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On the equivalence of ball conditions for simplicial finite elements in $\mathbf{R}^d$

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Abstract: We prove that the inscribed and circumscribed ball conditions, commonly used in finite element analysis, are equivalent in any dimension.

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A *simplex* S in $\mathbf{R}^d$, $d \in \{1, 2, 3, \ldots\}$, is the convex hull of $d+1$ vertices $A_1, A_2, \ldots, A_{d+1}$ that do not belong to the same $(d-1)$-dimensional hyperplane. We denote by h_S the length of the longest edge of S. Let F_i be the facet of S opposite to A_i for $i \in \{1, \ldots, d+1\}$. Assume that $\overline{\Omega} \subset \mathbf{R}^d$ is a closed domain (i.e. the closure of a domain). If its boundary $\partial\overline{\Omega}$ is contained in a finite number of $(d-1)$-dimensional hyperplanes, we say that $\overline{\Omega}$ is *polytopic.*

Next we define a *simplicial partition* $\mathcal{T}_h$ over a bounded closed domain $\overline{\Omega} \subset \mathbf{R}^d$ as follows. We subdivide $\overline{\Omega}$ into a finite number of simplices (called *elements*), so that their union is $\overline{\Omega}$, any two simplices have disjoint interiors and any facet of any simplex is a facet of another simplex from the partition or belongs to the boundary $\partial\overline{\Omega}$. The maximal diameter of all elements $S \in \mathcal{T}_h$ is the so-called discretization parameter h.

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The set $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ is called a *family of partitions* if for any $\varepsilon > 0$ there exists $\mathcal{T}_h \in \mathcal{F}$ with $h < \varepsilon$.

In this paper we generalize recent results for triangular and tetrahedral elements (see [4]) to simplicial elements of arbitrary dimension. We were inspired by the paper [7], where the ball conditions were actually replaced by a simpler condition on the measure of every element to guarantee convergence of the finite element method. By meas_d we denote the d-dimensional measure. In what follows, all constants C_i are independent of S and h, but can depend on the dimension d.

Condition 1: There exists $C_1 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S \in \mathcal{T}_h$ we have

$$\operatorname{meas}_d S \ge C_1 h_S^d \,. \tag{1}$$

Condition 2: There exists $C_2 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S \in \mathcal{T}_h$ we have

$$\operatorname{meas}_d b \ge C_2 h_S^d \,, \tag{2}$$

where $b \subset S$ is the inscribed ball of S.

Condition 3: There exists $C_3 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S \in \mathcal{T}_h$ we have

$$\operatorname{meas}_d S \ge C_3 \operatorname{meas}_d B, \tag{3}$$

where $B \supset S$ is the circumscribed ball about S.

Throughout the paper, we denote by r and R the radii of the inscribed and circumscribed ball of S, respectively.

Lemma: For any simplex S and any $i \in \{1, \dots, d+1\}$ we have

$$\operatorname{meas}_d S \le h_S^d, \qquad \operatorname{meas}_{d-1} F_i \le h_S^{d-1}. \tag{4}$$

P r o o f: Relations (4) follow from the fact that the distance between any two points of S is not larger than h_S. Thus, S and any of its facets F_i are contained in a hypercube of the corresponding dimension d or $d-1$ with edges of length h_S. □

Theorem: Conditions 1, 2, and 3 are equivalent.

P r o o f: **(1) $\Longrightarrow$ (2):** Let o be the center of the inscribed ball b of S. We decompose S into $d+1$ subsimplices $S_i = \operatorname{conv}\{o, F_i\}$, $i \in \{1, \dots, d+1\}$. All of them have the same altitude r with respect to the facet F_i. Using the formula

$$\operatorname{meas}_d S_i = \frac{1}{d}\, r \operatorname{meas}_{d-1} F_i$$

for the volume of each subsimplex S_i, we find that

$$r \sum_{i=1}^{d+1} \operatorname{meas}_{d-1} F_i = d \operatorname{meas}_d S.$$

Hence, by (4) and (1) we obtain

$$r(d+1) h_S^{d-1} \ge d \operatorname{meas}_d S \ge C_1 d\, h_S^d,$$

which implies that

$$r \ge \frac{C_1 d}{d+1} h_S.$$

From this and the formula for the volume of a d-dimensional ball, we finally get

$$\operatorname{meas}_d b = C_4 r^d \geq C_2 h_S^d,$$

where

$$C_4 = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}. \tag{5}$$

(2) $\Longrightarrow$ (3): By [2], [6], or [8, p. 125], the volume of S can be computed in terms of lengths of its edges using the so-called Cayley-Menger determinant of size $(d+2)\times(d+2)$

$$D_d = (-1)^{d+1} 2^d (d!)^2 (\operatorname{meas}_d S)^2 = \det \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & a_{12}^2 & \cdots & a_{1d}^2 & a_{1,d+1}^2 \\ 1 & a_{21}^2 & 0 & \cdots & a_{2d}^2 & a_{2,d+1}^2 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 1 & a_{d+1,1}^2 & a_{d+1,2}^2 & \cdots & a_{d+1,d}^2 & 0 \end{bmatrix}, \tag{6}$$

where a_{ij} is the length of the edge A_iA_j for $i \neq j$.

The radius of the circumscribed ball satisfies (see [1])

$$R^2 = -\frac{1}{2}\frac{\Delta_d}{D_d}, \tag{7}$$

where

$$\Delta_d = \det \begin{bmatrix} 0 & a_{12}^2 & \cdots & a_{1d}^2 & a_{1,d+1}^2 \\ a_{21}^2 & 0 & \cdots & a_{2d}^2 & a_{2,d+1}^2 \\ \vdots & \vdots & & \ddots & \vdots \\ a_{d+1,1}^2 & a_{d+1,2}^2 & \cdots & a_{d+1,d}^2 & 0 \end{bmatrix}.$$

From this, (7), (6), and (2) we find that

$$R^2 = \frac{1}{2}\left|\frac{\Delta_d}{D_d}\right| = \frac{|\Delta_d|}{2^{d+1}(d!)^2(\operatorname{meas}_d S)^2} < \frac{|\Delta_d|}{2^{d+1}(d!)^2(\operatorname{meas}_d b)^2} \leq \frac{C_5 h_S^{2d+2}}{2^{d+1}(d!)^2 C_2^2 h_S^{2d}}.$$

Thus, there exists $C_6 > 0$ such that for any S from any $\mathcal{T}_h \in \mathcal{F}$ we have

$$R \leq C_6 h_S\,. \tag{8}$$

Using (2) once again, (8), and (5), we immediately see that

$$\operatorname{meas}_d S > \operatorname{meas}_d b \geq C_2 h_S^d \geq C_2 \frac{R^d}{C_6^d} = \frac{C_2}{C_6^d\, C_4} \operatorname{meas}_d B. \tag{9}$$

(3) $\Longrightarrow$ (1): Since $B \supset S$, we obtain $2R \geq h_S$. Hence, in view of (3) and (5) we observe that

$$\operatorname{meas}_d S \geq C_3 \operatorname{meas}_d B = C_3 C_4 R^d \geq \frac{C_3 C_4}{2^d} h_S^d, \tag{10}$$

which implies Condition 1. $\square$

Definition: A family of simplicial partitions is called *regular* if Condition 1 or 2 or 3 holds.

Remark 1: From (4) and (3) it follows that

$$\operatorname{meas}_d B \leq {C_3}^{-1} h_S^d.$$

This condition is equivalent to (3) if (1) holds.

Remark 2: Formula (1) seems to be simpler than the ball conditions (2) and (3) from [5, 3] and therefore, it may be preferred in theoretical finite element analysis and also in computer implementations.

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