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TOEPLITZ OPERATORS AND LOCALIZATION OPERATORS

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Abstract. We show that for any localization operator on the Fock space with polynomial window, there exists a constant coefficient linear partial differential operator D such that the localization operator with symbol f coincides with the Toeplitz operator with symbol Df. An analogous result also holds in the context of Bergman spaces on bounded symmetric domains. This verifies a recent conjecture of Coburn and simplifies and generalizes recent results of Lo.

1. Introduction

Let $\mathcal{F}$ be the Fock, or Segal-Bargmann, space of all entire functions on $\mathbf{C}^n$ square-integrable with respect to the Gaussian

$$d\mu(z) := e^{-\|z\|^2/2} \,\frac{dz}{(2\pi)^n},$$

dz being the Lebesgue volume measure on $\mathbf{C}^n$. It is well known (and easy to check) that the Weyl operators

$$W_a f(z) := e^{\langle z,a\rangle/2-\|a\|^2/4}\, f(z-a), \qquad a \in \mathbf{C}^n, \tag{1}$$

are unitary on $L^2(\mathbf{C}^n, d\mu)$ and on $\mathcal{F}$. For $w \in \mathcal{F}$ and $f \in L^\infty(\mathbf{C}^n)$, the Gabor-Daubechies localization operator $L_f^{(w)}$ with "window" w and "symbol" f is the operator on $\mathcal{F}$ defined by

$$\langle L_f^{(w)} u, v\rangle = (2\pi)^{-n} \int_{\mathbf{C}^n} f(a)\, \langle u, W_a w\rangle \langle W_a w, v\rangle \, da, \qquad u, v \in \mathcal{F}. \tag{2}$$

On the other hand, for $f \in L^\infty(\mathbf{C}^n)$, the Toeplitz operator T_f with symbol f is the operator on $\mathcal{F}$ defined by

$$T_f u = P(fu), \qquad u \in \mathcal{F}, \tag{3}$$

where $P : L^2(\mathbf{C}^n, d\mu) \to \mathcal{F}$ is the orthogonal projection. Using the fact that the exponentials

$$K_y(z) := K(z, y) := e^{\langle z,y\rangle/2}$$

serve as the reproducing kernel for $\mathcal{F}$, in the sense that

$$f(x) = \langle f, K_x\rangle = \int_{\mathbf{C}^n} f(y)\, K(x, y)\, d\mu(y) \qquad \forall f \in \mathcal{F},\ \forall x \in \mathbf{C}^n,$$

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we can also express T_f as an integral operator

$$T_f u(x) = \int_{\mathbf{C}^n} f(y)\, u(y)\, K(x,y)\, d\mu(y), \qquad u \in \mathcal{F},\ x \in \mathbf{C}^n. \tag{4}$$

It is immediate from (3) that for $f \in L^\infty(\mathbf{C}^n)$, T_f is bounded and

$$\|T_f\| \le \|f\|_\infty. \tag{5}$$

In principle, it is possible to define T_f by the formula (3) or (4) even for some unbounded symbols f — for instance, for all f such that $fK_y \in L^2(\mathbf{C}^n, d\mu)$ for all $y \in \mathbf{C}^n$. Then T_f is a densely defined, closed operator on $\mathcal{F}$. Similarly, (2) can be extended also to some unbounded symbols f as a densely defined operator.

It was observed by Coburn [C2], [C3] that for $w = \mathbf{1}$,

$$L_f^{(w)} = T_f$$

for all $f \in L^\infty(\mathbf{C}^n)$, while for $w(z) = 2^{-1/2} z_1$ and $w(z) = 2^{-3/2} z_1^2$, respectively,

$$L_f^{(w)} = T_{f + 2\partial_1\overline{\partial}_1 f},$$
$$L_f^{(w)} = T_{f + 4\partial_1\overline{\partial}_1 f + 2(\partial_1\overline{\partial}_1)^2 f},$$

for any f which is either a polynomial in $z, \overline{z}$ or belongs to the algebra $B_a(\mathbf{C}^n)$ of Fourier-Stieltjes transforms of compactly supported complex measures on $\mathbf{C}^n$. (Here $\partial_1 = \partial/\partial z_1$ and $\overline{\partial}_1 = \partial/\partial \overline{z}_1$.) This allows the amalgamation of the substantial work already done in studying T_f [Be] [BC1] [BC2] [BC3] [C1] [Ja] [Zh] and $L_f^{(w)}$ [D1] [D2] [FN] [Wo]. Coburn's most general result was that for any polynomial $w \in \mathcal{F}$ there exists a linear partial differential operator $D = D^{(w)}$, whose coefficients are polynomials in z and $\overline{z}$, such that

$$L_f^{(w)} p = T_{Df} p \tag{6}$$

for any polynomial $p \in \mathcal{F}$ and any polynomial f in z and $\overline{z}$. He also conjectured that D was actually a constant coefficient linear differential operator and (6) held also for all $f \in B_a(\mathbf{C}^n)$. This conjecture was verified by M.-L. Lo [Lo], who showed that (6) holds for any polynomials $p, w \in \mathcal{F}$ and any $f \in E(\mathbf{C}^n)$, where

$$E(\mathbf{C}^n) := \{f \in C^\infty(\mathbf{C}^n)\text{: for any multiindex } k, \text{ there exist } M, \alpha > 0 \text{ such that } |D^k f(z)| \le M\, e^{\alpha\|z\|}\ \forall z \in \mathbf{C}^n\} \tag{7}$$

contains both $B_a(\mathbf{C}^n)$ and all polynomials in z and $\overline{z}$.

Lo's proof went by a brute-force computation to establish the result for polynomials f (in z and $\overline{z}$), and then an approximation argument was used to extend it to all $f \in E(\mathbf{C}^n)$.

In this note, we present a simpler proof of these results, which also yields a bit more precise information for "nicer" symbols f.

Theorem 1. *For any polynomial $w \in \mathcal{F}$, there exists a constant coefficient linear partial differential operator $D = D^{(w)}$ such that for any $f \in BC^\infty(\mathbf{C}^n)$ (the space of all C^∞ functions on $\mathbf{C}^n$ whose partial derivatives of all orders are bounded),*

$$L_f^{(w)} = T_{Df} \qquad \text{on } \mathcal{F}. \tag{8}$$

Explicitly, the operator D is given by

$$D^{(w)} = \Big[e^{\Delta/2} |w(z)|^2\Big]_{\substack{z \mapsto -2\overline{\partial} \\ \overline{z} \mapsto -2\partial}}. \tag{9}$$

Here $e^{\Delta/2}$ should be understood as the infinite series

$$e^{\Delta/2} = \sum_{k=0}^{\infty} \frac{\Delta^k}{k!2^k}.$$

This infinite sum makes sense since, as w is assumed to be a polynomial, $\Delta^k|w|^2$ vanishes as soon as $k > \deg w$, thus there are only finitely many nonzero terms. Note also that for $f \in BC^\infty$ both sides of (8) are bounded operators, so the validity is not restricted to polynomials p as in (6). In fact, the left-hand side in (8) is a bounded operator for any $f \in L^\infty(\mathbf{C}^n)$ (see Proposition 2), so (8) tells us that Toeplitz operators can even be defined and nice (i.e. bounded) for the fairly wild symbols Df, $f \in L^\infty$ (which are distributions at best).

One more virtue of our proof is that it uses solely harmonic analysis methods, and thus easily extends also to other situations than the Segal-Bargmann space on $\mathbf{C}^n$ — for instance, to the standard weighted Bergman spaces on bounded symmetric domains, thus making contact with the work of Arazy and Upmeier [AU], de Mari and Nowak [MN], and others.

The paper is organized as follows. In Section 2, we review some preliminaries from Segal-Bargmann analysis. In Section 3, Theorem 1 is proved, and also extended to a wider class of functions f (including the polynomials, the algebra $B_a(\mathbf{C}^n)$, and the space $E(\mathbf{C}^n)$ from (7)). Generalizations to bounded symmetric domains are described in the final Section 4.

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2. Berezin symbols

In addition to K_a, we also consider the normalized reproducing kernels

$$k_a(z) := \frac{K_a(z)}{\|K_a\|} = e^{\langle z,a\rangle/2 - \|a\|^2/4}.$$

Note that the Weyl operators (1) can then be written simply as

$$W_a f(z) = k_a(z) f(z-a).$$

In particular, as $k_0 = \mathbf{1}$ (the function constant one),

$$k_a = W_a \mathbf{1}, \qquad \forall a \in \mathbf{C}^n. \tag{10}$$

One checks easily that W_a satisfy the composition law

$$W_a W_b = e^{(\overline{a}b - a\overline{b})/4}\, W_{a+b}, \qquad \forall a, b \in \mathbf{C}^n. \tag{11}$$

Consequently, $W_a^* = W_{-a}$ and

$$\begin{aligned} W_a k_b &= e^{(\overline{a}b - a\overline{b})/4}\, k_{a+b}, \\ W_a^* k_b &= e^{(a\overline{b} - \overline{a}b)/4}\, k_{b-a}. \end{aligned} \tag{12}$$

In particular, for $w=\mathbf{1}$ we get for any $u,v\in\mathcal{F}$,

$$\begin{aligned}\langle L_f^{(\mathbf{1})}u,v\rangle &= (2\pi)^{-n}\int_{\mathbf{C}^n} f(a)\langle u,k_a\rangle\langle k_a,v\rangle\,da\\ &= \int_{\mathbf{C}^n} f(a)\langle u,K_a\rangle\langle K_a,v\rangle\,d\mu(a)\\ &= \int_{\mathbf{C}^n} f(a)u(a)\overline{v(a)}\,d\mu(a)\\ &= \langle fu,v\rangle\\ &= \langle T_f u,v\rangle,\end{aligned}$$

so that indeed

$$L_f^{(\mathbf{1})} = T_f. \tag{13}$$

The next proposition is thus an analogue of (5) for an arbitrary window w. An analogous assertion is valid even in the much more general context of any square-integrable irreducible unitary representation of a unimodular group, see for instance Wong [Wo], Proposition 12.2, or [E] for an even further generalization; in the very special case that we have here, it is possible to give a simple direct proof based on the Fourier transform.

Proposition 2. *For any $w\in\mathcal{F}$ and $f\in L^\infty(\mathbf{C}^n)$, the localization operator $L_f^{(w)}$ is bounded, and*

$$\|L_f^{(w)}\| \le \|f\|_\infty\,\|w\|^2.$$

Proof. It is more convenient to pass from $\mathcal{F}$ to $L^2(\mathbf{R}^n)$, via the Bargmann transform

$$\beta f(z) := c_n\int_{\mathbf{R}^n} f(x)\,e^{xz-x^2/2-z^2/4}\,dx.$$

With the proper choice of the constant c_n, this is a unitary isomorphism of $L^2(\mathbf{R}^n)$ onto $\mathcal{F}$; see e.g. Folland [Fo]. (Here $x^2=x_1^2+\dots+x_n^2$ for $x\in\mathbf{R}^n$, and similarly for xz and z^2.) Its inverse is given by

$$\beta^{-1}F(x) = c_n'\int_{\mathbf{C}^n} F(z)\,e^{x\overline{z}-x^2/2-\overline{z}^2/4}\,e^{-\|z\|^2/2}\,dz,$$

and the Weyl operators (1) satisfy $W_{u+iv}=\beta U_{u,v}\beta^{-1}$, where the unitary operators $U_{u,v}$ on $L^2(\mathbf{R}^n)$ are given by

$$U_{u,v}f(x) = e^{iuv/2-ivx}\,f(x-u),\qquad x,u,v\in\mathbf{R}^n.$$

It follows that

$$\beta^{-1}L_f^{(w)}\beta = (2\pi)^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(u+iv)\,\langle\,\cdot\,,U_{u,v}H\rangle\langle U_{u,v}H,\,\cdot\,\rangle\,du\,dv,$$

where $H=\beta^{-1}w$. To prove the proposition, it therefore suffices to show that

$$\Big|(2\pi)^{-n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(u+iv)\,\langle F,U_{u,v}H\rangle\langle U_{u,v}H,G\rangle\,du\,dv\Big| \le \|f\|_\infty\,\|H\|^2\,\|F\|\,\|G\|$$

for all $F,G\in L^2(\mathbf{R}^n)$.

By the Cauchy-Schwarz inequality, the left-hand side is bounded by

$$(2\pi)^{-n}\|f\|_\infty\Big(\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|\langle F,U_{u,v}H\rangle|^2\,du\,dv\Big)^{1/2}\Big(\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}|\langle G,U_{u,v}H\rangle|^2\,du\,dv\Big)^{1/2}.$$

It is therefore enough to prove that

$$(2\pi)^{-n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\langle F, U_{u,v} H\rangle|^2 \, du \, dv \le \|F\|^2 \, \|H\|^2 \tag{14}$$

for any $F, H \in L^2(\mathbf{R}^n)$. However,

$$\langle F, U_{u,v} H\rangle = \int_{\mathbf{R}^n} F(x)\, e^{-iuv/2}\, e^{ivx}\, \overline{H(x-u)}\, dx = (2\pi)^{n/2} e^{-iuv/2} \hat{h}_u(v),$$

where $\hat{h}_u$ is the Fourier transform of the function $h_u(x) = F(x)\overline{H(x-u)}$. Thus by Parseval

$$\begin{aligned}
(2\pi)^{-n} & \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\langle F, U_{u,v} H\rangle|^2 \, du \, dv \\
&= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |\hat{h}_u(v)|^2 \, du \, dv \\
&= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |h_u(x)|^2 \, du \, dx \\
&= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} |F(x)|^2 \, |H(x-u)|^2 \, du \, dx \\
&= \int_{\mathbf{R}^n} |F(x)|^2 \Big[\int_{\mathbf{R}^n} |H(x-u)|^2 \, du \Big] \, dx \\
&= \int_{\mathbf{R}^n} |F(x)|^2 \Big[\int_{\mathbf{R}^n} |H(y)|^2 \, dy \Big] \, dx \qquad (y := x-u) \\
&= \|F\|^2 \, \|H\|^2, \qquad \text{q.e.d.}
\end{aligned}$$

□

Remark. We see that we have in fact an equality in (14). On the general level of square-integrable irreducible representations of an arbitrary unimodular group, this is of course just an immediate consequence of the Schur orthogonality relations. □

Recall that for a bounded linear operator T on $\mathcal{F}$, the *Berezin symbol* of T is the function $\widetilde{T}$ on $\mathbf{C}^n$ defined by

$$\widetilde{T}(x) := \langle T k_x, k_x \rangle.$$

Again, the definition makes sense even for unbounded operators, as long as the reproducing kernels k_x are in the domain of T, for all x. The following proposition records some properties of the Berezin symbol which we will need.

Proposition 3. (a) *The function* $\widetilde{T}$ *is real-analytic;*
(b) $\widetilde{T}$ *vanishes identically only if* $T = 0$*;*
(c) $\|\widetilde{T}\|_\infty \le \|T\|$*;*
(d) *for any* $a \in \mathbf{C}^n$*,*

$$(W_a^* T W_a)^{\sim} = \widetilde{T}(\,\cdot + a). \tag{15}$$

Proof. All this is well known, but here is the proof for completeness. Note that $\widetilde{T}(x)$ is the restriction to the diagonal $x = y$ of the function

$$\begin{aligned}
\frac{\langle T K_y, K_x\rangle}{\langle K_y, K_x\rangle} = e^{-\langle x,y\rangle/2} \langle T e^{\langle \cdot\,, y\rangle/2}, e^{\langle \cdot\,, x\rangle/2}\rangle &= e^{-\langle x,y\rangle/2} (T e^{\langle \cdot\,, y\rangle/2})(x) \\
&= e^{-\langle x,y\rangle/2} \overline{(T^* e^{\langle \cdot\,, x\rangle/2})(y)}
\end{aligned}$$

which is holomorphic in x and $\overline{y}$; in particular, $\widetilde{T}$ is a real-analytic function. Further, it is known that such functions are uniquely determined by their restriction to the diagonal (see e.g. Folland [Fo], Proposition 1.69); hence $\widetilde{T} \equiv 0$ only if $\langle TK_y, K_x \rangle = TK_y(x) = 0 \ \forall x, y$, which implies that $T = 0$ since the linear combinations of K_y, $y \in \mathbf{C}^n$, are dense in $\mathcal{F}$. Finally, (c) is immediate from the Schwarz inequality, and the covariance property (15) is immediate from (12). □

3. Main results

Proof of Theorem 1. From the definition of the localization operators $L_F^{(w)}$, we have for any $c \in \mathbf{C}^n$

$$
\begin{aligned}
L^{(w)}_{f(\,\cdot\,+c)} &= (2\pi)^{-n} \int f(a+c)\, \langle\,\cdot\,, W_a w\rangle W_a w\, da \\
&= (2\pi)^{-n} \int f(x)\, \langle\,\cdot\,, W_{x-c} w\rangle W_{x-c} w\, dx \\
&= W_c^* L_f^{(w)} W_c,
\end{aligned} \tag{16}
$$

by (11). In particular, for $w = \mathbf{1}$ we get by (13)

$$
T_{f(\,\cdot\,+c)} = W_c^* T_f W_c.
$$

By Proposition 2, and parts (a), (c) and (d) of Proposition 3, we thus see that the two maps

$$
f \mapsto \widetilde{L_f^{(w)}}, \qquad f \mapsto \widetilde{T_f},
$$

both map $L^\infty(\mathbf{C}^n)$ continuously into bounded real-analytic functions on $\mathbf{C}^n$, and commute with translations. Recall now (see e.g. [Ru], Theorem 6.33) that for any continuous linear map V from $\mathcal{D}(\mathbf{C}^n)$ into $C(\mathbf{C}^n)$ which commutes with translations there is a unique distribution $v \in \mathcal{D}'(\mathbf{C}^n)$ such that $Vf = v * f$ for all $f \in \mathcal{D}$. Thus there exist distributions $k = k^{(w)}$ and $h = k^{(\mathbf{1})}$ on $\mathbf{C}^n$ such that

$$
\begin{aligned}
\widetilde{L_f^{(w)}} &= k * f, \\
\widetilde{T_f} &= h * f,
\end{aligned} \tag{17}
$$

for all $f \in \mathcal{D}(\mathbf{C}^n)$. To find what k and h are, note that for any $f \in L^\infty(\mathbf{C}^n)$ and $z \in \mathbf{C}^n$,

$$\begin{aligned}
\widetilde{L_f^{(w)}}(z) &= \langle L_f^{(w)} k_z, k_z \rangle \\
&= (2\pi)^{-n} \int f(a)\, \langle k_z, W_a w \rangle \langle W_a w, k_z \rangle \, da \\
&= (2\pi)^{-n} \int f(a)\, |\langle W_a^* k_z, w \rangle|^2 \, da \\
&= (2\pi)^{-n} \int f(a)\, |\langle k_{z-a}, w \rangle|^2 \, da \qquad \text{by (12)} \\
&= (2\pi)^{-n} \int f(z-y)\, |\langle k_y, w \rangle|^2 \, dy \\
&= (2\pi)^{-n} \int f(z-y)\, |\langle K_y, w \rangle|^2 \, e^{-\|y\|^2/2} \, dy \\
&= (2\pi)^{-n} \int f(z-y)\, |w(y)|^2 \, e^{-\|y\|^2/2} \, dy \\
&= (f * (2\pi)^{-n} |w|^2 e^{-\|\cdot\|^2/2})(z).
\end{aligned}$$

Thus k is not only a distribution but a function, given by

$$k(z) = (2\pi)^{-n} |w(z)|^2 \, e^{-\|z\|^2/2}, \tag{18}$$

and, taking $w = \mathbf{1}$,

$$h(z) = (2\pi)^{-n} \, e^{-\|z\|^2/2}. \tag{19}$$

It also follows from the last computation that (17) holds not only for $f \in \mathcal{D}(\mathbf{C}^n)$, but for any $f \in L^\infty(\mathbf{C}^n)$.

Observe now that for any multiindices j, k, the Leibniz formula implies that

$$\partial^j \overline{\partial}^k e^{-\|z\|^2/2} = e^{-\|z\|^2/2} \Big[\Big(-\frac{1}{2} \Big)^{|j+k|} \overline{z}^j z^k + \text{lower order terms} \Big]. \tag{20}$$

By a straightforward induction argument, it follows that there exists a unique differential operator $D = D^{(w)}$ with constant coefficients such that

$$D e^{-\|\cdot\|^2/2} = |w|^2 \, e^{-\|\cdot\|^2/2},$$

i.e. $Dh = k$. By the properties of convolution,

$$h * Df = Dh * f = k * f \tag{21}$$

for any reasonable f (for instance, whenever all derivatives of f up to the order of D are bounded). Consequently,

$$\widetilde{T_{Df}} = h * Df = k * f = \widetilde{L_f^{(w)}}$$

for any $f \in BC^\infty(\mathbf{C}^n)$. By part (b) of Proposition 3, this implies that

$$T_{Df} = L_f^{(w)},$$

thus completing the proof of (8).

It remains to show that the operator D is given by the formula (9). To this end, write out the "lower order terms" in (20) explicitly:

$$\begin{aligned}
\partial^j\overline{\partial}^k e^{-\|z\|^2/2} &= \partial^j\Big[\Big(-\frac{z}{2}\Big)^k e^{-\|z\|^2/2}\Big] \\
&= \sum_{l\subset j}\binom{j}{l}\Big(-\frac12\Big)^{|k|}\frac{k!}{(k-l)!}z^{k-l}\Big(-\frac{\overline{z}}{2}\Big)^{j-l}e^{-\|z\|^2/2} \\
&= \sum_l \frac{j!}{(j-l)!}\overline{z}^{j-l}\frac{k!}{(k-l)!}z^{k-l}\Big(-\frac12\Big)^{|j+k-l|}\frac{e^{-\|z\|^2/2}}{l!} \\
&= \Big(-\frac12\Big)^{|j+k|}e^{-\|z\|^2/2}\sum_l(\overline{\partial}^l\overline{z}^j)\cdot(\partial^l z^k)\,\frac{(-2)^{|l|}}{l!} \\
&= \Big(-\frac12\Big)^{|j+k|}e^{-\|z\|^2/2}\sum_{L=0}^\infty\frac{(-2)^L}{L!}\sum_{|l|=L}\binom{L}{l}\partial^l\overline{\partial}^l\overline{z}^j z^k \\
&= \Big(-\frac12\Big)^{|j+k|}e^{-\|z\|^2/2}\sum_{L=0}^\infty\frac{(-2)^L}{L!}\Big(\frac{\Delta}{4}\Big)^L\overline{z}^j z^k \\
&= \Big(-\frac12\Big)^{|j+k|}e^{-\|z\|^2/2}e^{-\Delta/2}\overline{z}^j z^k.
\end{aligned}$$

It follows that for any polynomial p in two variables with complex coefficients,

$$p(-2\partial,-2\overline{\partial})e^{-\|z\|^2/2} = e^{-\|z\|^2/2}e^{-\Delta/2}p(\overline{z},z).$$

Thus if we choose

$$p(\overline{z},z) = e^{\Delta/2}|w(z)|^2$$

then $p(-2\partial,-2\overline{\partial}) = D$. This completes the proof of Theorem 1. □

Corollary 4. *Let $w_1, w_2 \in \mathcal{F}$ be polynomials. Then the following two assertions are equivalent:*

(a) *There exists a constant coefficient linear differential operator D such that*

$$L_f^{(w_2)} = L_{Df}^{(w_1)} \tag{22}$$

for all $f \in \mathcal{D}(\mathbf{C}^n)$.

(b) *The polynomial $e^{\Delta/2}|w_2|^2$ is divisible by the polynomial $e^{\Delta/2}|w_1|^2$.*

Further, if (a) or (b) are fulfilled, then D is of order $2(\deg w_2 - \deg w_1)$ and (22) holds for all $f \in BC^\infty(\mathbf{C}^n)$.

Proof. Immediate from (8) and (9). □

Note that we have proved (8) not only for $f \in BC^\infty$, but in fact for any $f \in L^\infty$ whose derivatives up to the order of D are bounded. Going through the above arguments with some care, it is not difficult to extend this even further. Let r be the degree of w and denote

$$\begin{aligned}\mathcal{M}_r := \{f \in C^{2r}(\mathbf{C}^n)\colon &\text{ for any multiindices } j,k \text{ with } |j|,|k| \le r \\ &\text{and any } a>0,\ e^{a\|\cdot\|}|\partial^j\overline{\partial}^k f|\,e^{-\|\cdot\|^2/2} \in L^\infty(\mathbf{C}^n)\}.\end{aligned} \tag{23}$$

Observe that the condition implies that for any $m \geq 0$ and $|j|, |k| \leq r$, $\|z\|^m |\partial^j \overline{\partial}^k f| \cdot e^{-\|z\|^2/2}$ belongs to L^1 and vanishes at the infinity. Integrating by parts in

$$\int f(z-x)\, De^{-\|x\|^2/2}\, dx$$

it therefore follows that

$$f * Dh = Df * h \qquad \forall f \in \mathcal{M}_r,$$

i.e. (21) still holds for $f \in \mathcal{M}_r$. Thus again

$$\widetilde{T_{Df}} = \widetilde{L_f^{(w)}}.$$

Since now T_{Df} and $L_f^{(w)}$ need no longer be bounded in general, it is not clear whether this implies $T_{Df} = L_f^{(w)}$; however, from the proof of part (b) of Proposition 3 it is clear at least that $T_{Df} K_z = L_f^{(w)} K_z$ for any $z \in \mathbf{C}^n$. Thus we arrive at the following strengthening of Theorem 1.

Theorem 5. *Let $w \in \mathcal{F}$ be a polynomial of degree r, and let $\mathcal{M}_r$ be as in* (23). *Then for any $f \in \mathcal{M}_r$, T_{Df} and $L_f^{(w)}$ coincide on the linear span of K_z, $z \in \mathbf{C}^n$.*

Note that $E(\mathbf{C}^n) \subset \mathcal{M}_r$ for any r; thus, in particular, the last theorem covers completely the main result of [Lo] (except that the polynomials p are replaced by linear combinations of K_z).

We conclude this section by a generalization in a different direction. It may seem a little artificial at first sight, but becomes very natural after we pass to the bounded symmetric domains in the next section. For any bounded linear operator A on $\mathcal{F}$, we may define a “localization operator” with symbol f and “window” A by

$$L_f^{(A)} := (2\pi)^{-n} \int_{\mathbf{C}^n} f(a)\, W_a A W_a^*\, da. \tag{24}$$

The localization operators $L_f^{(w)}$ considered so far are recovered upon choosing $A = \langle\,\cdot\,, w\rangle w$.

We then have the following generalizations of Proposition 2 and Theorem 1.

Proposition 6. *If A is trace-class, then the integral* (24) *converges in the weak operator topology for any $f \in L^\infty(\mathbf{C}^n)$, and*

$$\|L_f^{(A)}\| \leq \|f\|_\infty \, \|A\|_{tr},$$

where $\|\cdot\|_{tr}$ denotes the trace norm.

Theorem 7. *Let A be a finite sum*

$$A = \sum_j \langle\,\cdot\,, u_j\rangle v_j,$$

where $u_j, v_j \in \mathcal{F}$ are polynomials. Then there exists a unique linear partial differential operator $D = D^{(A)}$ such that

$$L_f^{(A)} = T_{Df} \qquad \forall f \in BC^\infty(\mathbf{C}^n).$$

The proof of Proposition 6 can (again in a much more general setup) be found in [E], or carried out directly along the lines of the proof of Proposition 2. Similarly, Theorem 7 can be proved either by mimicking the proof of Theorem 1, or from Theorem 1 directly using the linearity in A and the familiar polarization identity

$$\langle\,\cdot\,,w_1\rangle w_2=\sum_{k=0}^{3} i^{-k}\langle\,\cdot\,,w_1+i^k w_2\rangle(w_1+i^k w_2).$$

4. Bounded symmetric domains

Throughout this section we let Ω be an irreducible bounded symmetric domain in $\mathbf{C}^n$ (i.e. a Cartan domain) in its Harish-Chandra realization (so Ω is circular with respect to the origin and convex). Let G be the group of all biholomorphic self-maps of Ω; then G acts transitively on Ω, so denoting by K the stabilizer of the origin $0\in\Omega$ in G, Ω can be identified with the coset space G/K. For each $z\in\Omega$, there exists a unique so-called *geodesic symmetry* $g_x\in G$ interchanging x and the origin, i.e. g_x is an involution (that is, $g_x=g_x^{-1}$), $g_x(0)=x$, $g_x(x)=0$, and g_x has only isolated fixed-points. We refer e.g. to [Ar], [Ko] or [Up] for an overview of bounded symmetric domains.

Let dz be the Lebesgue measure on Ω normalized so that Ω has total mass one. Abusing the notation a little, we will denote by the same letter K also the Bergman kernel $K_y(x)=K(x,y)$ of Ω, i.e. the reproducing kernel of the subspace $\mathcal{H}=L^2_{\text{hol}}(\Omega,dz)$ of all holomorphic functions in $L^2(\Omega,dz)$. We will also use the same notation $k_z=K_z/\|K_z\|$ as before for the normalized reproducing kernels.

From the familiar formula for the change of variables, it is immediate that the operators

$$U_g:\ f\mapsto j_{g^{-1}}\cdot(f\circ g^{-1}),\qquad g\in G, \tag{25}$$

are unitary on $L^2(\Omega)$ and $\mathcal{H}$; here j_g denotes the complex Jacobian of the mapping g. From the chain rule for derivatives it follows that

$$U_{g_1}U_{g_2}=U_{g_1g_2},\qquad \forall g_1,g_2\in G,$$

so that $g\mapsto U_g$ is a unitary representation of G in $\mathcal{H}$. In particular, $U_g^*=U_{g^{-1}}$. From the computation

$$\begin{aligned}\langle f,U_gk_z\rangle&=\langle U_{g^{-1}}f,k_z\rangle=K(z,z)^{-1/2}(U_{g^{-1}}f)(z)\\&=K(z,z)^{-1/2}\,j_g(z)\,f(g(z))\\&=K(g(z),g(z))^{1/2}\,K(z,z)^{-1/2}\,j_g(z)\,\langle f,k_{g(z)}\rangle,\qquad\forall f\in\mathcal{H},\end{aligned}$$

it follows that $U_gk_z=\text{const}\cdot k_{g(z)}$; since U_g is unitary and $k_z,k_{g(z)}$ are both unit vectors, the constant must be unimodular, i.e.

$$U_gk_z=\epsilon_{g,z}k_{g(z)},\qquad |\epsilon_{g,z}|=1, \tag{26}$$

which is an analogue of (12).

Yet another consequence of the change-of-variable formula is the equality

$$K(x,y)=j_{g^{-1}}(x)\,K(g^{-1}(x),g^{-1}(y))\,\overline{j_{g^{-1}}(y)},$$

from which it follows that the measure

$$d\mu(z):=K(z,z)\,dz,\qquad z\in\Omega,$$

is G-invariant.

Denoting by dg the Haar measure on G, we may now define for any bounded linear operator ("window") A on $\mathcal{H}$ and any function ("symbol") f on G the "localization operator"

$$\mathcal{L}_f^{(A)} := \int_G f(g)\, U_g A U_g^*\, dg.$$

Comparing this with (24), we immediately see the drawback that our symbols f now live on G, not on Ω. As shown in [AU] and [E], this can be resolved by restricting to operators A which are K-*invariant*, in the sense that

$$AU_k = U_k A \qquad \forall k \in K.$$

Indeed, then for any $g \in G$ we have

$$U_{gk} A U_{gk}^* = U_g U_k A U_k^* U_g^* = U_g A U_g^*.$$

Thus $U_g A U_g^*$ depends only on the coset gK of g in G/K, i.e. only on $g(0) \in \Omega$. We can therefore define unambiguously the operator A_z, for any $z \in \Omega$, by

$$A_z := U_g A U_g^* \quad \text{for any } g \in G \text{ such that } g(0) = z,$$

and the localization operator

$$L_f^{(A)} := \int_\Omega f(z)\, A_z\, d\mu(z). \tag{27}$$

Such operator calculi were studied in [E]. It was shown there, for instance, that (27) converges in the weak operator topology whenever f is bounded and A is trace-class, and

$$\|L_f^{(A)}\| \le \|f\|_\infty\, \|A\|_{\mathrm{tr}},$$

an analogue of Propositions 2 and 6. Our goal in the rest of this section will be to establish also an analogue of Theorems 1 and 7. Before stating the latter, we need to review some facts about the structure of K-invariant operators.

It is known that under the action U_k of the group K, the space $\mathcal{H}$ decomposes into an orthogonal direct sum of irreducible subspaces (Peter-Weyl decomposition)

$$\mathcal{H} = \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}.$$

Here $\mathbf{m}$ ranges over all *signatures,* i.e. r-tuples $\mathbf{m} = (m_1, \dots, m_r)$ of integers satisfying $m_1 \ge m_2 \ge \cdots \ge m_r \ge 0$; the number r is the *rank* of Ω. One has $\mathcal{P}_{(0,\dots,0)} = \{\text{the constant functions}\}$, $\mathcal{P}_{(1,0,\dots,0)} = \{\text{the linear functions}\}$, and, in general, the elements of $\mathcal{P}_{\mathbf{m}}$ are homogeneous polynomials of degree $|\mathbf{m}| := m_1 + \cdots + m_r$. Let $P_{\mathbf{m}}$ be the orthogonal projection in $\mathcal{H}$ onto $\mathcal{P}_{\mathbf{m}}$. By construction, $P_{\mathbf{m}}$ is a K-invariant operator. Conversely, if A is any K-invariant operator, then it follows from Schur's lemma that the restriction of A to each $\mathcal{P}_{\mathbf{m}}$ is a multiple of the identity. Thus, K-invariant operators on $\mathcal{H}$ are precisely the operators of the form

$$A = \sum_{\mathbf{m}} c_{\mathbf{m}} P_{\mathbf{m}}, \qquad c_{\mathbf{m}} \in \mathbf{C}.$$

Clearly A is bounded if and only if $\{c_{\mathbf{m}}\}$ is a bounded sequence, and A is trace-class if and only if $\sum_{\mathbf{m}} c_{\mathbf{m}} \dim \mathcal{P}_{\mathbf{m}} < \infty$.

The simplest K-invariant operator is thus

$$A = P_{(0,\dots,0)} = \langle \cdot\,, \mathbf{1} \rangle \mathbf{1},$$

the projection onto the constants. By (26), in that case

$$A_z = \langle\,\cdot\,, k_z\rangle k_z$$

and

$$\begin{aligned} L_f^{(A)} &= \int_\Omega f(z)\,\langle\,\cdot\,, k_z\rangle k_z\,d\mu(z) \\ &= \int_\Omega f(z)\,\langle\,\cdot\,, K_z\rangle K_z\,dz \\ &= T_f, \end{aligned}$$

the Toeplitz operator with symbol f.

We now have the following analogue of Theorems 1 and 7.

Theorem 8. *Let A be a K-invariant operator on $\mathcal{H}$ of the form*

$$A = \sum_{\textit{finite}} c_{\mathbf{m}} P_{\mathbf{m}}.$$

Then there exists a unique G-invariant linear partial differential operator $D = D^{(A)}$ on Ω such that

$$L_f^{(A)} = T_{Df} \qquad \forall f \in \mathcal{D}(\Omega).$$

Proof. The proof is completely parallel to that of Theorem 1, so we will be brief. Using linearity, it is enough to prove the theorem for $A = P_{\mathbf{m}}$, which we will assume from now on. For any bounded linear operator T on $\mathcal{H}$, we again define its Berezin symbol $\widetilde{T}$ by

$$\widetilde{T}(z) = \langle Tk_z, k_z\rangle, \qquad z \in \Omega.$$

The proof of Proposition 3 extends to the present setting without any changes, so that again $\|\widetilde{T}\|_\infty \le \|T\|$, $\widetilde{T}$ is real-analytic, and $\widetilde{T} \equiv 0$ only if $T = 0$. By a similar computation as for the Fock space, for any $f \in L^\infty(\Omega)$,

$$\widetilde{L_f^{(A)}}(z) = \langle L_f^{(A)} k_z, k_z\rangle = \int_\Omega f(x)\,\langle A_x k_z, k_z\rangle\,d\mu(x).$$

Let $g_x \in G$ be the geodesic symmetry interchanging x and the origin, so that $g_x = g_x^{-1}$, $g_x(0) = x$ and $g_x(x) = 0$. Then $\langle A_x k_z, k_z\rangle = \langle AU^*_{g_x} k_z, U^*_{g_x} k_z\rangle = \langle Ak_{g_x(z)}, k_{g_x(z)}\rangle$, by (26). Since $g_x(g_z(0)) = g_x(z) = g_{g_x(z)}(0)$, there exists $k \in K$ such that $g_x g_z = g_{g_x(z)} k$; taking inverses gives $k g_z g_x = g_{g_x(z)}$, whence $g_x(z) = g_{g_x(z)}(0) = k(g_z(g_x(0))) = k(g_z(x))$. As A is K-invariant, $\langle Ak_{g_x(z)}, k_{g_x(z)}\rangle = \langle AU_k k_{g_z(x)}, U_k k_{g_z(x)}\rangle = \langle Ak_{g_z(x)}, k_{g_z(x)}\rangle = \widetilde{A}(g_z(x))$. Thus

$$\widetilde{L_f^{(A)}}(z) = \int_\Omega f(x)\,\widetilde{A}(g_z(x))\,d\mu(x).$$

The last integral is the definition of convolution (in G) of f and $\widetilde{A}$ [H]:

$$\widetilde{L_f^{(A)}} = f * \widetilde{A}.$$

As $A = P_{\mathbf{m}}$ we have

$$\begin{aligned} \widetilde{A}(z) = \langle P_{\mathbf{m}} k_z, k_z\rangle &= K(z,z)^{-1}\,(P_{\mathbf{m}} K_z)(z) \\ &= K(z,z)^{-1}\,K_{\mathbf{m}}(z,z), \end{aligned}$$

where $K_{\mathbf{m}}(x, y)$ is the reproducing kernel of the subspace $\mathcal{P}_{\mathbf{m}} \subset \mathcal{H}$. In particular, for $\mathbf{m} = (0, \dots, 0)$, we have $\widetilde{P_{(0,\dots,0)}}(z) = K(z, z)^{-1}$.

Now it was shown by Ørsted and Zhang [OZ], Proposition 3.15, that there exists a unique G-invariant linear partial differential operator $D = D^{\mathbf{m}}$ on Ω such that

$$DK(z, z)^{-1} = K_{\mathbf{m}}(z, z)K(z, z)^{-1}.$$

Arguing as in the Fock-space case, it follows that

$$\begin{aligned} \widetilde{L_f^{(A)}} &= f * \widetilde{P_{\mathbf{m}}} = f * D\widetilde{P_{(0,\dots,0)}} \\ &= Df * \widetilde{P_{(0,\dots,0)}} \\ &= \widetilde{T_{Df}}, \end{aligned} \tag{28}$$

whence $L_f^{(A)} = T_{Df}$ by part (b) of Proposition 3. This completes the proof. □

Remark. Again, it is evident from the proof that (28) holds not only for $f \in \mathcal{D}(\Omega)$, but for any $f \in C^\infty(\Omega)$ whose derivatives do not grow too fast at the boundary, so that the partial integration implicit in the third equality in (28) is legitimate. □

References

[Ar] J. Arazy: *A survey of invariant Hilbert spaces of analytic functions on bounded symmetric domains,* Multivariable operator theory (R.E. Curto, R.G. Douglas, J.D. Pincus, N. Salinas, editors), Contemporary Mathematics **185**, Amer. Math. Soc., Providence, 1995, pp. 7–65.

[AU] J. Arazy, H. Upmeier: *Invariant symbolic calculi and eigenvalues of invariant operators on symmetric domains,* Function spaces, interpolation theory, and related topics (Lund, 2000) (A. Kufner, M. Cwikel, M. Engliš, L.-E. Persson, G. Sparr, editors), Walter de Gruyter, Berlin, 2002, pp. 151–211.

[BC1] C.A. Berger, L.A. Coburn: *Toeplitz operators and quantum mechanics,* J. Funct. Anal. **68** (1986), 273-299.

[BC2] C.A. Berger, L.A. Coburn: *Toeplitz operators on the Segal-Bargmann space,* Trans. Amer. Math. Soc. **301** (1987), 813–829.

[BC3] C.A. Berger and L.A. Coburn: *Heat flow and Berezin-Toeplitz estimates,* Amer. J. Math. **116** (1994), 563–590.

[Be] F.A. Berezin: *Covariant and contravariant symbols of operators,* Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 1134–1167 (in Russian); English translation in Math. USSR Izv. **6** (1972), 1117–1151.

[C1] L. Coburn: *On the Berezin-Toeplitz calculus,* Proc. Amer. Math. Soc. **129** (2001), 3331–3338.

[C2] L. Coburn: *The Bargmann isometry and Gabor-Daubechies wavelet localization operators,* Systems, approximation, singular integrals and related topics (A. Borichev, N. Nikolski, editors), Operator Theory: Advances and Applications 129, Birkhäuser Verlag, Basel, 2001, pp. 169–178.

[C3] L. Coburn: *Symbol calculus for Gabor-Daubechies windowed Fourier localization operators,* preprint.

[D1] I. Daubechies: *Time frequency localization operators: A geometric phase space approach,* IEEE Trans. Inform. Theory **34** (1988), 605–612.

[D2] I. Daubechies, *Ten lectures on wavelets,* CBMS-NSF Regional Conference Series 6, SIAM, Philadelphia, 1992.

[E] M. Engliš: *Toeplitz operators and group representations,* J. Fourier Anal. Appl., to appear.

[FN] H.G. Feichtinger, K. Nowak: *A Szegö type theorem for Gabor-Toeplitz localization operators,* Michigan Math. J. **49** (2001), 13–21.

[Fo] G.B. Folland, *Harmonic analysis in phase space,* Annals of Mathematics Studies 122, Princeton University Press, Princeton, 1989.

[H] S. Helgason, *Groups and geometric analysis,* Academic Press, Orlando, 1984.

[Ja] J. Janas: *Unbounded Toeplitz operators in the Segal-Bargmann space III,* Math. Scand. **88** (2001), 126–136.

[Ko] A. Korányi: *Function spaces on bounded symmetric domains,* Analysis and geometry on complex homogeneous domains (J. Faraut, S. Kaneyuki, A. Korányi, Q.-K. Lu, G. Roos, editors), Progress in Mathematics 185, Birkhäuser, Basel-Boston-Berlin, 2000.

[Lo] M.-L. Lo: *The Bargmann transform and windowed Fourier localization,* preprint.

[MN] F. de Mari, K. Nowak: *Localization type Berezin-Toeplitz operators on bounded symmetric domains,* J. Geom. Anal. **12** (2002), 9–27.

[OZ] B. Ørsted, G. Zhang: *Weyl quantization and tensor products of Fock and Bergman spaces,* Indiana Univ. Math. J. **43** (1994), 551–583.

[Ru] W. Rudin, *Functional analysis,* McGraw-Hill, 1973.

[Up] H. Upmeier, *Toeplitz Operators and Index Theory in Several Complex Variables,* Operator Theory: Advances and Applications 81, Birkhäuser, Basel, 1996.

[Wo] M.W. Wong, *Wavelet transforms and localization operators,* Operator Theory: Advances and Applications 136, Birkhäuser Verlag, Basel, 2002.

[Zh] K. Zhu, *Operator theory in function spaces,* Marcel Dekker, New York, 1990.

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