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Anisotropic L^2–estimates of weak solutions to the stationary Oseen-type equations in 3D-exterior domain for a rotating body

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Abstract

We study the Oseen problem with rotational effect in exterior three-dimensional domains. Using a variational approach we prove existence and uniqueness theorems in anisotropically weighted Sobolev spaces in the whole three-dimensional space. As the main tool we derive and apply an inequality of the Friedrichs-Poincaré type and the theory of Calderon-Zygmund kernels in weighted spaces. For the extension of results to the case of exterior domains we use a localization procedure.

1 Introduction

1.1 Formulation of the problem

In a three-dimensional exterior domain $\Omega \subset \mathbb{R}^3$, the classical Oseen problem [28] describes the velocity vector $\mathbf{v}$ and the associated pressure π by a linearized version of the incompressible Navier-Stokes equations as a perturbation of $\mathbf{v}_\infty$ the velocity at infinity; $\mathbf{v}_\infty$ is generally assumed to be constant in a fixed direction, say the first axis, $\mathbf{v}_\infty = |\mathbf{v}_\infty|\,\mathbf{e}_1$. In the next we denote $|\mathbf{v}_\infty|$ by k, and we will write the Oseen operator $k\,\partial_1\mathbf{v}$. On the other hand it is known that for various flows past a rotating obstacle, the Oseen operator appears with some concrete non-constant coefficient functions, e.g. $\mathbf{a}(\mathbf{x}) = \omega \times \mathbf{x}$, where ω is a given vector, see [16, 27]; in view of industrial applications $\mathbf{a}(\mathbf{x})$ can also play the role of an "experimental" known velocity field, see [18].

This paper is devoted to the study of the following problem in Ω for general non-solenoidal vector function $\mathbf{u} = \mathbf{u}(\mathbf{x})$ and scalar function $p = p(\mathbf{x})$:

$$-\nu\,\Delta\mathbf{u} + k\partial_1\mathbf{u} - (\omega \times \mathbf{x}) \cdot \nabla\mathbf{u} + \omega \times \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \tag{1.1}$$

$$\operatorname{div}\mathbf{u} = g \quad \text{in } \Omega \tag{1.2}$$

$$\mathbf{u} \to \mathbf{0} \quad \text{as} \quad |\mathbf{x}| \to \infty \tag{1.3}$$

$$\mathbf{u} = (\omega \times \mathbf{x}) - k\mathbf{e}_1 \quad \text{on } \partial\Omega \tag{1.4}$$

where $\omega = (\widetilde{\omega},\, 0,\, 0)$ is a constant vector, ν, k and $\widetilde{\omega}$ are some positive constants, and $\mathbf{f} = \mathbf{f}(\mathbf{x})$ a given vector function, $g = g(\mathbf{x})$ a given scalar function.

We restrict ourselves to the assumption of compact support of g. The system arises from the Navier-Stokes system modelling viscous fluid around a rotating body which is moving with a given non-zero velocity in the direction of its axis of rotation. An appropriate coordinate transform and a linearization yield in the stationary case equations (1.1) and (1.2), for details see [3, 16]. The third term together with the forth one (the Coriolis force $\omega \times \mathbf{u}$) in (1.1) arise from the influence of rotation of the body.

Let us begin with some comments and relevant process of analysis of the problem (1.1)–(1.4).

- The governing equations of fluid motion are stationary and linear, but in unbounded domains the convective operators, $k\,\partial_1$ and $(\omega \times \mathbf{x}) \cdot \nabla$, *cannot be treated as perturbations* of lower order of the Laplacian.
- The fundamental tensor (similarly as the fundamental tensor to the Oseen problem) exhibits *the anisotropic behavior* in the three-dimensional space.To reflect the decay properties near the infinity we introduce the following weight functions:

$$\eta_\beta^\alpha(\mathbf{x}) = \eta_{\beta,\varepsilon}^{\alpha,\delta}(\mathbf{x}) = (1+\delta r)^\alpha (1+\varepsilon s)^\beta ,$$

  with $r = r(\mathbf{x}) = |\mathbf{x}| = (\sum_{i=1}^3 x_i^2)^{1/2}$, $s = s(\mathbf{x}) = r - x_1$, $\mathbf{x} \in \mathbb{R}^3$, $\varepsilon, \delta > 0$, $\alpha, \beta \in \mathbb{R}$. Discussing the range of the exponents α and β the corresponding weighted spaces $L^q\left(\mathbb{R}^3; \eta_\beta^\alpha\right)$ give the appropriate framework to test the solutions to (1.1)-(1.3). This paper is concerned with $q = 2$. Let us mention also that η_β^α belongs to the Muckenhoupt class A_2 of weights in $\mathbb{R}^3$ if $-1 < \beta < 1$ and $-3 < \alpha + \beta < 3$.
- In this paper we will prefer *the variational approach.* To avoid the difficulties with the pressure part of the solution p we solve firstly the problem in $\mathbb{R}^3$. Using the theory of Calderon-Zygmund integrals in corresponding weighted spaces we determine the pressure p of the problem in $\mathbb{R}^3$ to be from the same space as the right-hand side of (1.1). This first step cannot be done directly in an exterior domain. Then we apply the variational approach for the velocity part of the solution.
- For the extension of the results to the case of exterior domains we use *the localization procedure*, see [20].

1.2 Short bibliographical remarks

The weighted estimates of the solution to the stationary classical Oseen problem were firstly obtained by Finn in 1959, see [9]. *The variational approach* to the model equation $-\nu\,\Delta u + k\partial_1 u = f$ in an exterior domain in anisotropically weighted L^2-spaces was applied by Farwig, see [1]. The same variational viewpoint has been also applied in [25, 26] by Kračmar and Penel to solve the generic scalar model equation $-\nu\,\Delta u + k\partial_1 u - \mathbf{a} \cdot \nabla u = f$ with a given non-constant and, in general, non-solenoidal vector function **a** in an exterior domain. Both model equations are assumed with boundary conditions $u = 0$ on $\partial\Omega$ and $u \to 0$ as $|\mathbf{x}| \to \infty$.

Another common approach to study the asymptotic properties of the solutions to the Dirichlet problem of the classical steady Oseen flow is the use of *the potential theory*, i.e. convolutions with Oseen fundamental tensor and its first and second gradients for the velocity (or with the fundamental solution of Laplace equation for the pressure): the L^2-estimates in anisotropically weighted Sobolev spaces in $\mathbb{R}^3$ were derived by Farwig [2], the L^q-estimates in these spaces were proved in $\mathbb{R}^3$ and $\mathbb{R}^n$ by Kračmar, Novotný and Pokorný in [23] and [24], respectively. Different approach was used by Kobayashi and Shibata [19].

The fundamental solution to rotating Oseen problem in the time dependent case is known due to Guenther and Thomann, see [30], but, unfortunately, the respective stationary kernel is not seem to be of Calderon-Zygmund type. *The Littlewood-Paley decomposition technique* offers another approach for an L^q-analysis: Thus, L^q-estimates in non-weighted spaces were derived for the rotating Stokes problem by Farwig, Hishida, and Müller [5], and for the rotating Oseen problem in $\mathbb{R}^3$ by Farwig [3, 4]. L^q-setting with non-integrable right-hand side in non-homogeneous case was investigated by Kračmar, Nečasová and Penel in [22]. The Littlewood-Paley decomposition technique for L^q-weighted estimates with anisotropic weight functions was used by Farwig, Krbec and Nečasová [7, 8].

Another approach based on the use of the *non-stationary equations* in both the linear and also non-linear cases is proposed by Galdi and Silvestre in [12, 11, 13].

We would like also to mention that the problem was solved by the *semigroup theory* in L^2-setting in particular by Hishida [17], and then the respective results were extended to L^q case by Geissert, Heck and Hieber [14].

1.3 Basic notations and elementary properties

Let us outline our notations. Let $\mathcal{S}'$ be the space of the moderate distributions in $\mathbb{R}^3$. Let Ω be an exterior domain with a boundary of the class $\mathcal{C}^2$, and

$$\widehat{W}^{m,q}(\Omega) = \left\{u \in L^1_{loc}(\Omega) : \ D^l u \in L^q(\Omega), \quad |l| = m\right\}$$

with the seminorm $|u|_{m,q} = \left(\sum_{|l|=m} \int_\Omega |u|^q\right)^{1/q}$. It is known that $\widehat{W}^{m,q}(\Omega)$ is a Banach space (and if $q = 2$ the space $\widehat{H}^m(\Omega) = \widehat{W}^{m,2}(\Omega)$ a Hilbert space), provided we identify two functions u_1, u_2 whenever $|u_1 - u_2|_{m,q} = 0$, i.e. u_1, u_2 differ (at most) on the polynomial of the degree $m-1$. As usual, we denote by $\widehat{W}^{m,q}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in $\widehat{W}^{m,q}(\Omega)$.

Let $(L^2(\Omega; w))^3$ be the set of measurable vector functions $\mathbf{f} = (f_1, f_2, f_3)$ in Ω such that

$$\|\mathbf{f}\|^2_{2,\Omega;\,w} = \int_\Omega |\mathbf{f}|^2\, w\, d\mathbf{x} < \infty.$$

We will use the notation $\mathbf{L}^2_{\alpha,\beta}(\Omega)$ instead of $\left(L^2\left(\Omega;\, \eta^\alpha_\beta\right)\right)^3$ and $\|\cdot\|_{2,\alpha,\beta}$ instead of $\|\cdot\|_{\left(L^2\left(\Omega;\,\eta^\alpha_\beta\right)\right)^3}$. Let us define the weighted Sobolev space $\mathbf{H}^1\left(\Omega; \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)$ as the set of functions $\mathbf{u} \in \mathbf{L}^2_{\alpha_0,\beta_0}(\Omega)$ with the weak derivatives $\partial_i \mathbf{u} \in \mathbf{L}^2_{\alpha_1,\beta_1}(\Omega)$. The norm of $\mathbf{u} \in \mathbf{H}^1\left(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)$ is given by

$$\|\mathbf{u}\|_{\mathbf{H}^1\left(\Omega;\,\eta^{\alpha_0}_{\beta_0},\eta^{\alpha_1}_{\beta_1}\right)} = \left(\int_\Omega |\mathbf{u}|^2\, \eta^{\alpha_0}_{\beta_0}\, d\mathbf{x} + \int_\Omega |\nabla \mathbf{u}|^2\, \eta^{\alpha_1}_{\beta_1}\, d\mathbf{x}\right)^{1/2}.$$

As usual, $\overset{\circ}{\mathbf{H}}{}^1\left(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)$ will be the closure of $\mathbf{C}^\infty_0(\Omega)$ in $\mathbf{H}^1\left(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)$, where $\mathbf{C}^\infty_0(\Omega)$ is $\left(C^\infty_0(\Omega)\right)^3$, and $\overset{\circ}{\mathbf{H}}{}^1\left(\overline{\Omega};\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)$ will be the closure of $\mathbf{C}^\infty_0\left(\overline{\Omega}\right)$ in $\mathbf{H}^1\left(\Omega;\, \eta^{\alpha_0}_{\beta_0}, \eta^{\alpha_1}_{\beta_1}\right)$.

For simplicity, we shall use the following abbreviations:

$\mathbf{L}^2_{\alpha,\beta}(\Omega)$	instead of	$\left(L^2\left(\Omega;\, \eta^\alpha_\beta\right)\right)^3$
$\|\cdot\|_{2,\alpha,\beta;\,\Omega}$	instead of	$\|\cdot\|_{\left(L^2\left(\Omega;\,\eta^\alpha_\beta\right)\right)^3}$
$\overset{\circ}{\mathbf{H}}{}^1_{\alpha,\,\beta}(\Omega)$	instead of	$\overset{\circ}{\mathbf{H}}{}^1\left(\Omega;\, \eta^{\alpha-1}_{\beta-1}, \eta^\alpha_\beta\right)$
$\mathbf{V}_{\alpha,\beta}(\Omega)$	instead of	$\overset{\circ}{\mathbf{H}}{}^1\left(\Omega;\, \eta^{\alpha-1}_{\beta}, \eta^\alpha_\beta\right)$
$\mathbf{V}_{\alpha,\beta}\left(\overline{\Omega}\right)$	instead of	$\overset{\circ}{\mathbf{H}}{}^1\left(\overline{\Omega};\, \eta^{\alpha-1}_{\beta}, \eta^\alpha_\beta\right)$

We shall use these last two Hilbert spaces for $\alpha \geq 0,\ \beta > 0,\ \alpha + \beta < 3$. If no confusion can occur, we omit the domain in the notation of the norm $\|\cdot\|_{2,\alpha,\beta;\,\Omega}$. The notation $\mathbf{H}^1(\Omega)$ and $\overset{\circ}{\mathbf{H}}{}^1(\Omega)$ mean, as usual, the non-weighted spaces $(H^1(\Omega;\, 1,\, 1))^3$ and $(\overset{\circ}{H}{}^1(\Omega;\, 1,\, 1))^3$, respectively. As usual, omitting the domain Ω in the notation of spaces will indicate that $\Omega = \mathbb{R}^3$, so e.g. $\mathbf{H}^1 = \mathbf{H}^1(\mathbb{R}^3)$.

Concerning the weight functions η^α_β, we will use two notations $\eta^\alpha_\beta(x)$ and $\eta^{\alpha,\delta}_{\beta,\varepsilon}(x)$ taking the advantages of the following remark:

Remark 1.1 Let us note that for $\eta^{\alpha,\delta}_{\beta,\varepsilon}$ and for any $\delta_1, \delta_2, \varepsilon_1, \varepsilon_2 > 0$ one has

$$c_{\min} \cdot \eta^{\alpha,\delta_2}_{\beta,\varepsilon_2} \leq \eta^{\alpha,\delta_1}_{\beta,\varepsilon_1} \leq c_{\max} \cdot \eta^{\alpha,\delta_2}_{\beta,\varepsilon_2},$$

$c_{\min} = \min\left(1, (\delta_1/\delta_2)^\alpha\right) \cdot \min(1, (\varepsilon_1/\varepsilon_2)^\beta)$, $c_{\max} = \max\left(1, (\delta_1/\delta_2)^\alpha\right) \cdot \max(1, (\varepsilon_1/\varepsilon_2)^\beta)$. The parameters δ and ε are useful to re-scale separately the isotropic and anisotropic parts of the weight function η^α_β.

We also use the notation of sets $B_R = \left\{\mathbf{x} \in \mathbb{R}^3; |\mathbf{x}| \leq R\right\}$, $B^R = \left\{\mathbf{x} \in \mathbb{R}^3; |\mathbf{x}| \geq R\right\}$, $\Omega_R = B_R \cap \Omega$, $\Omega^R = B^R \cap \Omega$, $B^{R_1}_{R_2} = B^{R_1} \cap B_{R_2}$, $\Omega^{R_1}_{R_2} = B^{R_1}_{R_2} \cap \Omega$, for positive numbers $R,\ R_1,\ R_2$.

1.4 Main results

In the first part of the paper (chapters 2–4) we study the problem in $\mathbb{R}^3$. Let us assume for a moment that pressure p is known. In solving the problem (1.1)–(1.3) with respect to $\mathbf{u}$ and p by means of a pure variational approach, we shall deal with the following equation:

$$\nu \int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2\, w\, d\mathbf{x} + \nu \int_{\mathbb{R}^3} \mathbf{u} \nabla\, \mathbf{u} \cdot \nabla w\, d\mathbf{x} - \frac{k}{2} \int_{\mathbb{R}^3} |\mathbf{u}|^2\, \partial_1 w\, d\mathbf{x} \tag{1.5}$$
$$-\frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{u}|^2\, \mathrm{div}\left(w\, [\omega \times \mathbf{x}]\right) d\mathbf{x} = \int_{\mathbb{R}^3} \mathbf{f}\, \mathbf{u}\, w\, d\mathbf{x} - \int_{\mathbb{R}^3} \nabla p \cdot \mathbf{u}\, w\, d\mathbf{x}$$

as we get integrating formally the product of (1.1) by $w\,\mathbf{u}$ with w an appropriate weight function. First, let us note that div $\left(\eta^\alpha_\beta\, [\omega \times \mathbf{x}]\right)$ equals zero for $w = \eta^\alpha_\beta$. The left hand side can be estimated from below by:

$$\frac{\nu}{2}\int_{\mathbb{R}^3} |\nabla \mathbf{u}|^2\, w\, d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{u}|^2 \left(-\nu\, |\nabla w|^2/w - k\partial_1 w\right) d\mathbf{x} \tag{1.6}$$

Because the term $-\nu\,|\nabla w|^2/w - k\partial_1 w$ is known explicitly, we have the possibility to evaluate it from below by a small negative quantity in the form $-C\,\eta^{\alpha-1}_{\beta-1}$ without any constraint in $s(\cdot)$ (see Lemma 2.5).

An improved weighted Friedrichs-Poincaré type inequality in $\overset{\circ}{\mathbf{H}}{}^1_{\alpha,\beta}$ is necessary. The obtained inequality allows us to compensate by the viscous Dirichlet integral the "small" negative contribution in the second integral of (1.6). We finally prove the existence of a weak solution (1.1) - (1.3) in $\mathbf{V}_{\alpha,\beta}$ by the Lax-Milgram theorem.

The main results of the first part of the paper can be summarized in the following theorems (parameters α, β, δ, ε are specified in Section 1.3):

Theorem 1.2 *Let $\beta > 0$. There are positive constants R_0, c_0, c_1 depending on α, β, δ, ε (explicit expressions of these constants are given by Lemma 2.3, essentially $c_0 = O(\varepsilon^{-2} + \delta^{-2})$ and $c_1 = O(\varepsilon^{-1}\delta^{-1})$ for δ and ε tending to zero) such that for all $\mathbf{v} \in \overset{\circ}{\mathbf{H}}{}^1_{\alpha,\beta}$*

$$\|\mathbf{v}\|^2_{2,\alpha-1,\beta-1} \;\le\; c_0 \int_{B_{R_0}} |\nabla \mathbf{v}|^2\, \eta^\alpha_\beta\, d\mathbf{x} + c_1 \int_{B^{R_0}} |\nabla \mathbf{v}|^2\, \eta^\alpha_\beta\, d\mathbf{x}. \tag{1.7}$$

Theorem 1.3 *(Existence and uniqueness) Let $0 < \beta \le 1$, $0 \le \alpha < y_1\beta$, $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}$, $g \in W^{1,2}_0$ with $\operatorname{supp} g = K \subset\subset \mathbb{R}^3$, and $\int_{\mathbb{R}^3} g\, d\mathbf{x} = 0$; y_1 will be given in Lemma 4.3. Then there exists a unique weak solution $\{\mathbf{u},\, p\}$ of the problem (1.1) - (1.3) such that $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}$, $p \in L^2_{\alpha,\beta-1}$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}$ and*

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla \mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \le C\,\left(\|\mathbf{f}\|_{2,\alpha+1,\beta} + \|g\|_{1,2}\right).$$

In the second part of the paper (chapters 5, 6) we extend the results of the first part onto exterior domains.

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain and $0 < \beta \le 1$, $0 \le \alpha < y_1 \cdot \beta$; y_1 is given in Lemma 4.3, $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$, $g \in W^{1,2}_0(\Omega)$, with $\operatorname{supp} g = K \subset\subset \Omega$ and $\int_\Omega g\, d\mathbf{x} = 0$. Then there exists a weak solution $\{\mathbf{u}, p\}$ of the problem (1.1) - (1.4) such that $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}\left(\overline{\Omega}\right)$, $p \in L^2_{\alpha,\beta-1}(\Omega)$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$ and*

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla \mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \le C\,\left(\|\mathbf{f}\|_{2,\alpha+1,\beta} + \|g\|_{1,2}\right).$$

2 Friedrichs-Poincaré inequality

In this section we derive an inequality of the Friedrichs-Poincaré type in weighted Sobolev spaces. We also recall some necessary technical assertions, for more details see Kračmar and Penel [25].

Proposition 2.1 *For arbitrary $\alpha, \beta \geq 0$ and $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{x} \neq \mathbf{0}$:*

$$\Delta\eta_{\beta}^{\alpha}(\mathbf{x}) \geq 2\beta \min(1,\beta)\, \varepsilon\, \delta\, \eta_{\beta-1}^{\alpha-1}(\mathbf{x})$$

Proof. We introduce $\beta^* = \min(\beta, 1)$ in an explicit expression of $\Delta\eta_{\beta}^{\alpha}$:

$$\begin{aligned}\Delta\eta_{\beta}^{\alpha} \;=\; & \left\{\left(\alpha^2\delta^2\frac{1+\varepsilon s}{1+\delta r} - \alpha\delta^2\frac{1+\varepsilon s}{1+\delta r}\right) + 2\alpha\beta\delta\varepsilon\frac{s}{r}\right. \\ & +2\beta(\beta-1)\frac{\varepsilon}{r}(1+\delta r)\frac{\varepsilon s}{1+\varepsilon s} \\ & \left. +2\alpha\,\delta^2(1+\varepsilon s)\frac{1}{\delta r} + (1-\beta^*+\beta^*)\,2\beta\frac{\varepsilon}{r}(1+\delta r)\right\}\eta_{\beta-1}^{\alpha-1},\end{aligned}$$

for $r > 0$. We denote the five terms in $\{\ \}$ by $T_1, T_2, \ldots, T_5$, and overwrite the previous relation as $\Delta\eta_{\beta}^{\alpha} = \{[T_1 + T_4] + T_2 + [T_3 + (1-\beta^*)\,T_5] + \beta^* T_5\}\ \eta_{\beta-1}^{\alpha-1}$. Observing that $T_5 \geq 2\beta\varepsilon\delta$, the proposition is trivial. □

Proposition 2.2 *Let $\alpha \geq 0$, $\beta \geq 0$, $\delta > 0$, $\varepsilon > 0$ and $\kappa > 1$. Then for $\mathbf{x} \in \mathbb{R}^3$, $|\mathbf{x}| \geq \left|\delta^{-1} - (2\varepsilon)^{-1}\right|(\kappa-1)^{-1}$:*

$$\left|\nabla\eta_{\beta}^{\alpha}(\mathbf{x})\right|^2 \leq 2\,\kappa\,\delta\,\varepsilon\,(\alpha+\beta)^2\left(\eta_{\beta-1/2}^{\alpha-1/2}(\mathbf{x})\right)^2 \tag{2.8}$$

Let $\alpha \geq 0$, $\beta \geq 0$, $\delta > 0$, $\varepsilon > 0$ and $(\beta-\alpha)(2\varepsilon-\delta) \geq 0$. Then for $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{x} \neq 0$:

$$\left|\nabla\eta_{\beta}^{\alpha}(\mathbf{x})\right|^2 \leq (\alpha\delta + 2\beta\varepsilon)^2\left(\eta_{\beta-1/2}^{\alpha-1/2}(\mathbf{x})\right)^2 \tag{2.9}$$

Proof. If $\beta = 0$ and $\alpha = 0$ then both inequalities (2.8) and (2.9) are valid. Let us concentrate on the nontrivial cases:

For $r > 0$, $s \in [0, 2r]$, we have that $\partial g/\partial s > 0$, where g is a function defined by relations:

$$\begin{aligned}\left|\nabla\eta_{\beta}^{\alpha}(\mathbf{x})\right|^2 &= g(s(\mathbf{x}), r(\mathbf{x}))\left(\eta_{\beta-1/2}^{\alpha-1/2}(\mathbf{x})\right)^2, \\ g(s,r) &\equiv \alpha^2\delta^2\left(\frac{1+\varepsilon s}{1+\delta r}\right) + 2\alpha\beta\delta\varepsilon\frac{s}{r} + 2\beta^2\varepsilon^2\left(\frac{1+\delta r}{1+\varepsilon s}\right)\frac{s}{r}.\end{aligned}$$

So, $g(s, r)$ is increasing as a function of s and

$$\begin{aligned}G(r) &\equiv \max_{s\in[0,2r]} g(s,r) = g(2r, r) \\ &= \alpha^2\delta^2\frac{1+2\varepsilon r}{1+\delta r} + 4\alpha\beta\delta\varepsilon + 4\beta^2\varepsilon^2\frac{1+\delta r}{1+2\varepsilon r} \;\leq\; 2\kappa(\alpha+\beta)^2\delta\varepsilon\end{aligned} \tag{2.10}$$

for $\kappa > 1$ and $r \geq \left|\delta^{-1} - (2\varepsilon)^{-1}\right|(\kappa-1)^{-1}$. So, inequality (2.8) is proved.

To justify the second inequality (2.9), we observe that for the given values of α, β, δ, ε and for $r > 0$, $G(r) \le G(0)$. □

Next we derive an inequality of the Friedrichs-Poincaré type in the space $\overset{\circ}{\mathbf{H}}{}^1_{\alpha,\beta}$. It is necessary for our aim to get expressions of constants in this inequality. It follows from Proposition 2.1

Lemma 2.3 *Let $\alpha \ge 0$, $\beta > 0$, $\alpha + \beta < 3$, $\kappa > 1$. Let δ and ε be arbitrary positive constants, such that $(\beta - \alpha)(2\varepsilon - \delta) \ge 0$. Then for all $\mathbf{u} \in \overset{\circ}{\mathbf{H}}{}^1_{\alpha,\beta}$*

$$\|\mathbf{u}\|^2_{2,\alpha-1,\beta-1} \le c_0 \left\| \nabla\mathbf{u}\right| B_{R_0}\|^2_{2,\alpha,\beta} + c_1 \left\| \nabla\mathbf{u}\right| B^{R_0}\|^2_{2,\alpha,\beta}, \tag{2.11}$$

where $c_0 = \left[(\alpha\delta + 2\beta\varepsilon)/(\beta\beta^*\delta\varepsilon)\right]^2$, $c_1 = [(2\kappa)/(\delta\varepsilon)] \cdot [(\alpha+\beta)/(\beta\beta^*)]^2$ *and* $R_0 \ge \left|\delta^{-1} - (2\varepsilon)^{-1}\right| (\kappa - 1)^{-1}$.

Remark 2.4 Let us observe that if additionally $\delta < 2\varepsilon$ and $1 < \kappa \le 2\varepsilon/\delta + \delta/(2\varepsilon) - 1$ then $c_0 \ge c_1$.

Proof of Lemma 2.3 Due to the density of $\mathbf{C}_0^\infty$ in $\overset{\circ}{\mathbf{H}}{}^1_{\alpha,\beta}$ it is sufficient to prove the inequality for all $\mathbf{u} \in \mathbf{C}_0^\infty$. From Proposition 2.1 it follows that for $\mathbf{v} \in \mathbf{C}_0^\infty$

$$\begin{aligned} & 2\beta\beta^*\delta\varepsilon \int_{\mathbb{R}^3\setminus B_\rho} \mathbf{v}^2 \eta^{\alpha-1}_{\beta-1}\, d\mathbf{x} \le \int_{\mathbb{R}^3\setminus B_\rho} \mathbf{v}^2 \Delta\eta^\alpha_\beta \, d\mathbf{x} \\ = \; & -2\int_{\mathbb{R}^3\setminus B_\rho} \mathbf{v}\nabla\, \mathbf{v}\cdot\nabla\eta^\alpha_\beta \, d\mathbf{x} + \int_{\partial B_\rho} \mathbf{v}^2 \nabla\eta^\alpha_\beta\cdot\mathbf{n}\, dS \\ \le \; & \beta\beta^*\delta\varepsilon \int_{\mathbb{R}^3\setminus B_\rho} \mathbf{v}^2\eta^{\alpha-1}_{\beta-1}\, d\mathbf{x} + \frac{1}{\beta\beta^*\delta\varepsilon} \int_{\mathbb{R}^3\setminus B_\rho} |\nabla\mathbf{v}|^2 \left|\nabla\eta^\alpha_\beta\right|^2 \eta^{-\alpha+1}_{-\beta+1}\, d\mathbf{x} \\ & + \int_{\partial B_\rho} \mathbf{v}^2\, \nabla\eta^\alpha_\beta\cdot\mathbf{n}\, dS. \end{aligned}$$

Hence, because the surface integral is a value of the order $O(\rho^2)$, we have:

$$\beta\beta^*\delta\varepsilon \int_{\mathbb{R}^3} \mathbf{v}^2\eta^{\alpha-1}_{\beta-1}\, d\mathbf{x} \le \frac{1}{\beta\beta^*\delta\varepsilon} \int_{\mathbb{R}^3} |\nabla\mathbf{v}|^2 \left|\nabla\eta^\alpha_\beta\right|^2 \eta^{-\alpha+1}_{-\beta+1}\, d\mathbf{x} \tag{2.12}$$

By means of the Cauchy-Schwarz inequality and from Proposition 2.2 with $R_0 \ge \left|\delta^{-1} - (2\varepsilon)^{-1}\right| / (\kappa - 1)$ we finally get (2.11). □

We will need some technical lemmas. Let us define $F_{\alpha,\beta}(s, r;\, \nu)$ by the relation:

$$F_{\alpha,\beta}(s, r;\, \nu) \cdot \eta^{\alpha-1}_{\beta-1} \equiv -\nu \left|\nabla\eta^\alpha_\beta\right|^2 / \eta^\alpha_\beta - k\, \partial_1\, \eta^\alpha_\beta \tag{2.13}$$

The following lemma gives the evaluation of $F_{\alpha,\beta}(s, r;\, \nu)$ from below

Lemma 2.5 *Let $0 \le \alpha < \beta$, $\kappa > 1$, $0 < \varepsilon \le (1/(2\kappa))\cdot(k/\nu)\cdot((\beta-\alpha)/\beta^2)$ and δ, ν, $k > 0$. Then*

$$F_{\alpha,\beta}(s, r; \nu) - \left(1 - \kappa^{-1}\right) k\delta\varepsilon\,(\beta - \alpha)\, s \ge -\alpha\delta k \left(1 + \nu k^{-1}\alpha\delta\right) \tag{2.14}$$

for all $r > 0$ and $s \in [0, 2r]$.

Proof. Expressing the function $F_{\alpha,\beta}(s,r;\nu)$ explicitly we get:

$$\begin{aligned} F_{\alpha,\beta}(s,r;\nu) &= -\nu\alpha^2\delta^2\left(\frac{1+\varepsilon s}{1+\delta r}\right)-2\nu\alpha\beta\delta\varepsilon\frac{s}{r}-2\nu\beta^2\varepsilon^2\left(\frac{1+\delta r}{1+\varepsilon s}\right)\frac{s}{r} \\ &\quad -k\alpha\delta(1+\varepsilon s)\frac{r-s}{r}+k\beta\varepsilon(1+\delta r)\frac{s}{r} \end{aligned}$$

For convenient use we subtract $(1-\kappa^{-1})\,k\delta\varepsilon(\beta-\alpha)\,s$ from $F_{\alpha,\beta}(s,r;\nu)$. We observe (see Appendix A) that, for the given α, β, ε, κ, for all δ, ν, $k>0$ and for $r>0$, $F_{\alpha,\beta}(s,r;\,\nu)-(1-\kappa^{-1})\,k\delta\varepsilon(\beta-\alpha)\,s\;\geq\;F_{\alpha,\beta}(0,r;\,\nu)$, which immediately gives inequality (2.14). □

3 Uniqueness in $\mathbb{R}^3$

In this section, we will start with the question about the unique weak solvability of the problem (1.1)–(1.3) in $\Omega=\mathbb{R}^3$. The presented approach will be also used in the next section, in the proof of existence of a solution verifying solenoidality of the constructed function $\mathbf{u}$.

Theorem 3.1 *(Uniqueness in $\mathbb{R}^3$) Let $\{\mathbf{u},p\}$ be a distributional solution of the problem (1.1)–(1.3) with $\mathbf{f}=\mathbf{0}$, $g=0$ such that $\mathbf{u}\in\widehat{\mathbf{H}}_0^{1,2}$ and $p\in L^2_{loc}$. Then $\mathbf{u}=\mathbf{0}$ and $p=const$.*

Proof. From the condition $\mathbf{u}\in\widehat{\mathbf{H}}_0^{1,2}$ we get $\nabla\mathbf{u}\in\mathbf{L}^2$, $\mathbf{u}\in\mathbf{L}^6$, $\mathbf{u}\in\mathcal{S}'$. Because $\operatorname{div}((\omega\times\mathbf{x})\cdot\nabla\mathbf{u}-\omega\times\mathbf{u})=(\omega\times\mathbf{x})\cdot\nabla\operatorname{div}\mathbf{u}=0$, we have $\triangle p=0$. Hence, applying Laplacian and the Fourier transform we get

$$\triangle\left(-\nu\,\Delta\mathbf{u}+k\,\partial_1\mathbf{u}-(\omega\times\mathbf{x})\cdot\nabla\mathbf{u}+\omega\times\mathbf{u}\right)=\mathbf{0},$$

$$|\xi|^2\left(\nu\,|\xi|^2\,\widehat{\mathbf{u}}+i\,k\,\xi_1\widehat{\mathbf{u}}-(\omega\times\xi)\cdot\nabla_\xi\widehat{\mathbf{u}}+\omega\times\widehat{\mathbf{u}}\right)=\mathbf{0}\qquad\text{in }\mathcal{S}'.$$

Assuming the equation in cylindrical coordinates (ξ_1,ρ,φ), and denoting $T(\varphi)\,\widehat{\mathbf{v}}=\widehat{\mathbf{u}}(\xi_1,\rho,\varphi)$, where

$$T(\varphi)=\begin{bmatrix}1, & 0, & 0\\ 0, & \cos(\varphi), & -\sin(\varphi)\\ 0, & \sin(\varphi), & \cos(\varphi)\end{bmatrix},$$

we get

$$|\xi|^2\left\{-\partial_\varphi\widehat{\mathbf{v}}+\left[(\nu/\widetilde{\omega})\,|\xi|^2+i\,(k/\widetilde{\omega})\,\xi_1\right]\widehat{\mathbf{v}}\right\}=\mathbf{0}\quad\text{in }\mathcal{S}'. \tag{3.15}$$

We will show that from this equation follows that $\operatorname{supp}\widehat{\mathbf{v}}\subset\{0\}$, and due to the definition of $\widehat{\mathbf{v}}$ we will have also $\operatorname{supp}\widehat{\mathbf{u}}\subset\{\mathbf{0}\}$. This means that $\mathbf{u}$ is a polynomial of x_1, x_2, x_3. Because $\mathbf{u}\in\mathbf{L}^6$ we get $\mathbf{u}=0$. Substituting into (1.1) we get $\nabla p=0$ and $p=const$.

So, we have to prove that for an arbitrary real vector function $\Psi\in\mathbf{C}_0^\infty(\mathbb{R}^3\setminus\{\mathbf{0}\})$ defined for $[\xi_1,\xi_2,\xi_3]\in\mathbb{R}^3$ we have $\langle\widehat{\mathbf{v}},\Psi\rangle=0$. If for each $\Psi\in\mathbf{C}_0^\infty(\mathbb{R}^3\setminus\{\mathbf{0}\})$ there is a function $\Phi\in\mathbf{C}_0^\infty(\mathbb{R}^3\setminus\{\mathbf{0}\})$ such that

$$\partial_\varphi\left(|\xi|^2\,\Phi\right)+\left[(\nu/\widetilde{\omega})\,|\xi|^2+i\,(k/\widetilde{\omega})\,\xi_1\right]\left(|\xi|^2\,\Phi\right)=\Psi \tag{3.16}$$

then from (3.15) follows:

$$0 = \left\langle |\xi|^2 \left\{ -\partial_\varphi \widehat{\mathbf{v}} + \left[(\nu/\widetilde{\omega}) |\xi|^2 + i\ (k/\widetilde{\omega}) \xi_1 \right] \widehat{\mathbf{v}} \right\},\ \Phi \right\rangle$$
$$= \left\langle \widehat{\mathbf{v}},\ \partial_\varphi \left(|\xi|^2 \Phi \right) + \left[(\nu/\widetilde{\omega}) |\xi|^2 + i\ (k/\widetilde{\omega}) \xi_1 \right] \left(|\xi|^2 \Phi \right) \right\rangle = \langle \widehat{\mathbf{v}},\ \Psi \rangle$$

Hence, the proof of $\operatorname{supp} \widehat{\mathbf{v}} \subset \{0\}$ is reduced to the solvability of (3.16). First we note that it is sufficient to solve the equation

$$\partial_\varphi \zeta + \left((\nu/\widetilde{\omega})\ |\xi|^2 + i\ (k/\widetilde{\omega})\ \xi_1 \right) \zeta = \Psi \tag{3.17}$$

because the division on the expression $|\xi|^2$ defines the one-to-one correspondence of the space $\mathbf{C}_0^\infty (\mathbb{R}^3 \setminus \{\mathbf{0}\})$ onto $\mathbf{C}_0^\infty (\mathbb{R}^3 \setminus \{\mathbf{0}\})$.

Let us analyze the equation (3.17) in cylindrical coordinates $[\xi_1, \rho, \varphi]$, where $\rho = (\xi_2^2 + \xi_3^2)^{1/2}$. For an arbitrary real vector function $\Psi \in \mathbf{C}_0^\infty (\mathbb{R}^3 \setminus \{\mathbf{0}\})$ defined for $[\xi_1, \xi_2, \xi_3] \in \mathbb{R}^3$ we define $f(t) := \Psi(\xi_1,\ \rho \cos t,\ \rho \sin t)$, $a := (\nu/\widetilde{\omega}) |\xi|^2 + i (k/\widetilde{\omega}) \xi_1$, assuming $\widetilde{\omega} > 0$.

Now, we will use the following technical proposition about the existence of a solution of an ordinary differential equation in a space of periodical functions (and later also in the proof of existence of a solution of the problem for checking solenoidality of a constructed solution, see the proof of Theorem 4.4):

Proposition 3.2 *Let $a \in \mathbb{C}$, $Re\ a > 0$. Let $f \in C^\infty(\mathbb{R})$ be a 2π-periodical complex function. Then there is unique 2π-periodical solution $g \in C^\infty(\mathbb{R})$ of the equation*

$$g' + a\, g = f$$

and the solution g can be expressed in the following form:

$$g(\varphi) = \left(e^{2\pi a} - 1 \right)^{-1} \int_0^{2\pi} e^{at} f(\varphi + t)\ dt = e^{-a\varphi} \int_{-\infty}^{\varphi} e^{at} f(t)\ dt$$

Proof of the proposition follows from standard computations.

Using the Proposition 3.2 we get the solution of (3.17) in the form

$$\zeta(\xi_1, \rho, \varphi) = \left\{ \exp \left[2\pi \left(\frac{\nu}{\widetilde{\omega}} |\xi|^2 + i \frac{k}{\widetilde{\omega}} \right) \right] - 1 \right\}^{-1}$$
$$\cdot \int_0^{2\pi} \exp \left[\left(\frac{\nu}{\widetilde{\omega}} |\xi|^2 + i \frac{k}{\widetilde{\omega}} \xi_1 \right) t \right]\ \Psi(\xi_1,\ \rho \cos(t + \varphi),\ \rho \sin(t + \varphi))\ dt.$$

It is easy to see that function ζ as the function of $[\xi_1, \xi_2, \xi_3]$ is infinitely differentiable with respect to these variables and $\zeta \in \mathbf{C}_0^\infty (\mathbb{R}^3 \setminus \{\mathbf{0}\})$. Finally we put $\Phi = \zeta / |\xi|^2$.
□

4 Existence of a solution in $\mathbb{R}^3$

In this section we will construct a weak solution of the problem (1.1)–(1.3) assuming that $g = 0$.

4.1 Existence of the pressure in $\mathbb{R}^3$ for a solenoidal solution

If there exist distributions $\mathbf{u}, p$ satisfying

$$\begin{aligned} -\nu\,\Delta\mathbf{u} + k\,\partial_1\mathbf{u} - (\omega\times\mathbf{x})\cdot\nabla\mathbf{u} + \omega\times\mathbf{u} + \nabla p &= \mathbf{f} \qquad \text{in } \mathbb{R}^3 \\ \operatorname{div}\mathbf{u} &= 0 \qquad \text{in } \mathbb{R}^3 \end{aligned}$$

then pressure p satisfies the equation

$$\triangle p = \operatorname{div}\mathbf{f} \tag{4.18}$$

because $\operatorname{div}\left((\omega\times\mathbf{x})\cdot\nabla\mathbf{u} - \omega\times\mathbf{u}\right) = (\omega\times\mathbf{x})\cdot\nabla\operatorname{div}\mathbf{u} = 0$, and $\operatorname{div}(\Delta\mathbf{u} + k\,\partial_1\mathbf{u}) = 0$ provided $\operatorname{div}\mathbf{u} = 0$.

Let $\mathcal{E}$ be the fundamental solution of the Laplace equation, i.e. $\mathcal{E} = -1/\left(4\pi r\right)$. Assuming firstly $\mathbf{f} \in\mathbf{C}_0^\infty$ we have $p = \mathcal{E}\star\operatorname{div}\mathbf{f}$ and $\nabla p = \nabla\mathcal{E}\star\operatorname{div}\mathbf{f}$ and so, $p = \nabla\mathcal{E}\star\mathbf{f}$ and $\nabla p = \nabla^2\mathcal{E}\star\mathbf{f}$. It is well known that both formulas can be extended for $\mathbf{f}\in\mathbf{L}^2_{\alpha+1,\,\beta}$ with $0<\beta<1$ and $-2<\alpha+\beta<2$ (the last convolution $\nabla p = \nabla^2\mathcal{E}\star\mathbf{f}$ due to the fact that $\nabla^2\mathcal{E}$ is the singular kernel of the Calderon-Zygmund type and that $\eta_\beta^{\alpha+1}$ belongs to the Muckenhoupt class of weights A_2), see [2, Thm. 3.2, Thm 5.5], [24, Thm. 4.4, Thm 5.4], where the theorems are formulated for the pressure part $\mathcal{P}$ of the fundamental solution of the classical Oseen problem, so $\mathcal{P} = \nabla\mathcal{E}$ and $\nabla\mathcal{P} = \nabla^2\mathcal{E}$. For $\mathbf{f}\in\mathbf{L}^2_{\alpha+1,\,\beta}$ we get $p\in L^2_{\alpha,\,\beta-1}$ and $\nabla p\in\mathbf{L}^2_{\alpha+1,\,\beta}$, and there are positive constants C_1, C_2 such that the following estimates are satisfied:

$$\|p\|_{2,\alpha,\beta-1} \le C_1\,\|\mathbf{f}\|_{2,\alpha+1,\beta}\ , \quad \|\nabla p\|_{2,\alpha+1,\beta} \le C_2\,\|\mathbf{f}\|_{2,\alpha+1,\beta} \tag{4.19}$$

4.2 The problem in B_R - solenoidal solutions

We will study in this section the existence of a weak solution of the following problem in a bounded domain B_R, pressure p is assumed here to be known, the right hand side $\mathbf{f}-\nabla p = \widetilde{\mathbf{f}}\in\mathbf{L}^2_{\alpha+1,\,\beta}$:

$$-\nu\Delta\mathbf{u} + k\,\partial_1\mathbf{u} - (\omega\times\mathbf{x})\cdot\nabla\mathbf{u} + \omega\times\mathbf{u} = \widetilde{\mathbf{f}} \quad \text{in } B_R \tag{4.20}$$

$$\mathbf{u} = 0 \quad \text{on } \partial B_R \tag{4.21}$$

We show the existence of a weak solution $\mathbf{u}_R\in\overset{\circ}{\mathbf{H}}(B_R)$. Following (1.5), (1.6) again with $w=\eta^0_{\beta_0}$, $\beta_0\in(0,1]$, using notation (2.13), let us introduce a continuous bilinear form $\widetilde{Q}\left(\cdot,\cdot\right)$ on $\overset{\circ}{\mathbf{H}}(B_R)\times\overset{\circ}{\mathbf{H}}(B_R)$:

$$\begin{aligned} \widetilde{Q}\left(\mathbf{u},\mathbf{v}\right) &= \int_{B_R}\nu\nabla\mathbf{u}\cdot\nabla\left(\mathbf{v}\cdot\eta^0_{\beta_0}\right)\,d\mathbf{x} + k\int_{B_R}\partial_1\mathbf{u}\cdot\left(\mathbf{v}\eta^0_{\beta_0}\right)\,d\mathbf{x} \\ &\quad + \int_{B_R}(\omega\times\mathbf{x})\cdot\nabla\mathbf{u}\,\left(\mathbf{v}\eta^0_{\beta_0}\right)\,d\mathbf{x}, \end{aligned}$$

$$\widetilde{Q}\left(\mathbf{v},\mathbf{v}\right) \ge 2^{-1}\nu\int_{B_R}|\nabla\mathbf{v}|^2\,\eta^0_{\beta_0}d\mathbf{x} + 2^{-1}\int_{B_R}\mathbf{v}^2F_{0,\beta_0}\left(s,r;\,\nu\right)\eta^{-1}_{\beta_0-1}d\mathbf{x}. \tag{4.22}$$

Lemma 4.1 *Let $0 < \beta_0 \le 1$. Then, for all $\widetilde{\mathbf{f}} \in \mathbf{L}^2_{1,\beta_0}(B_R)$, $\varepsilon_0 < (1/2)\cdot(k/\nu)\cdot(1/\beta_0)$, $\eta^\alpha_{\beta_0} \equiv \eta^{\alpha,\varepsilon_0}_{\beta_0,\varepsilon_0}$, there exists unique $\mathbf{u}_R \in \overset{\circ}{\mathbf{H}}(B_R)$ such that for all $\mathbf{v} \in \overset{\circ}{\mathbf{H}}(B_R)$*

$$\widetilde{Q}(\mathbf{u}_R, \mathbf{v}) = \int_{B_R} \widetilde{\mathbf{f}}\cdot\mathbf{v}\eta^0_{\beta_0}\, d\mathbf{x}. \tag{4.23}$$

Proof. Bilinear form $\widetilde{Q}$ is coercive, i.e. there exists a constant $C_R > 0$ such that $\widetilde{Q}(\mathbf{v},\mathbf{v}) \ge C_R \|\mathbf{v}\|^2$, where $\|\cdot\|$ is the norm in the space $\overset{\circ}{\mathbf{H}}(B_R)$. Indeed, we get

$$\widetilde{Q}(\mathbf{v},\mathbf{v}) \ge \frac{\nu}{2}\int_{B_R} |\nabla \mathbf{v}|^2 \eta^0_{\beta_0} d\mathbf{x} + \frac{1}{2}\int_{B_R} \mathbf{v}^2 F_{0,\beta_0}(s,r;\,\nu)\,\eta^{-1}_{\beta_0-1} d\mathbf{x}$$

Because $\varepsilon_0 < (1/2)\cdot(k/\nu)\cdot(1/\beta_0)$ there is a constant κ satisfying all previous conditions and additionally $\varepsilon_0 \le (1/2\kappa)\cdot(k/\nu)\cdot(1/\beta_0)$. Because $\alpha = 0$ we get from Lemma 2.5

$$\int_{B_R} \mathbf{v}^2 F_{0,\beta_0}(s,r;\,\nu)\,\eta^{-1}_{\beta_0-1} d\mathbf{x} \ge \left(1-\kappa^{-1}\right) k\varepsilon_0^2\beta_0 \int_{B_R} \mathbf{v}^2 \eta^{-1}_{\beta_0-1} s\, d\mathbf{x},$$

$$\widetilde{Q}(\mathbf{v},\mathbf{v}) \ge \frac{\nu}{2}\int_{B_R} |\nabla \mathbf{v}|^2 \eta^0_{\beta_0} d\mathbf{x} + \frac{1}{2}\left(1-\frac{1}{\kappa}\right) k\varepsilon_0\beta_0 \int_{B_R} \mathbf{v}^2 \eta^{-1}_{\beta_0-1}(\varepsilon_0 s)\; d\mathbf{x}.$$

Using Lemma 2.3 and Remark 2.4 we derive:

$$\begin{aligned}
\widetilde{Q}(\mathbf{v},\mathbf{v}) \;\; &\ge \;\; \frac{\nu}{4}\int_{B_R} |\nabla \mathbf{v}|^2 \eta^0_{\beta_0}\, d\mathbf{x} + \frac{\nu}{16}\varepsilon_0^2\beta_0^2 \int_{B_R} \mathbf{v}^2\eta^{-1}_{\beta_0-1} d\mathbf{x} \\
&\quad + \frac{1}{2}\left(1-\frac{1}{\kappa}\right) k\varepsilon_0\beta_0 \int_{B_R} \mathbf{v}^2\eta^{-1}_{\beta_0-1}(\varepsilon_0 s)\; d\mathbf{x} \\
&\ge \;\; \left(1-\frac{1}{\kappa}\right)\frac{\nu}{4}\min\left\{1,\, \frac{1}{4}\varepsilon_0^2\beta_0^2,\, 2\frac{k}{\nu}\beta_0\varepsilon_0\right\} \\
&\qquad \cdot\left(\int_{B_R} |\nabla v|^2 \eta^0_{\beta_0} d\mathbf{x} + \int_{B_R} \mathbf{v}^2\eta^{-1}_{\beta_0} d\mathbf{x}\right)
\end{aligned} \tag{4.24}$$

$$\widetilde{Q}(\mathbf{v},\mathbf{v}) \ge C_R\left(\int_{B_R} |\nabla \mathbf{v}|^2 d\mathbf{x} + \int_{B_R} \mathbf{v}^2 d\mathbf{x}\right) = C_R \|\mathbf{v}\|^2, \tag{4.25}$$

where $C_R = (\nu/4)\cdot(1-\kappa^{-1})\cdot\min\{1,\, \varepsilon_0^2\beta_0^2/4,\, 2\,(k/\nu)\,\beta\varepsilon_0\}\cdot(1+\varepsilon_0 R)$. Using Lax-Milgram theorem we get that there is $\mathbf{u}_R \in \overset{\circ}{\mathbf{H}}(B_R)$ such that (4.23) is satisfied. □

Remark 4.2 *An arbitrary function $\mathbf{\Phi} \in \overset{\circ}{\mathbf{H}}(B_R)$ can be expressed in the form $\mathbf{\Phi} = \phi\,\eta^0_{\beta_0}$, where ϕ is a function from $\overset{\circ}{\mathbf{H}}(B_R)$. Therefore we have for $\mathbf{u}_R$*

$$Q(\mathbf{u}_R, \mathbf{\Phi}) = \int_{B_R} \widetilde{\mathbf{f}}\cdot\mathbf{\Phi}\, d\mathbf{x}, \tag{4.26}$$

for all $\mathbf{\Phi} \in \overset{\circ}{\mathbf{H}}(B_R)$ where by the definition $Q(\mathbf{u}_R,\mathbf{\Phi}) \equiv Q\left(\mathbf{u}_R, \phi\cdot\eta^0_{\beta_0}\right) \equiv \widetilde{Q}(\mathbf{u}_R,\phi)$.

4.3 Uniform estimates of $\mathbf{u}_R$ in $\mathbb{R}^3$ - solenoidal solutions

Our next aim is to prove that the weak solutions $\mathbf{u}_R$ of (4.23) are uniformly bounded in $\mathbf{V}_{\alpha,\beta}$ as $R \to +\infty$.

Let y_1 be the unique real solution of the algebraic equation $4y^3+8y^2+5y-1=0$. It is easy to verify that $y_1 \in (0,1)$. We will explain later, why the control of α/β by y_1 is necessary.

Lemma 4.3 *Let $0<\beta\le 1$, $0\le\alpha<y_1\beta$, $\widetilde{\mathbf{f}}\in\mathbf{L}^2_{\alpha+1,\beta}$. Then, as $R\to+\infty$, the weak solutions $\mathbf{u}_R$ of (4.23) given by Lemma 4.1 are uniformly bounded in $\mathbf{V}_{\alpha,\beta}$. There is a constant $c>0$, which does not depend on R such that*

$$\int_{\mathbb{R}^3} \tilde{\mathbf{u}}_R^2 \eta_\beta^{\alpha-1} d\mathbf{x} + \int_{\mathbb{R}^3} |\nabla \tilde{\mathbf{u}}_R|^2 \eta_\beta^\alpha d\mathbf{x} \le c \int_{\mathbb{R}^3} \left|\widetilde{\mathbf{f}}\right|^2 \eta_\beta^{\alpha+1} d\mathbf{x} \tag{4.27}$$

for all R greater than some $R_0>0$, $\tilde{\mathbf{u}}_R$ being extension by zero of $\mathbf{u}_R$ on $\mathbb{R}^3\setminus B_R$.

Proof. First, we derive estimate of $\mathbf{u}_R$ on a bounded subdomain $B_{R_0}\subset B_R$; The choice of R_0 will be given in the next part of the proof. Our aim is to get an estimate with a constant not depending on R. Let us substitute $\phi=\mathbf{u}_R$ into (4.23). Hence, we get from (4.24):

$$\widetilde{Q}(\mathbf{u}_R,\mathbf{u}_R) = \int_{B_R} \widetilde{\mathbf{f}}\, \mathbf{u}_R \eta_{\beta_0}^0 d\mathbf{x} \ge C_1 \left(\int_{B_R} |\nabla \mathbf{u}_R|^2 \eta_{\beta_0}^0 d\mathbf{x} + \int_{B_R} \mathbf{u}_R^2 \eta_{\beta_0}^{-1} d\mathbf{x} \right),$$

with the constant $C_1>0$ stated in (4.24). Let R_0 be some fixed positive number such that $0<R_0<R$. We get

$$\int_{B_{R_0}} |\nabla \mathbf{u}_R|^2 \eta_\beta^\alpha d\mathbf{x} + \int_{B_{R_0}} \mathbf{u}_R^2 \eta_\beta^{\alpha-1} d\mathbf{x} \le C_2 \int_{B_R} \left|\widetilde{\mathbf{f}}\right| |\mathbf{u}_R| \eta_\beta^\alpha d\mathbf{x}, \tag{4.28}$$

where the constant $C_2 = C_1^{-1}(1+\varepsilon_0 R_0)^\alpha (1+\varepsilon_0\, 2\, R_0)^{|\beta-\beta_0|}$ depend on k, ν, α, β, β_0, ε_0, R_0, κ, but does not depend on R.

Now, we are going to derive an estimate of $\mathbf{u}_R$ on domain B_R. Using the test function $\mathbf{\Phi}=\mathbf{u}_R\eta_\beta^\alpha=\mathbf{u}_R(1+\delta r)^\alpha(1+\varepsilon s)^\beta \in \overset{\circ}{\mathbf{H}}(B_R)$ in (4.26) we get after integration by parts:

$$\nu \int_{B_R} |\nabla \mathbf{u}_R|^2 \eta_\beta^\alpha d\mathbf{x} + \nu \int_{B_R} \mathbf{u}_R \nabla \mathbf{u}_R \cdot \nabla \eta_\beta^\alpha \, d\mathbf{x} - \frac{k}{2} \int_{B_R} \mathbf{u}_R^2\, \partial_1 \eta_\beta^\alpha d\mathbf{x}$$
$$= \int_{B_R} \widetilde{\mathbf{f}} \mathbf{u}_R \eta_\beta^\alpha d\mathbf{x}$$

So, we get for some $\kappa>1$:

$$\frac{\nu}{2} \int_{B_R} |\nabla \mathbf{u}_R|^2 \eta_\beta^\alpha d\mathbf{x} + \frac{1}{2} \int_{B_R} \mathbf{u}_R^2 F_{\alpha,\beta}(s,r;\,\nu) \eta_{\beta-1}^{\alpha-1} d\mathbf{x} \le \int_{B_R} \left|\widetilde{\mathbf{f}}\right| |\mathbf{u}_R| \eta_\beta^\alpha d\mathbf{x}$$

Let $R_0 \geq \left|\delta^{-1} - (2\varepsilon)^{-1}\right| (\kappa - 1)^{-1}$. Using Lemma 2.5 (with $0 \leq \alpha < \beta$, $\varepsilon \leq (1/(2\kappa))(k/\nu)((\beta-\alpha)/\beta^2)$) and Lemma 2.3 (with $\delta < 2\varepsilon$), the second term in the previous estimate can be evaluated from below:

$$\begin{aligned}
&\int_{B_R} \mathbf{u}_R^2\, F_{\alpha,\beta}(s,r;\,\nu))\eta_{\beta-1}^{\alpha-1}\, d\mathbf{x} \\
\geq\; & -\alpha\delta k \left(1 + \frac{\nu\kappa}{k}\alpha\delta\right) \frac{2\kappa}{\delta\varepsilon} \left(\frac{\alpha+\beta}{\beta\beta^*}\right)^2 \int\limits_{B_R^{R_0}} |\nabla \mathbf{u}_R|^2\, \eta_\beta^\alpha d\mathbf{x} \\
& + \left(1-\kappa^{-1}\right) k\delta\varepsilon\,(\beta-\alpha) \int_{B_R^{R_0}} \mathbf{u}_R^2 \eta_{\beta-1}^{\alpha-1} s\, d\mathbf{x} - 2C_4 \int_{B_{R_0}} |\nabla \mathbf{u}_R|^2\, \eta_\beta^\alpha d\mathbf{x}
\end{aligned}$$

Denote $C_5 = \alpha\delta k\,(1+\kappa\,(\nu/\kappa)\,\alpha\delta)\,(\kappa/(\delta\,\varepsilon))\,((\alpha+\beta)/(\beta\beta^*))^2$. It is clear that $C_5 \leq \nu/(2\kappa^2) < \nu/(2\kappa)$ if $1 + \nu\kappa\alpha\delta/k \leq \kappa$ (i.e. $\delta \leq (k/\nu)\cdot((\kappa-1))/(\kappa\beta)$) and $\alpha \leq (1/(2\kappa^4))\cdot(\nu/k)\cdot((\beta\,\beta^*)/(\alpha+\beta))^2\,\varepsilon$. We have

$$\begin{aligned}
&\frac{\nu}{2\kappa}\int_{B_R} |\nabla \mathbf{u}_R|^2\ \eta_\beta^\alpha\, d\mathbf{x} + \frac{1}{2}\left(1-\frac{1}{\kappa}\right) k\delta\varepsilon\,(\beta-\alpha)\int_{B_R} \mathbf{u}_R^2 \eta_{\beta-1}^{\alpha-1}\, s\, d\mathbf{x} \\
&-C_6 \int_{B_{R_0}} \mathbf{u}_R^2 \eta_{\beta-1}^{\alpha-1} d\mathbf{x} - C_7 \int_{B_{R_0}} |\nabla \mathbf{u}_R|^2\, \eta_\beta^\alpha d\mathbf{x} \leq \int_{B_R} \left|\widetilde{\mathbf{f}}\right| |\mathbf{u}_R|\, \eta_\beta^\alpha d\mathbf{x}.
\end{aligned}$$

We use now relation (4.28) in order to estimate the integrals computed on the domain B_{R_0}. Before using the mentioned inequality we should re-scale it with respect to new values ε, δ, see Remark 1.1. The new constant in (4.28) after rescaling we denote C_2'.

$$\frac{\nu}{\kappa}\int_{B_R} |\nabla \mathbf{u}_R|^2\, \eta_\beta^\alpha d\mathbf{x} + k\delta\varepsilon\,(\beta-\alpha)\int_{B_R} \mathbf{u}_R^2 \eta_{\beta-1}^{\alpha-1} s d\mathbf{x} \leq C_8 \int_{B_R} \left|\widetilde{\mathbf{f}}\right| |\mathbf{u}_R|\, \eta_\beta^\alpha d\mathbf{x},$$

where $C_8 = \{1 + C_2' \max(C_6,\, C_7)\} \cdot 2 \cdot (1-\kappa^{-1})^{-1}$. We use Lemma 2.3 and Remark 2.4. So, if $\delta < 2\varepsilon$ and $1 < \kappa \leq 2\varepsilon/\delta + \delta/(2\varepsilon) - 1$ we get

$$\frac{\nu}{2\,\kappa}\left(\frac{\beta\,\beta^*\delta\,\varepsilon}{\alpha\delta + 2\beta\varepsilon}\right)^2 \int_{B_R} \mathbf{u}_R^2\, \eta_{\beta-1}^{\alpha-1}\, d\mathbf{x} \;\leq\; \frac{\nu}{2\kappa}\int_{B_R} |\nabla \mathbf{u}_R|^2\, \eta_\beta^\alpha\, d\mathbf{x},$$

$$\begin{aligned}
&\frac{\nu}{2\kappa}\int_{B_R} |\nabla \mathbf{u}_R|^2\, \eta_\beta^\alpha dx + \frac{\nu}{2\kappa}\left(\frac{\beta\,\beta^*\delta\,\varepsilon}{\alpha\delta + 2\beta\varepsilon}\right)^2 \int_{B_R} \mathbf{u}_R^2 \eta_{\beta-1}^{\alpha-1}\, d\mathbf{x} \\
&+k\delta\varepsilon\,(\beta-\alpha)\int_{B_R} \mathbf{u}_R^2 \eta_{\beta-1}^{\alpha-1} s\, d\mathbf{x} \leq C_8 \int_{B_R} \left|\widetilde{\mathbf{f}}\right| |\mathbf{u}_R|\, \eta_\beta^\alpha d\mathbf{x}.
\end{aligned}$$

So we get

$$\begin{aligned}
&\int_{B_R} |\nabla \mathbf{u}_R|^2\, \eta_\beta^\alpha d\mathbf{x} + 2\int_{B_R} \mathbf{u}_R^2 \eta_{\beta-1}^{\alpha-1}\, d\mathbf{x} + 2\varepsilon \int_{B_R} \mathbf{u}_R^2 \eta_{\beta-1}^{\alpha-1} s\, d\mathbf{x} \\
&= \int_{B_R} |\nabla \mathbf{u}_R|^2\, \eta_\beta^\alpha d\mathbf{x} + 2\int_{B_R} \mathbf{u}_R^2 \eta_\beta^{\alpha-1}\, d\mathbf{x} \leq C_{10} \int_{B_R} \left|\widetilde{\mathbf{f}}\right| |\mathbf{u}_R|\, \eta_\beta^\alpha d\mathbf{x},
\end{aligned}$$

$C_9 = \min\left(\nu/(2\kappa), (\nu/(2\kappa))\ (\beta\beta^*\delta\varepsilon/(\alpha\delta + 2\beta\varepsilon))^2, k\delta(\beta-\alpha)\ /2\right)$ and $C_{10} = C_8/C_9$. We have also:

$$\int_{B_R} \left|\widetilde{\mathbf{f}}\right| |\mathbf{u}_R| \eta_\beta^\alpha dx \leq \frac{t}{2}\int_{B_R} \mathbf{u}_R^2 \eta_\beta^{\alpha-1} d\mathbf{x} + \frac{1}{2t}\int_{B_R} \left|\widetilde{\mathbf{f}}\right|^2 \eta_\beta^{\alpha+1} dx$$

So, if we choose $t = 2 \cdot C_{10}^{-1}$ then we get :

$$\int_{B_R} |\nabla \mathbf{u}_R|^2 \eta_\beta^\alpha d\mathbf{x} + \int_{B_R} \mathbf{u}_R^2 \eta_\beta^{\alpha-1} d\mathbf{x} \leq c \int_{\mathbb{R}^3} \left|\widetilde{\mathbf{f}}\right|^2 \eta_\beta^{\alpha+1} d\mathbf{x},$$

It can be easily shown that the all conditions on α, β, δ, ε, κ used in the proof are compatible if $0 \leq \alpha < y_1\beta$, see Appendix B. □

4.4 The problem in $\mathbb{R}^3$ - solenoidal solutions

Let y_1 be the same as in Lemma 4.3.

Theorem 4.4 *(Existence and uniqueness in $\mathbb{R}^3$) Let $0 < \beta \leq 1$, $0 \leq \alpha < y_1\beta$, $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}$. Then there exists a unique weak solution $\{\mathbf{u}, p\}$ of the problem*

$$\begin{aligned} -\nu\,\Delta\mathbf{u} + k\partial_1\mathbf{u} - (\omega\times\mathbf{x})\cdot\nabla\mathbf{u} + \omega\times\mathbf{u} + \nabla\mathbf{p} &= \mathbf{f} \ \ in\ \mathbb{R}^3, &(4.29)\\ div\,\mathbf{u} &= 0 \ \ in\ \mathbb{R}^3 &(4.30)\end{aligned}$$

such that $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}$, $p \in L^2_{\alpha,\beta-1}$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}$ and

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla\mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \leq C\,\|\mathbf{f}\|_{2,\alpha+1,\beta}\,. \tag{4.31}$$

Proof. Existence. Let p be the same as in Subsection 4.1. Let $\{R_n\}$ be a sequence of positive real numbers, converging to $+\infty$. Let $\mathbf{u}_{R_n}$ be the weak solution of (4.20), (4.21) on B_{R_n}. Extending $\mathbf{u}_{R_n}$ by zero on $\mathbb{R}^3 \setminus B_{R_n}$ to a function $\tilde{\mathbf{u}}_\mathbf{n} \in \mathbf{V}_{\alpha,\beta}$ we get a bounded sequence $\{\tilde{\mathbf{u}}_\mathbf{n}\}$ in $\mathbf{V}_{\alpha,\beta}$. Thus, there is a subsequence $\tilde{\mathbf{u}}_{\mathbf{n_k}}$ of $\tilde{\mathbf{u}}_\mathbf{n}$ with a weak limit $\mathbf{u}$ in $\mathbf{V}_{\alpha,\beta}$. Obviously, $\mathbf{u}$ is a weak solution of (4.29) and

$$\begin{aligned}\|\mathbf{u}\|^2_{2,\alpha-1,\beta} + \|\nabla\mathbf{u}\|^2_{2,\alpha,\beta} &\leq \liminf_{k\in\mathbb{N}}\left(\int_{\mathbb{R}^3} \tilde{\mathbf{u}}^2_{n_k}\eta_\beta^{\alpha-1}\,d\mathbf{x} + \int_{\mathbb{R}^3} |\nabla\tilde{\mathbf{u}}_{n_k}|^2\,\eta_\beta^\alpha\,d\mathbf{x}\right)\\ &\leq c\left|\widetilde{\mathbf{f}}\right|^2 \eta_\beta^{\alpha+1}\,d\mathbf{x} = c\int_{\mathbb{R}^3} |\mathbf{f}-\nabla p|^2\,\eta_\beta^{\alpha+1}\,d\mathbf{x}.\end{aligned}$$

Taking into account also relation (4.19) we get (4.31).

Let us also check that for $\mathbf{u}$ the equation (4.30) is satisfied. Let us mention that $\mathbf{u} \in \mathbf{H}^2_{loc}$ because $\mathbf{f} - \nabla p \in \mathbf{L}^2_{\alpha+1,\,\beta}$. So, computing the divergence of (4.29) we get

$$-\nu\,\Delta\,(\mathrm{div}\,\mathbf{u}) + k\partial_1\,(\mathrm{div}\,\mathbf{u}) - (\omega\times\mathbf{x})\cdot\nabla\,(\mathrm{div}\,\mathbf{u}) = \mathrm{div}\,\mathbf{f} - \triangle\mathbf{p} \tag{4.32}$$

in distributional sense. From (4.18) and (4.31) we have

$$-\nu\,\Delta\gamma + k\partial_1\gamma - (\omega\times\mathbf{x})\cdot\nabla\gamma = 0$$

for $\gamma = \operatorname{div} \mathbf{u} \in L^2_{\alpha,\beta} \subset L^2$. Using Fourier transform we get

$$\left(\nu\, |\xi|^2 + i\, k\, \xi_1\right) \widehat{\gamma} - (\omega \times \xi) \cdot \nabla_\xi \widehat{\gamma} = 0 \quad \text{in } \ \mathcal{S}'.$$

Assuming $\widehat{\gamma}$ in cylindrical coordinates $[\xi_1, \rho, \varphi]$, $\rho = (\xi_2^2 + \xi_3^2)^{1/2}$, we can overwrite the equation in the form:

$$-\partial_\varphi \widehat{\gamma} + \left[(\nu/\widetilde{\omega})\ |\xi|^2 + i\ (k/\widetilde{\omega})\ \xi_1\right] \widehat{\gamma} = 0.$$

Using the same approach as in the proof of the uniqueness Theorem 3.1 we prove that $\operatorname{supp} \widehat{\gamma} \subset \{0\}$. The proof of this fact is reduced to the solvability of the equation (3.17) which was proved for arbitrary $\Psi \in C_0^\infty(\mathbb{R}^3 \setminus \{0\})$ in the proof of Theorem 3.1. So, by the same procedure we derive that γ is a polynomial in $\mathbb{R}^3$ and because $\gamma \in L^2$ we get $\gamma \equiv 0$, i.e. (4.30). The uniqueness of the solution follows from Theorem 3.1.

□

4.5 The problem in $\mathbb{R}^3$ with non-zero divergence

First of all let us formulate the lemma which will be used for the extension of our results to the case with nonzero divergence:

Lemma 4.5 *(M.E. Bogovski, G.P. Galdi, H. Sohr)*

Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, be a bounded Lipschitz domain, and $1 < q < \infty$, $n \in \mathbb{N}$. Then for each $g \in W_0^{k,q}(\Omega)$ with $\int_\Omega g\, dx = 0$, there exists $\mathbf{G} \in \left(W_0^{k+1,q}(\Omega)\right)^n$ satisfying

$$\mathit{div}\, \mathbf{G} = g\,, \qquad \|\mathbf{G}\|_{\left(W_0^{k+1,q}(\Omega)\right)^n} \leq C\, \|g\|_{W_0^{k,q}(\Omega)}$$

with some constant $C = C(q, k, \Omega) > 0$.

For the proof and further references see e.g. [29, Lemma 2.3.1]. We will prove the following theorem:

Theorem 4.6 *(Existence and uniqueness in $\mathbb{R}^3$) Let $0 < \beta \leq 1$, $0 \leq \alpha < y_1\beta$, $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}$, $g \in W_0^{1,2}$ with $\operatorname{supp} g = K \subset\subset \mathbb{R}^3$, and $\int_{\mathbb{R}^3} g\, dx = 0$. Then there exists a unique weak solution $\{\mathbf{u}, p\}$ of the problem*

$$\begin{aligned} -\nu\, \Delta \mathbf{u} + k\partial_1 \mathbf{u} - (\omega \times \mathbf{x}) \cdot \nabla \mathbf{u} + \omega \times \mathbf{u} + \nabla \mathbf{p} &= \mathbf{f} \ \ \mathit{in}\ \mathbb{R}^3, \\ \mathit{div}\, \mathbf{u} &= g \ \ \mathit{in}\ \mathbb{R}^3 \end{aligned}$$

such that $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}$, $p \in L^2_{\alpha,\beta-1}$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}$ and

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla \mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \leq C \left(\|\mathbf{f}\|_{2,\alpha+1,\beta} + \|g\|_{1,2}\right).$$

Proof. Using Lemma 4.5 we find $\mathbf{G} \in \mathbf{W}_0^{2,2}$, $\operatorname{supp} \mathbf{G} \subset \mathcal{K}$, where $\mathcal{K}$ is a bounded Lipschitz domain containing in $\varepsilon-$neighbourhood $\mathcal{K}_\varepsilon$ of compact set K for an arbitrary $\varepsilon > 0$, $\operatorname{div} \mathbf{G} = g$, $\|\mathbf{G}\|_{2,2} \leq C\, \|g\|_{1,2}$. Let us assume the following problem

$$\begin{aligned} -\nu\, \Delta \mathbf{U} + k\, \partial_1 \mathbf{U} - (\omega \times \mathbf{x}) \cdot \nabla \mathbf{U} + \omega \times \mathbf{U} + \nabla p &= \mathbf{F} \qquad \text{in } \mathbb{R}^3 \\ \operatorname{div} \mathbf{U} &= 0 \qquad \text{in } \mathbb{R}^3 \end{aligned}$$

where $\mathbf{U} = \mathbf{u} - \mathbf{G}$, $\mathbf{F} = \mathbf{f} + \nu\, \Delta \mathbf{G} - k\, \partial_1 \mathbf{G} + (\omega \times \mathbf{x}) \cdot \nabla \mathbf{G} - \omega \times \mathbf{G}$ with $\mathbf{G} \in \mathbf{W}_0^{2,2}$, function $\mathbf{G}$ has a compact support, and $\|\mathbf{G}\|_{2,2} \leq C\, \|g\|_{1,2}$. The assertion of Theorem 4.6 follows from Theorem 4.4. □

5 Uniqueness in an exterior domain $\Omega \subset \mathbb{R}^3$

The last two sections are devoted to the problem in an exterior domain. We start with the question of uniqueness. The uniqueness theorem proved in this section together with the uniqueness theorem in $\mathbb{R}^3$ from Section 3 will be used in the next section in the proof of the existence of a solution in an exterior domain, in the localization procedure. The homogenous Dirichlet boundary condition on $\partial\Omega$ for $\mathbf{u}$ in the next theorem follows from the assumption $\mathbf{u} \in \mathbf{V}_{0,0}(\Omega)$.

Theorem 5.1 *Let $\{\mathbf{u}, p\}$ be a distributional solution of the problem (1.1)–(1.3) with $\mathbf{f} = \mathbf{0}$ and $g = 0$ such that $\mathbf{u} \in \mathbf{V}_{0,0}(\Omega)$ and $p \in L^2_{-1,0}(\Omega)$. Then $\mathbf{u} = \mathbf{0}$ and $p = 0$.*

Proof. Let $\Phi = \Phi(z) \in C_0^\infty(\langle 0, +\infty))$ be a non-increasing cut-off function such that $\Phi(z) \equiv 1$ for $z < 1/2$ and $\Phi(z) \equiv 0$ for $z > 1$. Let $|\Phi'| \leq 3$. Let $\Phi_R \equiv \Phi_R(\mathbf{x}) \equiv \Phi(|\mathbf{x}|/R)$. We have $|\nabla \Phi_R| \leq 3/R$ and $|\partial_1 \Phi_R| \leq 3/R$ for $\mathbf{x} \in \mathbb{R}^3$, $R/2 \leq |\mathbf{x}| \leq R$. Let $\{R_j\} \in \mathbb{R}$ be an increasing sequence of radii with the limit $+\infty$. So we have that $\mathbf{u}_j \equiv \mathbf{u} \cdot \Phi_{R_j} \in \overset{\circ}{\mathbf{H}}{}^1(\Omega)$, and $\{\mathbf{u}_j\}$ is a sequence of functions with limit $\mathbf{u}$ in the space $\mathbf{V}_{0,0}(\Omega)$. Using the (non-solenoidal) test functions $\varphi = \mathbf{u}\,\Phi^2_{R_j} = \mathbf{u}_j\,\Phi_{R_j} \in \overset{\circ}{\mathbf{H}}{}^1(\Omega)$ for equation (1.1) we get:

$$\begin{aligned} &\nu \int_\Omega \nabla \mathbf{u} \cdot \nabla \left(\mathbf{u}\,\Phi^2_{R_j}\right) d\mathbf{x} + k \int_\Omega \partial_1 \mathbf{u} \cdot \mathbf{u}\,\Phi^2_{R_j}\, d\mathbf{x} \\ &+ \int_\Omega (\omega \times \mathbf{x}) \cdot \nabla \mathbf{u} \cdot \mathbf{u}\,\Phi^2_{R_j}\, d\mathbf{x} + \int_\Omega \nabla p \cdot \mathbf{u}\,\Phi^2_{R_j}\, d\mathbf{x} = 0 \end{aligned} \tag{5.33}$$

Using in (5.33) relation $\nabla \mathbf{u} \cdot \nabla \left(\mathbf{u}\,\Phi^2_{R_j}\right) = |\nabla \mathbf{u}_j|^2 - \nabla \Phi_{R_j} \cdot \nabla \Phi_{R_j}\, \mathbf{u}^2$, integrating by parts, we get after some evident rearrangements

$$\begin{aligned} &\nu \int_\Omega |\nabla \mathbf{u}_j|^2\, d\mathbf{x} - \frac{1}{2} \int_\Omega \operatorname{div}(\omega \times \mathbf{x})\, \mathbf{u}_j^2\, d\mathbf{x} \\ &- \frac{k}{2} \int_\Omega \mathbf{u}^2\, \partial_1 \Phi^2_{R_j} d\mathbf{x} - \frac{1}{2} \int_\Omega \mathbf{u}^2\, (\omega \times \mathbf{x}) \cdot \nabla \Phi^2_{R_j} d\mathbf{x} \\ &- \nu \int_\Omega \left|\nabla \Phi_{R_j}\right|^2\, \mathbf{u}^2\, d\mathbf{x} - \int_\Omega p\mathbf{u} \cdot \nabla \left(\Phi^2_{R_j}\right) d\mathbf{x} = 0. \end{aligned}$$

$$\nu \int_\Omega |\nabla \mathbf{u}_j|^2\, d\mathbf{x} \leq C \left(\int_{\Omega^{R_j/2}_{R_j}} \mathbf{u}^2 r^{-1}\, d\mathbf{x} + \int_{\Omega^{R_j/2}_{R_j}} |p|\, |\mathbf{u}|\, r^{-1}\, d\mathbf{x} \right).$$

$\mathbf{u} \in \mathbf{L}^2_{-1,0}(\Omega)$, $p \in L^2_{-1,0}(\Omega)$, $p\mathbf{u} \in \mathbf{L}^1_{-1,0}(\Omega)$. So, for $j \to \infty$ we get $\int_\Omega |\nabla \mathbf{u}|^2\, d\mathbf{x} \leq 0$. Hence, the function $\nabla \mathbf{u} = 0$ a.e. in Ω, and this means $\mathbf{u}$ is a constant a.e. in Ω. From $\mathbf{u} \in \mathbf{L}^2_{-1,0}(\Omega)$ follows that $\mathbf{u} = \mathbf{0}$ a.e. in Ω. Using now an arbitrary test function ϕ for equation (1.1), we get $\int_\Omega \nabla p\, \phi\, d\mathbf{x} = 0$. So, the function $\nabla p = 0$ a.e. in Ω, and this means p is a constant a.e. in Ω. From $p \in L^2_{-1,0}(\Omega)$ follows that $p = 0$ a.e. in Ω, and the uniqueness is proved. □

6 Existence of solution in exterior domains

In this section we assume problem (1.1)-(1.4) in an exterior domain Ω. First we assume the case of the homogenous Dirichlet boundary condition on $\partial\Omega$.

6.1 Homogenous Dirichlet boundary conditions

Function g is assumed to be zero, and $\mathbf{f} = \operatorname{div}\mathbf{F}$ with $\mathbf{F} \in C_0^\infty(\Omega)^9$. We will prove that the problem has a weak solution $\{\mathbf{u}, p\} \in \widehat{\mathbf{H}}_0^1(\Omega) \times L^2_{loc}(\Omega)$. So we assume the following sequence of problems on domains $\Omega_R = B_R \cap \Omega$:

$$-\nu\,\Delta\mathbf{u}_R + k\,\partial_1\mathbf{u}_R + (\omega\times\mathbf{x})\cdot\nabla\mathbf{u}_R - \omega\times\mathbf{u}_R + \nabla p_R = \operatorname{div}\mathbf{F} \quad \text{in } \Omega_R \tag{6.34}$$

$$\operatorname{div}\mathbf{u}_R = 0 \quad \text{in } \Omega_R \tag{6.35}$$

$$\mathbf{u}_R = 0 \quad \text{on } \partial\Omega_R \tag{6.36}$$

Using (Girault-Raviart [15]) mixed variational approach we formulate the problem in the following form: To find $\{\mathbf{u}_R, p_R\} \in \mathbf{W}_R \times \Pi_R$, such that for all $\mathbf{v} \in \mathbf{W}_R$, $\pi \in \Pi_R$:

$$a(\mathbf{u}_R,\,\mathbf{v}) + b(\mathbf{v},\,p_R) = \langle \operatorname{div}\mathbf{F}, \mathbf{v}\rangle \tag{6.37}$$

$$b(\mathbf{u}_R, \pi) = 0, \tag{6.38}$$

where $\mathbf{W}_R = \widehat{\mathbf{H}}_0^1(\Omega_R)$,

$$\Pi_R = \left\{\pi \in L^2(\Omega_R)\,;\ \int_{\Omega_R} \pi\,d\mathbf{x} = 0\right\}$$

with usual norms $|\phi|_{W_R} = \|\nabla\phi\|_2$, $\|\pi\|_{\Pi_R} = \|\pi\|_2$ and

$$\begin{aligned} a(\phi,\,\psi) &= \nu\int_{\Omega_R} \nabla\phi\cdot\nabla\psi\,d\mathbf{x} + k\int_{\Omega_R}\partial_1\phi\cdot\psi\,d\mathbf{x} \\ &\quad + \int_{\Omega_R} [(\omega\times\mathbf{x})\cdot\nabla\phi - \omega\times\phi]\cdot\psi\,d\mathbf{x} \\ b(\phi,\,\pi) &= -\int_{\Omega_R} \operatorname{div}\phi\cdot\pi\,d\mathbf{x}. \end{aligned}$$

These bilinear forms are continuous on $\mathbf{W}_R \times \mathbf{W}_R$ and $\mathbf{W}_R \times \Pi_R$, respectively. It is easy to see that $a(\phi,\,\phi) \geq \nu\,\|\phi\|^2_{\mathbf{W}_R}$, and it is known that

$$\sup_{\mathbf{v}\in\mathbf{W}_R} \frac{(\pi,\,\operatorname{div}\mathbf{v})}{|\mathbf{v}|_{\mathbf{W}_R}} \geq C_0\,\|\pi\|_{\Pi_R}$$

for some $C_0 = C_0(R) > 0$. Hence, there exists a weak solution $\{\mathbf{u}_R, p_R\}$ of the problem and $\|\mathbf{u}_R\|_{\mathbf{W}_R} + \|p_R\|_{\Pi_R} \leq C_1\,\|\operatorname{div}\mathbf{F}\|_{-1}$ for some $C_1 = C_1(R) > 0$. Testing now (6.37) by $\mathbf{v} = \mathbf{u}_R$ we get:

$$\nu\int_{\Omega_R} |\nabla\mathbf{u}_R|^2\,d\mathbf{x} = \int_{\Omega_R} (\operatorname{div}\mathbf{F})\cdot\mathbf{u}_R\,d\mathbf{x} = \int_{\Omega_R} \mathbf{F}\cdot\nabla\mathbf{u}_R\,d\mathbf{x} \leq \|\mathbf{F}\|_2\,\|\nabla\mathbf{u}_R\|_2$$

$$\|\nabla \mathbf{u}_R\|_2 \leq \nu^{-1} \|\mathbf{F}\|_2 \tag{6.39}$$

Since the a priori estimate (6.39) is available, where $\mathbf{u}_R$ is understood as its extension by setting zero in $\Omega \setminus \Omega_R$, there exists $\mathbf{u} \in \widehat{\mathbf{H}}_0^1(\Omega)$ and a sequence $\{R_n\} \to \infty$ so that $\mathbf{u}_{R_n} \rightharpoonup \mathbf{u}$ weakly in $\widehat{\mathbf{H}}_0^1(\Omega)$ as $n \to \infty$.

Let us show that $\operatorname{div} \mathbf{u} = 0$ in $L^2(\Omega)$. From the same inequality follows the weak convergence of $\operatorname{div} \mathbf{u}_{R_n}$ in $L^2(\Omega)$. From (6.38) we get $\operatorname{div} \mathbf{u}_{R_n} \equiv C_n$ on Ω_{R_n} for some real constant C_n depending on n. In spite of (6.39) we get that the weak limit of $\operatorname{div} \mathbf{u}_{R_n}$ is zero in $L^2(\Omega)$.

Finally, for all $\phi \in \mathbf{C}_0^\infty(\Omega)$ with $\operatorname{div} \phi = 0$ we have from (6.37) after $R_n \to \infty$

$$\langle L\mathbf{u} - \operatorname{div} \mathbf{F}, \phi \rangle = 0,$$

$$L\mathbf{u} \equiv -\nu \Delta \mathbf{u} + k \partial_1 \mathbf{u} + (\omega \times \mathbf{x}) \cdot \nabla \mathbf{u} - \omega \times \mathbf{u}.$$

By a result of de Rham, there is a distribution p such that $-\nabla p = L\mathbf{u} - \operatorname{div} \mathbf{F}$ in $\mathcal{D}'(\Omega)$. Because the right-hand side belongs to $H^{-1}(\Omega_R)$ for every sufficiently large $R > 0$ we have that $p \in L^2(\Omega_R)$ and so, $p \in L^2_{loc}(\Omega)$.

Now we use the following

Lemma 6.1 *(Kozono and Sohr [20, Lemma 2.2, Corollary 2.3]) Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be any domain and let $1 < q < \infty$. For all $g \in \widehat{W}^{-1,q}(\Omega)$, there is $G \in L^q(\Omega)^n$ such that*

$$\text{div } G = g, \qquad \|G\|_{q,\Omega} \leq C\|g\|_{-1,q,\Omega}$$

with some $C > 0$. As a result, the space $\{\text{div } G;\ G \in C_0^\infty(\Omega)^n\}$ is dense in $\widehat{W}^{-1,q}(\Omega)$.

Hence, we get the existence of solution $\{\mathbf{u}, p\} \in \widehat{\mathbf{H}}_0^1(\Omega) \times L^2_{loc}(\Omega)$ for an arbitrary function $\widetilde{\mathbf{f}} \in \widehat{\mathbf{H}}^{-1}(\Omega)$.

For the extension of Theorem 4.4 to the case of an exterior domain we use the localization procedure, see [20]. Let now $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$. We define for an arbitrary $R > 0$:

$$\mathbf{f}_R = \begin{cases} \mathbf{f}, & \mathbf{x} \in \Omega_R \\ \mathbf{0}, & \mathbf{x} \in \Omega \setminus \Omega_R \end{cases}$$

It can be shown that $\mathbf{f}_R$ belongs to $\widehat{\mathbf{H}}^{-1}(\Omega) \cap \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$. By use of cut-off function Ψ we decompose the solution $\{\mathbf{u}, p\}$ of the problem (1.1)-(1.4) (with the homogenous Dirichlet boundary condition) on the solution of the problem in $\mathbb{R}^3$ and the solution of the Stokes problem in a bounded domain:

$$\begin{aligned} \mathbf{u} &= \mathbf{U} + \mathbf{V} \quad \text{where } \mathbf{U} = (1 - \Psi)\mathbf{u}, \quad \mathbf{V} = \Psi \mathbf{u} \\ p &= \sigma + \tau \quad \text{where } \sigma = (1 - \Psi)p, \quad \tau = \Psi p, \end{aligned}$$

where $\Psi \in C_0^\infty$, $\operatorname{supp} \Psi \subset\subset B_{\rho_1}$ such that $\Psi \equiv 1$ on B_{ρ_0}, $0 < \rho_0 < \rho_1 < \rho$ so that $\mathbb{R}^3 \setminus \Omega \subset B_{\rho_0}$. We get that $\{\mathbf{U}, \sigma\}$ is a weak solution of the modified Oseen problem in $\mathbb{R}^3$

$$-\nu \triangle \mathbf{U} + k \partial_1 \mathbf{U} + (\omega \times \mathbf{x}) \cdot \nabla \mathbf{U} - \omega \times \mathbf{U} + \nabla \sigma = \mathbf{Z_1} \tag{6.40}$$

$$\operatorname{div} \mathbf{U} = -\nabla \Psi \cdot \mathbf{u} \tag{6.41}$$

and $\{\mathbf{V},\tau\}$ is weak solution of the Stokes problem in a bounded domain Ω_ρ

$$\begin{aligned} -\nu\triangle\mathbf{V}+\nabla\tau &= \mathbf{Z_2} \quad \text{in} \quad \Omega_\rho &(6.42)\\ \operatorname{div}\mathbf{V} &= \nabla\Psi\cdot\mathbf{u} \quad \text{in} \quad \Omega_\rho &(6.43)\\ \mathbf{V}|_{\partial\Omega_\rho} &= 0 &(6.44)\end{aligned}$$

where the right-hand sides are given by $\mathbf{Z}_1$ and $\mathbf{Z}_2$.

$$\begin{aligned}\mathbf{Z}_1 &= 2\nabla\Psi\cdot\nabla\mathbf{u}+\mathbf{u}\triangle\Psi-k\partial_1\Psi\,\mathbf{u}-(\nabla\Psi\cdot(\omega\times\mathbf{x}))\,\mathbf{u}-\nabla\Psi\,p+(1-\Psi)\,\mathbf{f}_R,\\ \mathbf{Z}_2 &= -2\nabla\Psi\cdot\nabla\mathbf{u}-\mathbf{u}\triangle\Psi+k\partial_1\Psi\,\mathbf{u}-\Psi\,[(\omega\times\mathbf{x})\cdot\nabla\mathbf{u}-\omega\times\mathbf{u}]+\nabla\Psi\,p+\Psi\,\mathbf{f}_R.\end{aligned}$$

Let us mention that $\mathbf{Z}_1\in\mathbf{L}^2_{\alpha+1,\beta}(\Omega)$. To solve the Stokes problem on the bounded domain we use the following lemma, see [20]:

Lemma 6.2 *(The Stokes problem on a bounded domain) Let Ω be a bounded domain of $\mathbb{R}^n$, $n\geq 2$, of class C^{m+2}, $m\geq 0$. For any*

$$\mathbf{f}\in\mathbf{W}^{m,q}(\Omega),\qquad g\in W^{m+1,q}(\Omega),\qquad \mathbf{v}_*\in\mathbf{W}^{m+2-1/q,q}(\partial\Omega),$$

$1<q<\infty$, *with*

$$\int_{\partial\Omega}\mathbf{v}_*\cdot\mathbf{n}\,dS=\int_\Omega g\,d\mathbf{x}, \tag{6.45}$$

there exists one and only one solution $\{\mathbf{V},\tau\}$ to the Stokes system

$$\begin{aligned}-\triangle\mathbf{V}+\nabla\tau &= \mathbf{f} \quad \textit{in} \quad \Omega\\ \textit{div}\,\mathbf{V} &= g \quad \textit{in} \quad \Omega\\ \mathbf{V} &= \mathbf{v}_* \quad \textit{on} \quad \partial\Omega\end{aligned}$$

such that $\mathbf{V}\in\mathbf{W}^{m+2,q}(\Omega)$, $\tau\in W^{m+1,q}(\Omega)$ and

$$\|\mathbf{V}\|_{m+2,q}+\|\tau-\overline{\tau}\|_{m+1,q}\leq c\left(\|\mathbf{f}\|_{m,q}+\|\mathbf{v}_*\|_{m+2-1/q,q}+\|g\|_{m+1,q}\right), \tag{6.46}$$

where $\overline{\tau}=|\Omega|^{-1}\int_\Omega\tau d\mathbf{x}$ and $c=c(m,n,q,\Omega)$.

Furthermore, for Ω of class C^2, for every

$$\mathbf{f}\in\mathbf{W}_0^{-1,q}(\Omega),\qquad g\in L^q(\Omega),\qquad \mathbf{v}_*\in\mathbf{W}^{1-1/q,q}(\partial\Omega),$$

$1<q<\infty$, *with (6.45) there exists one and only one q-generalized solution $\{\mathbf{V},\tau\}$ to the Stokes system such that $\mathbf{V}\in\mathbf{W}^{1,q}(\Omega)$, $\tau\in L^q(\Omega)$ and the estimate (6.46) is valid with $m=-1$.*

From the results about the existence and uniqueness of solutions of the Oseen problem in $\mathbb{R}^3$ (6.40), (6.41), i.e. from Theorem 4.4 and Theorem 3.1 follows , that a solution $\{\mathbf{U},\sigma\}$ is subject of the estimate (4.31), with $\mathbf{f}$ and g replaced by $\mathbf{Z}_1$ and $-\nabla\Psi\cdot\mathbf{u}$, respectively. Using also the respective results in a bounded domain for (6.42) - (6.44), see Lemma 6.2 with $m=0$ and bounded domain Ω_ρ, we get the following lemma for an exterior domain:

Lemma 6.3 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain and $0 < \beta \leq 1$, $0 \leq \alpha < y_1 \cdot \beta$; y_1 is given in Lemma 4.3. Then there exists a weak solution $\{\mathbf{u}, p\}$ of the problem (1.1) - (1.3) with the homogenous Dirichlet boundary condition, $\mathbf{f} := \mathbf{f}_R$ and $g = 0$, such that $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}(\Omega)$, $p \in L^2_{\alpha,\beta-1}(\Omega)$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$ and*

$$\begin{aligned} &\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla \mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \\ &\qquad \leq C_1 \left(\|\mathbf{f}_R\|_{2,\alpha+1,\beta} + \|\mathbf{u}\|_{1,2;\,A_\rho} + \|p\|_{0,2;\,\Omega_\rho} \right), \end{aligned} \tag{6.47}$$

where $A_\rho := B_\rho \setminus B_{\rho/2}$, and constant C_1 does not depend on R.

Now, we would like to show that the preceding estimate is valid (with another constant) also if we add to the left-hand side the L^2-norm of second gradient of $\mathbf{u}$ on some compact subset of Ω. Taking into account the assertion of Lemma 6.2 for $m = 0$, we get that $\mathbf{u} \in \mathbf{W}^{2,2}_{loc}(\Omega)$, $p \in W^{1,2}_{loc}(\Omega)$. Multiplying the relation (1.1) - (1.4) in an exterior domain Ω (with $g = 0$ and the homogenous Dirichlet boundary condition on $\partial\Omega$) by $\Delta \mathbf{u}$ and integrating over the compact set K_1 with $A_\rho \subset K_1 \subset \Omega$ we get

$$\|\Delta \mathbf{u}\|_{2;K_1} \leq C_2 \left(\|\mathbf{u}\|_{2;K_1} + \|\nabla \mathbf{u}\|_{2;K_1} + \|p\|_{2;K_1} + \|\nabla p\|_{2;K_1} \right). \tag{6.48}$$

Using (6.47), (6.48) and the known relation

$$\|\nabla^2 \mathbf{u}\|_{2;K} \leq c \left(\|\Delta \mathbf{u}\|_{2;K_1} + \|\nabla \mathbf{u}\|_{2;K_1} \right)$$

with $A_\rho \subset K \subset K_1$, we get

Corollary 6.4 *In conditions of Lemma 6.3 the following estimate is valid:*

$$\begin{aligned} &\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla \mathbf{u}\|_{2,\alpha,\beta} + \left\|\nabla^2 \mathbf{u}\right\|_{2;\,A_\rho} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \\ &\qquad \leq C \left(\|\mathbf{f}_R\|_{2,\alpha+1,\beta} + \|\mathbf{u}\|_{1,2;\,A_\rho} + \|p\|_{0,2;\Omega_\rho} \right) \end{aligned} \tag{6.49}$$

Now, we will prove that the estimate (6.49) is valid without the right-hand side terms containing $\mathbf{u}$ and p, i.e. we will prove:

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla \mathbf{u}\|_{2,\alpha,\beta} + \left\|\nabla^2 \mathbf{u}\right\|_{2;\,A_\rho} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \leq c \,\|\mathbf{f}_R\|_{2,\alpha+1,\beta} \tag{6.50}$$

Let us define the Hilbert spaces H_1, H_2 with norms $\|\cdot\|_{(1)}$, $\|\cdot\|_{(2)}$, respectively:

$$\begin{aligned} \|\,(\mathbf{v}, q)\,\|_{(1)} &:= \|\mathbf{v}\|_{1,2;A_\rho} + \|q\|_{0,2;\Omega_\rho} \\ \|\,(\mathbf{v}, q)\,\|_{(2)} &:= \|\mathbf{v}\|_{2,\alpha-1,\beta} + \|\nabla \mathbf{v}\|_{2,\alpha,\beta} + \|\nabla^2 \mathbf{v}\|_{2;\,A_\rho} + \|q\|_{2,\alpha,\beta-1} + \|\nabla q\|_{2,\alpha+1,\beta} \end{aligned}$$

We have $H_2 \hookrightarrow\hookrightarrow H_1$. Let us assume that the estimate (6.50) is not true. This means that there is a sequence of functions $\left\{\mathbf{f}^{(k)}_R\right\}_{k=1}^{\infty}$, a sequence of corresponding solutions $\{(\mathbf{u}_k, p_k)\}_{k=1}^{\infty}$ and a sequence of constants $\{c_k\}_{k=1}^{\infty} \to \infty$ such that

$$\begin{aligned} 1 &\equiv \|\mathbf{u}_k\|_{2,\alpha-1,\beta} + \|\nabla \mathbf{u}_k\|_{2,\alpha,\beta} + \|\nabla^2 \mathbf{u}_k\|_{2;\,A_\rho} + \|p_k\|_{2,\alpha,\beta-1} + \|\nabla p_k\|_{2,\alpha+1,\beta} \\ &\equiv \|(\mathbf{u}_k, p_k)\|_{(2)} \geq c_k \left\|\mathbf{f}^{(k)}_R\right\|_{2,\alpha+1,\beta} \end{aligned}$$

So we get $\left\{ \left\| \mathbf{f}_R^{(k)} \right\|_{2,\alpha+1,\beta} \right\} \to 0$. The sequence $\{(\mathbf{u}_k, p_k)\}_{k=1}^{\infty}$ is bounded in the norm $\|\cdot\|_{(2)}$, so there is a subsequence of this sequence (we will denote this subsequence using the same notation) with the weak limit $(\mathbf{u}, p)$ in the corresponding Hilbert space H_2. Because $H_2 \hookrightarrow\hookrightarrow H_1$, we have $\| (\mathbf{u}_k, p_k) \|_{(1)} \to 0$. So, $(\mathbf{u}, p)$ is a solution of the problem with the zero right-hand side. Due to uniqueness given by the Theorem 5.1 we conclude that $\| (\mathbf{u}, p) \|_{(2)} = 0$. From the Corollary 6.4 we also get

$$\| (\mathbf{u} - \mathbf{u}_k, p - p_k) \|_{(2)} \to 0,$$

i.e. $\{(\mathbf{u}_k, p_k)\}_{k=1}^{\infty}$ converges strongly in H_2. Because $\| (\mathbf{u}_k, p_k) \|_{(2)} = 1$ for $k \in \mathbb{N}$, so we also get $\| (\mathbf{u}, p) \|_{(2)} = 1$. This is the contradiction.

Let us also mention that the constant C does not depend on R, so we can also extend the result for an arbitrary $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$. So, we proved the following

Theorem 6.5 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain and $0 < \beta \le 1$, $0 \le \alpha < y_1 \cdot \beta$; y_1 is given in Lemma 4.3, $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$. Then there exists a weak solution $\{\mathbf{u}, p\}$ of the problem (1.1) - (1.3) with the homogenous Dirichlet boundary condition on $\partial\Omega$, $g = 0$, such that $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}(\Omega)$, $p \in L^2_{\alpha,\beta-1}(\Omega)$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$ and*

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla\mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \le C \|\mathbf{f}\|_{2,\alpha+1,\beta}.$$

As in the whole space we can prove the following extension for the case $g \neq 0$:

Corollary 6.6 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain and $0 < \beta \le 1$, $0 \le \alpha < y_1 \cdot \beta$; y_1 is given in Lemma 4.3, $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$, $g \in W_0^{1,2}(\Omega)$, with $\operatorname{supp} g = K \subset\subset \Omega$ and $\int_\Omega g \, d\mathbf{x} = 0$. Then there exists a weak solution $\{\mathbf{u}, p\}$ of the problem (1.1) - (1.3) with the homogenous boundary condition on $\partial\Omega$ such that $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}(\Omega)$, $p \in L^2_{\alpha,\beta-1}(\Omega)$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$ and*

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla\mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \le C \left(\|\mathbf{f}\|_{2,\alpha+1,\beta} + \|g\|_{1,2} \right).$$

6.2 Non-homogenous Dirichlet boundary conditions

We assume problem (1.1) - (1.4) in an exterior domain Ω with, in general, $g \neq 0$.

Theorem 6.7 *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain and $0 < \beta \le 1$, $0 \le \alpha < y_1 \cdot \beta$; y_1 is given in Lemma 4.3, $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$, $g \in W_0^{1,2}(\Omega)$, with $\operatorname{supp} g = K \subset\subset \Omega$ and $\int_\Omega g \, d\mathbf{x} = 0$. Then there exists a weak solution $\{\mathbf{u}, p\}$ of the problem (1.1) - (1.4) such that $\mathbf{u} \in \mathbf{V}_{\alpha,\beta}\left(\overline{\Omega}\right)$, $p \in L^2_{\alpha,\beta-1}(\Omega)$, $\nabla p \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$ and*

$$\|\mathbf{u}\|_{2,\alpha-1,\beta} + \|\nabla\mathbf{u}\|_{2,\alpha,\beta} + \|p\|_{2,\alpha,\beta-1} + \|\nabla p\|_{2,\alpha+1,\beta} \le C \left(\|\mathbf{f}\|_{2,\alpha+1,\beta} + \|g\|_{1,2} \right).$$

Proof. Let $\rho > 0$ is such that $\mathbb{R}^3 \setminus B_{\rho/2} \subset \Omega$. Let $\Phi = \Phi(z) \in C_0^\infty(\langle 0, +\infty))$ be a non-increasing cut-off function such that $\Phi(z) \equiv 1$ for $z < 1/2$ and $\Phi(z) \equiv 0$ for $z > 1$. Let $|\Phi'| \le 3$. Let $\Phi_\rho \equiv \Phi_\rho(\mathbf{x}) \equiv \Phi(|\mathbf{x}|/\rho)$. We have $|\nabla\Phi_\rho| \le 3/\rho$ and $|\partial_1\Phi_\rho| \le 3/\rho$ for $\mathbf{x} \in \mathbb{R}^3$, $\rho/2 \le |\mathbf{x}| \le \rho$. Let us define $\widetilde{\mathbf{u}} = \mathbf{u} - [(\omega \times \mathbf{x}) - k\mathbf{e}_1] \cdot \Phi_\rho(\mathbf{x})$.

Then function $(\widetilde{\mathbf{u}}, p)$ satisfies to (1.1) - (1.3) with the homogenous Dirichlet boundary condition, where $\mathbf{f} \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$ is replaced by some another function $\widetilde{\mathbf{f}} \in \mathbf{L}^2_{\alpha+1,\beta}(\Omega)$, and g by another function $\widetilde{g} \in C_0^\infty(\Omega)$ with $\operatorname{supp} \widetilde{g} = K \subset A_\rho := B_\rho \setminus \overline{B_{\rho/2}} \subset\subset \Omega$ and

$$\int_\Omega \widetilde{g}\, d\mathbf{x} = 0.$$

The Dirichlet boundary condition (1.4) is replaced by the homogenous Dirichlet boundary condition for $\widetilde{\mathbf{u}}$. So, using now Corollary 6.6 we get the assertion of Theorem 6.7.

Appendix A

Relation (2.14) follows from an estimate of the derivative of F_1 :

$$\begin{aligned}
\frac{\partial}{\partial s} F_1(s,r) &\equiv \frac{\partial}{\partial s}\left\{F_{\alpha,\beta}(s,r;\nu) - \left(1-\kappa^{-1}\right)k\delta\varepsilon(\beta-\alpha)s\right\} \\
&= -\nu\alpha^2\delta^2\varepsilon\frac{1}{1+\delta r} - 2\nu\alpha\beta\delta\varepsilon\frac{1}{r} - 2\nu\beta^2\varepsilon^2\frac{1+\delta r}{r}\frac{1}{(1+\varepsilon s)^2} \\
&\quad -k\alpha\delta\varepsilon + k\alpha\delta\frac{1}{r}(1+2\varepsilon s) + k\beta\varepsilon(1+\delta r)\frac{1}{r} \\
&\quad -\left(1-\kappa^{-1}\right)k\delta\varepsilon(\beta-\alpha) \\
&\geq \delta\varepsilon\left\{r^{-1}\left[k(\alpha/\varepsilon+\beta/\delta) - \nu\alpha^2 - 2\nu\alpha\beta - 2\nu\beta^2\varepsilon/\delta\right]\right. \\
&\quad \left.+\left[-2\nu\beta^2\varepsilon + k(\beta-\alpha)/\kappa\right]\right\} \geq 0
\end{aligned}$$

The last inequality follows from the fact that we have $k\alpha/\varepsilon \geq \nu\alpha^2 + 2\nu\alpha\beta$, $k\beta/\delta \geq 2\nu\beta^2\varepsilon/\delta$, $k(\beta-\alpha)/\kappa \geq 2\nu\beta^2\varepsilon$ if $\varepsilon \leq (1/(2\kappa))(k/\nu)((\beta-\alpha)/\beta^2)$. Hence, if the last inequality (which is included in the conditions of Lemma 2.5) is satisfied then $(\partial/\partial s)F_1(s,r) \geq 0$. So, we get immediately:

$$F_1(s,r) \geq F_1(0,r) \equiv -k\alpha\delta - \nu\alpha^2\delta^2(1+\delta r)^{-1} \geq -\alpha\delta k\left(1+\nu k^{-1}\alpha\delta\right)$$

Appendix B

Let us show that all conditions on α, β, δ, ε, κ used in the proof of Lemma 4.3 are compatible if $0 < \beta \leq 1$, $0 \leq \alpha < y_1\beta$. Let us collect these assumptions: $0 < \delta < 2\varepsilon$, $1 < \kappa \leq 2\varepsilon/\delta + \delta/(2\varepsilon) - 1$, $0 \leq \alpha < \beta$, $\varepsilon \leq (1/(2\kappa^2)) \cdot (k/\nu) \cdot ((\beta-\alpha)/\beta^2)$, $\delta \leq (k/\nu)\cdot(\kappa-1)/(\kappa\beta)$, $\alpha \leq (1/(2\kappa^4))\cdot(k/\nu)\cdot(\beta\beta^*/(\alpha+\beta))^2\varepsilon$.

From $\alpha \leq (1/(2\kappa^4))\cdot(k/\nu)\cdot(\beta\beta^*/(\alpha+\beta))^2\varepsilon$, and $\varepsilon \leq (1/(2\kappa^2))\cdot(k/\nu)\cdot((\beta-\alpha)/\beta^2)$ we get $\alpha \leq (1/(4\kappa^6))\cdot(\beta^*)^2(\beta-\alpha)/(\alpha+\beta)^2$. So we get ($\kappa > 1$, $\beta \leq 1$): $\alpha/\beta \leq (1/(4\kappa^6))(1-\alpha/\beta)/(1+\alpha/\beta)^2$. By substitution $y = \alpha/\beta$ we get the inequality

$$4y^3 + 8y^2 + 4y + \kappa^{-6}\cdot(y-1) \leq 0. \tag{6.51}$$

Taking into account the condition $0 \leq \alpha < \beta$ we seek for solutions from $[0,1)$. It is clear that the equation $4y^3 + 8y^2 + y + \kappa^{-6}(y-1) = 0$ has a unique real solution $y_\kappa \in (0,1)$ for $\kappa > 1$. It is also clear that arbitrary $y \in [0, y_\kappa)$ solves

(6.51). The value y_κ as a function of κ is decreasing. For $\kappa \to 1$ we get the inequality $4y^3 + 8y^2 + 5y - 1 \leq 0$. This respective equation has a unique solution $y_1 = \left(\sqrt{13}/\left(6\sqrt{6}\right) + 53/216\right)^{1/3} + (1/30)\left(\sqrt{13}/\left(6\sqrt{6}\right) + 53/216\right)^{-1/3}$. Approximately, with an error less than 10^{-8} we have $y_1 \doteq 0.1582981$, $(y_1 > 1/7)$. If $0 \leq \alpha < y_1\beta$ then there is $\kappa > 1$ sufficiently close to number 1, such that $0 \leq \alpha \leq y_\kappa\beta$, so the relation $\alpha \leq (1/(4\kappa^6)) \cdot (\beta^*)^2 (\beta - \alpha)/(\alpha + \beta)^2$ is satisfied. Then we can define $\varepsilon = 1/(2\kappa^2) \cdot (k/\nu) \cdot ((\beta - \alpha)/(\beta^2))$. The relation $\varepsilon \leq (1/(2\kappa)) \cdot (k/\nu) \cdot (1/\beta)$ is satisfied. Then we take sufficiently small $\delta > 0$ such that $0 < \delta < 2\varepsilon$ and $1 < \kappa \leq 2\varepsilon/\delta + \delta/(2\varepsilon) - 1$. Hence, all conditions which we assume in the proof of Lemma 4.3 are satisfied.

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