}/dt = DE(f)\tilde{f} + \tilde{h}$ from (6.8) into (6.9) and taking into account (6.10) we rewrite (6.9) as follows

$$\frac{d\tilde{g}}{dt} - M(f)\tilde{f} = \tilde{k}, \tag{6.11}$$

where $\tilde{k} = L_0(f)\tilde{h}$ and

$$M(f)\tilde{f} = DL(f)\{\tilde{f}, \frac{df}{dt}\} - DL(f)\{E(f), \tilde{f}\} + DQ(f)\tilde{f} =$$

$$\overset{(6.10)}{=} DL(f)\{\tilde{f}, \frac{df}{dt}\} + L_0(f)DE(f)\tilde{f}. \tag{6.12}$$

is a linear differential operator in $\tilde{f}$ of degree 1 whose coefficients depend smoothly on f and its derivatives.

If we choose a measure on X and inner product on the vector bundle F and G, we can form a differential operator $L_0^*(f)g = h$ of degree 1 in f and g which is the adjoint of $L_0(f)$. Let us write

$$P(f)h := DE(f)h + L_0^*(f)L(f)h.$$

We claim that the equation $d\tilde{f}/dt = P(f)\tilde{f}$ is parabolic (for a given f). To see this we must examine the symbol

$$\sigma P(f)\xi = \sigma DE(f)\sigma + \sigma L_0^*(f)(\xi) \cdot \sigma L_0(f)(\xi). \tag{6.13}$$

Suppose v is an eigenvector in F with eigenvalue λ. Then $\sigma P(f)(\xi)v = 0$. But $\sigma L_0(f)(\xi) \cdot \sigma DE(f)(\xi) = 0$, so applying $\sigma L_0(f)$ to the LHS and RHS of (6.13) we get

$$\sigma L_0(f)(\xi)\cdot\sigma L_0^*(f)\xi\cdot\sigma L_0(f)(\xi)v=\lambda\sigma L_0(f)(\xi)v.$$

Taking inner product of the above equality with $\sigma L_0(f)(x)v$ we get

$$|\sigma L_0^*(f)(\xi)\cdot\sigma L_0(f)(\xi)v|^2=\lambda|\sigma L_0(f)(\xi)v|^2.$$

Now if $\sigma L_0^*(f)\cdot\sigma L_0(f)(\xi)v=0$ then $\sigma L_0(f)(\xi)v=0$, and otherwise λ is real and strictly positive. When $\sigma L_0(f)(\xi)v=0$, then $\sigma DE(f)(\xi)v=\lambda v$ by (6.13) and λ has strictly positive real part by our hypothesis in Theorem 6.6. Thus $P(f)$ is parabolic.

We proceed to solve the system of equations

$$\frac{d\tilde f}{dt}-P(f)\tilde f+L_0^*(f)\tilde g=\tilde h, \tag{6.14.1}$$

$$\frac{d\tilde g}{dt}-M(f)\tilde f=\tilde k \tag{6.14.2}$$

for the unknown function $\tilde f$ and $\tilde g$ for given $\tilde h$ and $\tilde k$ and given f, with initial data $\tilde f=\tilde f_0$ and $\tilde g=\tilde g_0=L(f_0)\tilde f_0$ at $t=0$.

In Step 3 below we prove that the solution $(\tilde f,\tilde g)$ exists and is unique, and is a smooth tame function of $(f,\tilde h,\tilde k,\tilde f_0,\tilde g_0)$. Then putting $\tilde l=\tilde g-L(f)\tilde f$ and substituting $\tilde k=L_0(f)\tilde h$ we get

$$\begin{aligned}\frac{d\tilde l}{dt}&=\frac{d\tilde g}{dt}-L_0(f)\frac{d\tilde f}{dt}\\ &=L_0(f)DE(f)\tilde f+\tilde k-L_0(f)\frac{d\tilde f}{dt}\\ &\overset{(6.14.1)}{=}-L_0(f)DE(f)\tilde f-L_0(f)P(f)\tilde f+L_0(f)L_0^*(f)\tilde g\\ &\overset{(6.13)}{=}L_0(f)[-L_0^*(f)L(f)\tilde f+L_0^*(f)L(f)(\tilde l+L(f)\tilde f)]\\ &=L_0(f)L_0^*(f)\tilde l,\end{aligned} \tag{6.15}$$

and $\tilde l=0$ at $t=0$. But then (6.15) implies the obvious integral inequality

$$\frac{d}{dt}\int_X|\tilde l|^2d\mu+2\int_X|L_0^*(f)\tilde l|d\mu=0.$$

Hence $\tilde l=0$. Then it follows that $\tilde g=L(f)\tilde f$. Using this and we get from (6.14.1)

$$\frac{d\tilde f}{dt}-DE(f)\tilde f=\tilde h.$$

This completes Step 2.

STEP 3. *The system (6.14.1) and (6.14.2) is a weakly parabolic linear system whose smooth solution uniquely exists.*

Set $P_0(f)h := DE(f)h + L_0^*(f)L_0(f)h$. Then $P_0(f)h$ is a linear differential operator in h and $P(f)h = P_0(f)h + L_0^*(f)A(f)$. Set $h = \tilde{h} - L_0^*(f)A(f)$. Since f in the system of (6.14.1) and (6.14.2) is given, we shall re-denote a given constant $\tilde{k}$ by k, variables $\tilde{f}, \tilde{g}$, by f, g and linear differential operators $P_0(f), L_0^*(f), M(f)$ by P, L, M. Then the system of (6.14.1) and (6.14.2) is equivalent to the following system of linear evolution equations on $0 \leq t \leq T$ for sections f of F and g of G

$$\frac{df}{dt} = Pf + Lg + h, \qquad \frac{dg}{dt} = Mf + k. \tag{6.16}$$

Clearly the existence, uniqueness and smoothness of a solution of (6.16) is a consequence of Hamilton's theorem [Hamilton1982, Theorem 6]. He considered the following equation

$$\frac{df}{dt} = Pf + Lg + h, \qquad \frac{dg}{dt} = Mf + Ng + k \tag{6.17}$$

where P, L, M and N are linear differential operators involving only space derivatives whose coefficients are smooth functions of both space and time. He assumed that P has degree 2, L and M have degree 1 and N has degree 0.

6.18. Theorem ([4], Theorem 6). *Suppose the equation $df/dt = Pf$ is parabolic. Then for any given (f_0, g_0, h, k) there exists a unique smooth solution (f, g) of the system (6.17) with $f = f_0$ and $g = g_0$ at $t = 0$.*

The proof of this Theorem occupies the whole section 6 in Hamilton's paper.

Finally we formulate a conjecture which might be solved by using the Yang-Mills bar equation and might be helpful for understanding the Hodge conjecture. A unitary connection A on a Hermitian bundle E over a projective algebraic manifold M is holomorphic, if the L^p_q-norm of the component $F_A^{0,2}$ less than some positive constant $\varepsilon(M)$, where p, q are some integers depending on the dimension of M.

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