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DO PROJECTIONS STAY CLOSE TOGETHER?

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Abstract. We estimate the rate of convergence of products of projections on lines in $\mathbb{R}^d$.

Consider the orbit of a point under any sequence of orthogonal projections on K lines in $\mathbb{R}^d$. Assume that the sum of the squares of the distances of the consecutive iterates is less than ε. We show that if ε tends to zero, then the diameter of the orbit tends to zero uniformly for all families $\mathcal{L}$ of a fixed number K of lines.

We relate this result to questions concerning convergence of products of projections on finite families of closed subspaces of ℓ_2.

Introduction

Let K be a fixed natural number and let $\mathcal{L}$ be a family of K affine subspaces of $\mathbb{R}^d$. Let $z \in \mathbb{R}^d$ and $k_1, k_2, \dots \in \{1, 2, \dots, K\}$ be arbitrary. Consider the sequence of projections

$$z_1 = P_{k_1} z,$$
$$z_n = P_{k_n} z_{n-1},$$

where P_k denotes the orthogonal projection on the k-th space in $\mathcal{L}$. The orbit $\{z_i\}$ is always bounded according to [ADW], [BGP], and [Me].

In this note we consider an additional constraint on the distances of the consecutive iterates, namely that

$$\sum_{i=1}^{\infty} |z_{i+1} - z_i|^2 \leq \varepsilon \tag{1}$$

for some $\varepsilon > 0$. In Theorem 2.3 we show, that if ε goes to zero, then the diameter of the orbit $\{z_i\}$ goes to zero uniformly for all families $\mathcal{L}$ of a fixed number K of lines.

Let $\mathcal{L}$ be a family of K closed linear subspaces of ℓ_2. Any sequence $\{z_i\}$ of orthogonal projections on the spaces in $\mathcal{L}$ converges weakly according to [AA]. If $K = 2$ the sequence of projections even converges in

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norm [vN]. If $K \geq 3$, this is known only under additional assumptions, for example, if the sequence $\{k_i\}$ is (quasi) periodic [H, S].

In Theorem 3.2 we show that proving the norm convergence of the sequence for every $K \in \mathbb{N}$ is equivalent to proving a version of Theorem 2.3 with the family $\mathcal{L}$ of lines replaced by any family $\mathcal{L}$ of K closed linear subspaces of $\mathbb{R}^d$.

The paper is organized as follows. In the next section we point out the main ingredients of Theorem 2.3. In Section 1 we present some elementary estimates for almost parallel lines. In Section 2 we state and prove the main result, Theorem 2.3, after reducing it to the case of almost parallel and well separated lines. In the crucial Lemma 2.1, we construct a calibration function needed in the proof of the theorem. Section 3 is devoted to norm convergence of successive projections in ℓ_2.

Notation. For $K \in \mathbb{N}$, we denote the set $\{1, \dots, K\}$ by $[K]$. If $x \in \mathbb{R}^d$ we denote by $|x|$ the euclidean norm of x. As usual, S^{d-1} is the unit sphere of $\mathbb{R}^d$. The set $\{x \in \mathbb{R}^d : \operatorname{dist}(x, A) \leq \delta\}$ is denoted by $B(A, \delta)$. By aff A we denote the affine hull of the set A. If X is an affine subspace of $\mathbb{R}^d$, we denote by P_X the orthogonal projection on X. Let $w \in \mathbb{R}^d$, $a \in \mathbb{R}$ and let F be the affine function defined by $F(x) = \langle w, x \rangle + a$. We denote $F'(x) = w$.

Outline of the proof of Theorem 2.3

The goal of this paper is to approach the question of convergence of products of projections by methods somewhat different from those which have so far appeared in the literature. This section is a brief guide to the ingredients of our main Theorem 2.3. We will show that if $z_{i+1} = P_{k_{i+1}} z_i$ defines a sequence of projections on K lines $p_1, \dots, p_K$, then

$$|z_1 - z_m| \leq c(K) \sum_{i=1}^{m-1} |z_{i+1} - z_i|^2, \tag{2}$$

where $c(K) > 0$ depends on K only. The proof proceeds by contradiction in several steps.

• Assume the theorem is false. Then for each $\varepsilon > 0$ there exists a family $\mathcal{L}$ of K lines such that the corresponding projections satisfy

$$z_1 = 0, \ |z_m| = 1, \sum_{i=1}^{m-1} |z_{i+1} - z_i|^2 \leq \varepsilon. \tag{3}$$

We may assume in addition, that $|z_i| \leq 1$ for all i's, since otherwise we obtain a counterexample for $z_1, \dots, z_n$, $n < m$ with $|z_n| = \max |z_i|$

after rescaling by $1/\max |z_i|$.
• In Lemma 2.2 we will show that if such a family $\mathcal{L}$ exists, then one can already achieve (3) with a family of lines which are all almost parallel to $w = z_m - z_1 = z_m$, and which are well separated in the sense that the points where two lines p_i and p_j are closest lie well outside the unit ball. The precise conditions are described in the Setting in Section 2.

The proof of Lemma 2.2 uses a compactness argument and the following simple observation. If a curve γ of diameter one is contained in the union of K lines, then a "long" sub-curve of γ is contained in one of the lines.
• As each vector $z_i - z_{i+1}$ is orthogonal to one of the lines in $\mathcal{L}$ and these lines are almost parallel to w, it follows that $\langle w, z_{i+1} - z_i \rangle \approx 0$. This seemingly allows the following contradictory estimate:

$$1 = |z_m - z_1|^2 = \langle w, z_m - z_1 \rangle = \sum_{i=1}^{m-1} \langle w, z_{i+1} - z_i \rangle \approx \sum_{i=1}^{m-1} 0 \approx 0. \tag{4}$$

Since, however, we do not have any estimate of the number m of the iterates, the last step "$\sum_{i=1}^{m-1} 0 \approx 0$" requires a justification.
• Let $w_i \in S^{d-1}$ be the direction of the line p_i; recall that all $w_i's$ are close to w. The main point is to construct a "calibration function" Φ, with the following properties. If $v \in S^{d-1}$ is orthogonal to $p_j \in \mathcal{L}$, then

$$|\langle v, \Phi'(y) \rangle| \leq C \mathrm{dist}\,(y, p_j), \tag{5}$$

for $y \in B(0,1)$, and if we set $F = w - \Phi$, then

$$|F| \leq 1/5. \tag{6}$$

Condition (5) implies that

$$\Phi(P_j z) - \Phi(z) \leq C|z - P_j z|^2,$$

if $|z| \leq 1$. Indeed, for every $x \in p_j \cap B(0,1)$ and for any $v \in S^{d-1}$ orthogonal to p_j, we have

$$\Phi(x) - \Phi(x + tv) = \int_0^t \langle -v, \Phi'(x + sv) \rangle ds \leq \int_0^t Cs\, ds \leq Ct^2.$$

In particular,

$$\Phi(z_{i+1}) - \Phi(z_i) \leq C|z_{i+1} - z_i|^2. \tag{7}$$

Summation yields

$$\Phi(z_m) - \Phi(z_1) \leq C \sum_{i=1}^{m-1} |z_{i+1} - z_i|^2 \leq C\varepsilon.$$

Thus by (6),

$$1 = \langle z_m - z_1, w\rangle = \Phi(z_m) - \Phi(z_1) + F(z_m) - F(z_1) \le C\varepsilon + 2/5.$$

For ε sufficiently small this yields the desired contradiction.

• The key to the whole proof is the construction of the calibration function Φ in Lemma 2.1. We finish this guide with its description.

We actually construct a piecewise affine "replacement" Φ of w in (4) satisfying (5) and (6). It is very close to the linear function w, and its derivative Φ' is very close to the constant mapping equal to w. In particular, on each line p_i of $\mathcal{L}$, $\Phi' = w_i$ on p_i.

• Let $u_i = p_i \cap B(0,1)$. We define Φ only where the piecewise linear curve $z_1, z_2, \ldots, z_m$ might appear, that is, on $\bigcup \mathrm{conv}\,(u_i \cup u_j)$.

• The construction of Φ on $\mathrm{conv}\,(u_1 \cup u_2)$, say, is based on the following two observations.

Suppose $A_i = w_i + \eta_i$, with $\eta_i \in \mathbb{R}$ small, are two affine functions equal at the point of p_1 where the lines p_1 and p_2 are closest. Then setting $\Phi(x) = A_i(x)$ if $\mathrm{dist}\,(x, p_i) \le \mathrm{dist}\,(x, p_j)$ works. Moreover, in Lemma 1.2 we show that $|A_1(x) - A_2(x)| \le \mathrm{dist}\,(u_1, u_2)^2$, for any $x \in u_i$.

Conversely, assume p_1 and p_2 are two lines and $A_i = w_i + \eta_i$ two affine functions such that $|A_1(x) - A_2(x)| \le \mathrm{dist}\,(u_1, u_2)^2$ for any $x \in u_i$. By Lemma 1.4, there exists a calibration function Φ on $\mathrm{conv}\,(u_1 \cup u_2)$ as required above so that $\Phi = A_i$ on u_i.

• In the proof of Lemma 2.1, we consider the complete weighted graph G on the vertices $[K]$, where $\mathrm{dist}\,(u_i, u_j)$ stands for the weight of the edge $\{i, j\}$. Let T be a minimum spanning tree of G. We first go inductively through the edges of T and use the first observation above to determine all constants η_i and Φ on $\mathrm{conv}\,(u_i \cup u_j)$ where $\{i, j\}$ is an arbitrary edge of T. The minimality of T ensures that the second observation can be used to determine Φ on $\mathrm{conv}\,(u_i \cup u_j)$ for the remaining pairs $\{i, j\}$.

• Since the lines in $\mathcal{L}$ are skew, the different $\mathrm{conv}\,(u_i \cup u_j)$ intersect only in the line segments u, and the above construction results in no conflicts.

1. Piecewise affine functions

For this entire section let $d \ge 4$ and let a large $K \in \mathbb{N}$ be fixed. Key to the proof of Theorem 2.3 is the construction of a certain potential function. This section remains very elementary, though. We prepare here the two and three dimensional affine blocs of which the potential constructed in Lemma 2.1 consists. A reader confident in his three-dimensional linear imagination might want to skip the proofs.

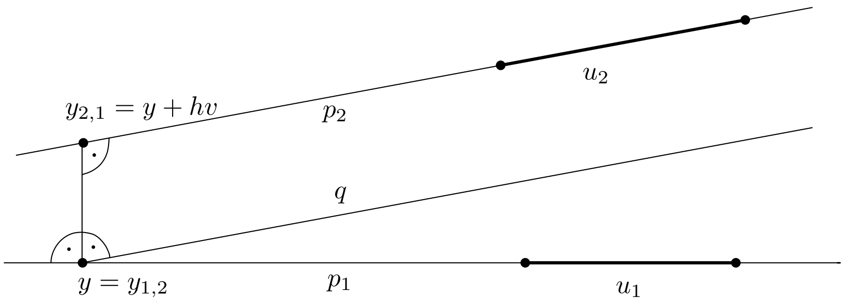

FIGURE 1. Skew lines p_1 and p_2 at distance h.

3-dimensional Setting: We consider two skew lines p_1 and p_2 at distance $h > 0$ (see Fig. 1). More precisely, for $i \in \{1, 2\}$, we assume that $w_i \in S^{d-1}$ and $x_i \in \mathbb{R}^d$ with $|w_1 - w_2| < 1/4$ and $|x_i| < 1/8$ are linearly independent and that $p_i = x_i + \operatorname{span} w_i$. Let $y_{i,j} \in p_i$ be the point for which $\operatorname{dist}(y_{i,j}, p_j) = h$. We assume, moreover, that both $|y_{1,2}| > K$ and $|y_{2,1}| > K$.

On the lines p_i we define the line segments $u_i = p_i \cap B(0, 1)$, and denote

$$m = \operatorname{dist}(u_1, u_2) = \min\{|x - y| : x \in u_1 \text{ and } y \in u_2\}.$$

We denote by X and Y the parallel two-dimensional affine subspaces of $\operatorname{aff}(p_1 \cup p_2)$ containing p_1 and p_2 respectively. Let $v \in S^{d-1}$ be such that $y_{2,1} = y_{1,2} + hv$. Notice that $Y = X + hv$, and that the linear function v is constant on X and also on Y; we denote the first constant by s. For brevity we also denote $y = y_{1,2}$; by q we denote the line $P_X(p_2)$.

We first show that the distance of any point of u_i from u_j is nearly equal to m.

Lemma 1.1. *Let $x \in u_i$ and $j \neq i$. Then $m \leq \operatorname{dist}(x, u_j) \leq 3m$. Moreover, $|w_1 - w_2| \leq 2m/|y|$.*

Proof. Let $x \in B(0, 1)$ be a point, p be a line, and $u = p \cap B(0, 1) \neq \emptyset$. It is easy to see that $\operatorname{dist}(x, u) \leq 2\operatorname{dist}(x, p)$.

Let $x \in u_1$ be given. To prove the lemma it is enough to show that

$$\operatorname{dist}(x, p_2) \leq 3m_1/2,$$

where $m_1 = \operatorname{dist}(u_1, p_2)$; because then

$$m \leq \operatorname{dist}(x, u_2) \leq 2\operatorname{dist}(x, p_2) \leq 3m_1 \leq 3m.$$

Indeed, let $x' = P_q(x)$ and $x'' = P_{p_2}(x') = P_{p_2}(x)$. Then $|x' - x''| = h$ and

$$\mathrm{dist}\,(x, p_2) = (|x - x'|^2 + |x' - x''|^2)^{1/2} = (|x - x'|^2 + h^2)^{1/2}.$$

Choose $a \in u_1$ so that $m_1 = \mathrm{dist}\,(a, p_2)$ and put $a' = P_q(a)$. By the similarity of the triangles yaa' and yxx' we have

$$\frac{|x - x'|}{|a - a'|} = \frac{|x - y|}{|a - y|} \le \frac{|a - y| + 2}{|a - y|} \le 1 + 2/(K - 1) \le \frac{3}{2},$$

since $|a - x| \le 2$ and $|a - y| \ge K - 1$. Hence

$$1 \le \frac{\mathrm{dist}\,(x, p_2)}{m_1} = \left(\frac{|x - x'|^2 + h^2}{|a - a'|^2 + h^2}\right)^{\frac{1}{2}} \le \frac{3}{2}.$$

The second inequality of the lemma follows again easily by similarity of suitable triangles. □

Let Q be the acute wedge

$$\{t_1 w_1 + t_2 w_2 + rv :\ t_i \ge 0, r \in [0, h]\}.$$

We define $W^+ = y + Q$ and $W^- = y - Q$, and the acute double-wedge $W = W^+ \cup W^-$. Notice that either $u_1 \cup u_2 \subset W^+$, or $u_1 \cup u_2 \subset W^-$; in particular, $\mathrm{conv}\,(u_1 \cup u_2) \subset W$.

For $\alpha_i \in \mathbb{R}$ we consider the affine functions $g_i = w_i + \alpha_i$. On $\mathbb{R}^d$ they define the piecewise affine function

$$G(x) = \begin{cases} g_1(x), & \text{if } \mathrm{dist}\,(x, p_1) \le \mathrm{dist}\,(x, q); \\ g_2(x), & \text{if } \mathrm{dist}\,(x, p_1) > \mathrm{dist}\,(x, q). \end{cases}$$

If $g_1(y) = g_2(y)$, then both g_1 and g_2 approximate G on $\mathrm{conv}\,(u_1 \cup u_2)$ very well.

Lemma 1.2. *Suppose $g_1(y) = g_2(y)$. Then for any $x \in \mathbb{R}^d$, $|g_1(x) - g_2(x)| \le m^2 + m\mathrm{dist}\,(x, u_1 \cup u_2)$. The function G is continuous, and for $x \in \mathbb{R}^d$,*

$$|g_i(x) - G(x)| \le m^2 + m\mathrm{dist}\,(x, u_1 \cup u_2).$$

If $x \in W \cap B(0, 1)$, then $|w_i - G'(x)| \le 2\mathrm{dist}\,(x, p_i)$.

Proof. The continuity of G is clear, and the estimate of its distance from g_i follows directly from the first inequality of the lemma. The derivative $G'(x)$ is w_1 or w_2 depending on whether x is closer to p_1 or to q. Assume $x \in X \cap W \cap B(0, 1)$ is such that $\mathrm{dist}\,(x, p_1) > \mathrm{dist}\,(x, q)$. Then

$$|w_1 - G'(x)| = |w_1 - w_2| \le 2\mathrm{dist}\,(x, p_1).$$

By the symmetry of W, this shows the second inequality in the lemma for all $x \in W \cap B(0,1)$.

Let $x \in \mathbb{R}^d$ be given. To show the estimate on $|g_1 - g_2|$, we can assume there is $b \in u_1$ so that $|x - b| = \operatorname{dist}(x, u_1 \cup u_2)$. Then

$$\begin{aligned}
&|g_1(x) - g_2(x)| = |\langle w_1 - w_2, x\rangle + \eta_1 - \eta_2| \\
&= |\langle w_1 - w_2, x - y\rangle + g_1(y) - g_2(y)| = |\langle w_1 - w_2, x - y\rangle| \\
&\leq |\langle w_1 - w_2, b - y\rangle| + |\langle w_1 - w_2, x - b\rangle| \leq |\langle w_1 - w_2, b - y\rangle| + m|x - b|,
\end{aligned}$$

since $|w_1 - w_2| \leq m$ by Lemma 1.1. Setting $b' = P_q(b)$, we have

$$\begin{aligned}
\pm\langle w_1, b - y\rangle &= |b - y| \\
\pm\langle w_2, b - y\rangle &= |b' - y|,
\end{aligned}$$

where the plus or minus signs depend on whether $\operatorname{conv}(u_1 \cup u_2)$ is contained in W^+ or in W^-. Hence

$$\begin{aligned}
|\langle w_1 - w_2, b - y\rangle| = |b - y| - |b' - y| &= \frac{|b - y|^2 - |b' - y|^2}{|b - y| + |b' - y|} \\
&\leq \frac{|b - b'|^2}{K - 1} \leq \frac{9m^2}{K - 1},
\end{aligned}$$

since $K - 1 \leq |b - y|$ and $|b - b'| \leq \operatorname{dist}(b, u_2) \leq 3m$ by Lemma 1.1. □

Now we construct a piecewise affine function on a strip. Let $\tilde{h} > 0$ and $\eta \in \mathbb{R}$ be given. We define a piecewise linear function φ on $\mathbb{R}$ as follows:

$$\varphi(t) = \begin{cases} 0 & \text{if } t \leq \tilde{h}/3\ ; \\ \eta(3t/\tilde{h} - 1) & \text{if } \tilde{h}/3 \leq t \leq 2\tilde{h}/3\ ; \\ \eta, & \text{if } 2\tilde{h}/3 \leq t. \end{cases}$$

For $\tilde{v} \in S^{d-1}$ and $\tilde{s} \in \mathbb{R}$ we define a piecewise affine function $H = \varphi \circ (\tilde{v} - \tilde{s})$ on $\mathbb{R}^d$. We denote by $\tilde{X}_i$ the two hyperplanes in $\mathbb{R}^d$ where $\tilde{v} = \tilde{s}$, or where $\tilde{v} = \tilde{s} + \tilde{h}$, respectively.

Lemma 1.3. *Suppose $|\eta| \leq c\tilde{h}^2$ for some $c > 0$. Then H is continuous, $|H| \leq |\eta|$ and $|H'(x)| \leq 9c \operatorname{dist}(x, \tilde{X}_i)$ for $x \in \mathbb{R}^d$.*

Proof. We check just the last inequality. If $\operatorname{dist}(x, \tilde{X}_1 \cup \tilde{X}_2) < \tilde{h}/3$, then $H'(x) = 0$. Otherwise

$$|H'(x)| \leq |3\eta/\tilde{h}\tilde{v}| = 3|\eta|/\tilde{h} \leq 3c\tilde{h} \leq 9c \operatorname{dist}(x, \tilde{X}_i).$$

□

Finally, we show a converse to Lemma 1.2.

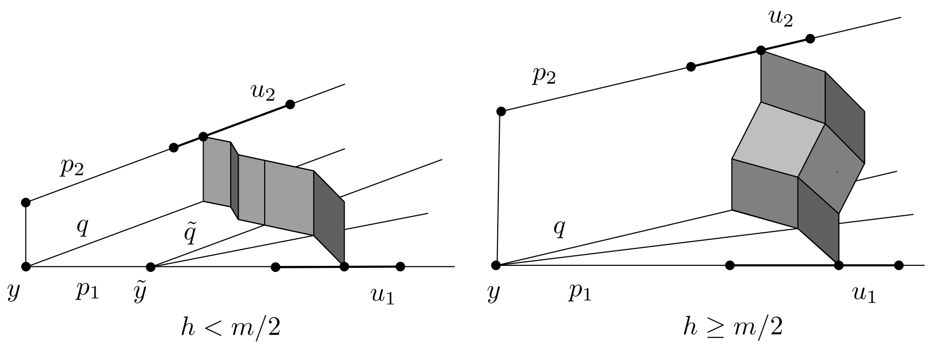

FIGURE 2. Level sets of the function A in the wedge W.

Lemma 1.4. *Let $c \geq 1$, $\eta_i \in \mathbb{R}$ and let $A_i = w_i + \eta_i$ be two affine functions such that*

$$|A_1(x) - A_2(x)| \leq cm^2$$

for all $x \in u_2$. Then there is a continuous piecewise affine function A so that $A = A_i$ on a neighborhood of u_i and for $x \in \operatorname{conv}(u_1 \cup u_2)$ we have $|A(x) - A_i(x)| \leq 6cm$ and

$$|A'(x) - w_i| \leq C \operatorname{dist}(x, p_i),$$

where $C > 0$ depends on c only.

Proof. We distinguish two cases. First suppose that $h = \operatorname{dist}(X, Y) \geq m/2$. Let $\alpha_1 = \eta_1$ and $\alpha_2 = A_1(y) - \langle w_2, y \rangle$ and $\eta = \eta_2 - \alpha_2$. Let G be defined as in Lemma 1.2 and H be as above, where we set $\tilde{h} = h$, $\tilde{v} = v$, and $\tilde{s} = s$. We define $A = G + H$, and fix some $b \in u_2$. Since $A_1 = g_1$,

$$|\eta| = |A_2(b) - G(b)| \leq |A_1(b) - A_2(b)| + |A_1(b) - G(b)| \leq (c+1)m^2$$

by Lemma 1.2. Let $x \in \operatorname{conv}(u_1 \cup u_2)$ be given. Then

$$\begin{aligned} |A(x) - A_i(x)| &\leq |G(x) - A_i(x)| + \max|\varphi| \leq |G(x) - g_i(x)| + 2\eta \\ &\leq 4m + 2\eta \leq 6cm. \end{aligned}$$

Also,

$$|A'(x) - w_i| \leq |G'(x) - w_i| + |H'(x)| \leq 11c \operatorname{dist}(x, p_i),$$

by Lemma 1.2 and Lemma 1.3.

If $h < m/2$, then we make A depend on $P_X(x)$ only. The construction of A is similar to the one above; therefore we just sketch it here (see also Fig. 2).

Let $\tilde{y} \in p_1$ be the midpoint of the line segment connecting y and u_1, and $\tilde{q} = \tilde{y} + \operatorname{span} w_2$ a line parallel to q. We define $\tilde{A}_2 = w_2 + \tilde{\eta}_2$, where $\tilde{\eta}_2 = A_1(\tilde{y}) - \langle w_2, \tilde{y} \rangle$. Let G be the piecewise affine continuous function

$$G(x) = \begin{cases} A_1(x), & \text{if } \operatorname{dist}(x, p_1) \leq \operatorname{dist}(x, \tilde{q}); \\ \tilde{A}_2(x), & \text{if } \operatorname{dist}(x, p_1) > \operatorname{dist}(x, \tilde{q}). \end{cases}$$

Let $\tilde{h} > 0$ and $\tilde{v} \in S^{d-1} \cap \operatorname{span}\{w_1, w_2\}$ orthogonal to w_2 be such that $q = \tilde{q} + \tilde{h}\tilde{v}$. Let $\eta = \eta_2 - \tilde{\eta}_2$. We set $H = \varphi \circ (\tilde{v} - \langle \tilde{v}, \tilde{y} \rangle)$ and define $A = G + H$. □

2. Projections on lines

In the plane $\mathbb{R}^2$ consider the unit line segment $[0, 1]$ on the x-axis and the family $\mathcal{L}$ of $K + 1$ lines parallel to the y-axis, intersecting $[0, 1]$ at the points $0, 1/K, 2/K, \ldots, 1$. The points $z_i = i/K$ form a sequence of projections on the lines in $\mathcal{L}$ and at the same time

$$|z_0 - z_K| = 1 \text{ and } \sum |z_{i+1} - z_i|^2 = 1/K < \varepsilon,$$

if K is large enough. In this section we will show, that with a fixed K number of lines, and very small $\varepsilon > 0$ this cannot occur.

Let $\mathcal{L}$ be a family of K lines in $\mathbb{R}^d$. Theorem 2.3 states that if $z_1, \ldots, z_m$ is a sequence of projections on the lines in $\mathcal{L}$, then

$$|z_1 - z_m|^2 \leq c(K) \sum_{i=1}^{m-1} |z_{i+1} - z_i|^2,$$

where $c(K) > 0$ depends on K only. In Lemma 2.2 we reduce the proof to families of almost parallel lines as in the Setting below.

Let K_0 be a large enough natural number (just how large can be in principle determined by an inspection of the estimates in the proof of Lemma 2.2). For a fixed $K \geq K_0$, we consider a family $\mathcal{L}$ of K line segments $u_1, \ldots, u_K$ in general position, which stay very close to a given unit line segment u, but do not intersect each other in a very strong sense.

Setting: Let $w \in S^{d-1}$, $K_0 \leq K \in \mathbb{N}$, and $0 < \delta < 1/10^4$ be given. Let u be the line segment $[-w, w]$ and $p_0 = \operatorname{span} w$. Consider a family $\mathcal{L}$ of K lines $p_i = x_i + \operatorname{span} w_i$, where the vectors $x_i \in \mathbb{R}^d$, $w_i \in S^{d-1} \cap B(w, \delta)$, $i \in [K]$, are linearly independent and such that:

(i) $u \subset B(p_i, \delta/K^2)$ for $i \in [K]$.

(ii) For $i \neq j$, let $y_{i,j} \in p_i$ be the point for which $\operatorname{dist}(y_{i,j}, p_j) = \operatorname{dist}(p_i, p_j)$. Then $|y_{i,j}| > K$.

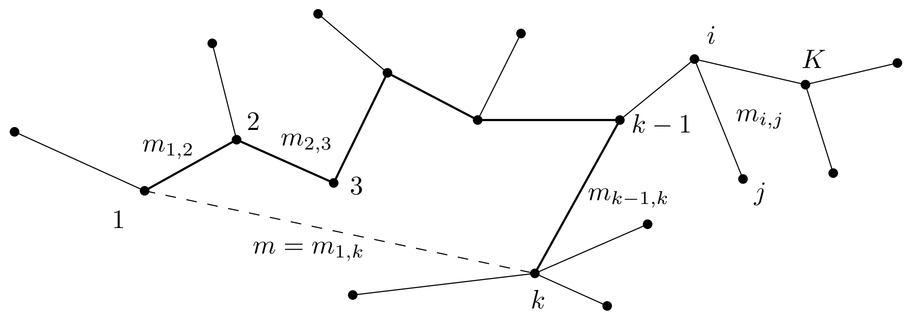

FIGURE 3. A spanning tree T of G.

We set $u_i = p_i \cap B(0,1)$. For $i \neq j$ we set $\mathcal{C}_{i,j} = \operatorname{conv}(u_i \cup u_j)$ and $\mathcal{C} = \bigcup \mathcal{C}_{i,j}$. We put also $\mathcal{Y} = \{y_{i,j} : i \neq j\}$.

Key to the proof of Theorem 2.3 is the following construction of an "error" function F. The potential F is small in absolute value, so that the function $\Phi = w - F$ is a small perturbation of the linear function w. Moreover, its derivative Φ' is a useful extension of the mapping equal to w_i on each p_i. In particular, it is a small perturbation of the constant mapping which equals w.

Lemma 2.1. *There exists a constant $C > 0$, depending only on K, such that for every family $\mathcal{L}$ of lines as in the Setting, there is a continuous piecewise affine function F on $\mathcal{C}$ so that $|F| \leq 1/5$ and*

$$|\langle v, w - F'(x)\rangle| \leq C \operatorname{dist}(x, p_i)$$

for all $v \in S^{d-1}$ orthogonal to p_i, and all $x \in \mathcal{C}_{i,j}$, $i, j \in [K]$.

Proof. We consider the complete weighted graph G on the K vertices $[K]$ with the weight function

$$m_{i,j} = \operatorname{dist}(u_i, u_j).$$

Let T be a minimum spanning tree of G; we denote by E the edges of T (see Fig. 3). For $i \in K$ we define affine functions

$$A_i = w_i + \eta_i$$

where the constants $\eta_i \in \mathbb{R}$ are defined inductively through the edges of E. We set $\eta_1 = 0$. Suppose $\{i, j\} \in E$. If η_i has already been defined but η_j is not, we put

$$\eta_j = A_i(y_{i,j}) - \langle w_j, y_{i,j}\rangle.$$

By Lemma 1.1,

$$|\eta_j| = |\langle w_i - w_j, y_{i,j}\rangle + \eta_i| \le |\eta_i| + |w_i - w_j| \cdot |y_{i,j}| \le |\eta_i| + 2m_{i,j}$$
$$\le |\eta_i| + 4\delta/K^2,$$

since $m_{i,j} \le 2\delta/K^2$ by (i) of the Setting. By induction we get then $|\eta_i| \le 4\delta$ for all $i \in [K]$, since $\eta_1 = 0$.

For $\{i, j\} \in E$, let X and Y be the parallel two-dimensional subspaces of aff $\{p_i \cup p_j\}$ containing p_i and p_j respectively. For $x \in \mathcal{C}_{i,j}$, we define

$$\Phi(x) = \begin{cases} A_i(x), & \text{if dist}\,(x, p_i) \le \text{dist}\,(x, P_X(p_j)); \\ A_j(x), & \text{if dist}\,(x, p_i) \ge \text{dist}\,(x, P_X(p_j)). \end{cases}$$

Hence Φ is continuous and $\Phi = A_i$ on u_i for each $i \in [K]$. By Lemma 1.2, if $\{i, j\} \in E$, then

$$|w_i - \Phi'(x)| \le 2\text{dist}\,(x, p_i),$$

for $x \in \mathcal{C}_{i,j}$, and

$$|\Phi(x) - \langle w_i, x\rangle| \le |w_i - w_j| + \max\{\eta_i, \eta_j\} \le 2\delta + 4\delta \le 1/10.$$

Now let a pair of indices which is not in E be given. For further easier indexing, we can assume that it is of the form $\{1, k\}$. Later we will show that if $x \in u_k$, then

$$|A_1(x) - A_k(x)| \le 10K^2m^2, \tag{8}$$

where $m = m_{1,k}$. Lemma 1.4 then implies that there exists a continuous piecewise affine function Φ with the following property. If $j \in \{1, k\}$, then $\Phi = A_j$ on u_j and for $x \in \mathcal{C}_{1,k}$,

$$|w_j - \Phi'(x)| \le C\text{dist}\,(x, p_j),$$

where $C > 0$ is a constant depending on K only. Moreover,

$$|\Phi(x) - \langle w_j, x\rangle| = |\Phi(x) - A_j(x) + \eta_j| \le 60K^2m + 4\delta \le 124\delta \le 1/10.$$

In order to show (8), we choose the unique path from 1 to k in T. We can assume that it corresponds to the vertices $1, 2, \dots, k$. Since T is a minimum spanning tree,

$$m_{i,i+1} \le m \text{ for } 1 \le i < k.$$

We choose an arbitrary $x_1 \in u_1$ and then by Lemma 1.1 inductively choose $x_i \in u_i$ so that $|x_i - x_{i+1}| \le 3m_{i,i+1}$. The triangle inequality implies that

$$m_{i,k} \le |x_i - x_k| \le |x_i - x_{i+1}| + \cdots + |x_{k-1} - x_k|$$
$$\le 3(m_{i,i+1} + \cdots + m_{k-1,k}) \le 3Km.$$

Since $\{i, i+1\} \in E$, if $x \in u_k$ then by Lemma 1.2,

$$|A_i(x) - A_{i+1}(x)| \le m_{i,i+1}^2 + m_{i,i+1}\mathrm{dist}\,(x, u_i) \le m_{i,i+1}^2 + 3m_{i,i+1}m_{i,k} \le (9K+1)m^2,$$

since $\mathrm{dist}\,(x, u_i) \le 3m_{i,k}$ by Lemma 1.1. To get inequality (8) for $x \in u_k$ we estimate

$$|A_1(x) - A_k(x)| \le |A_1(x) - A_2(x)| + \cdots + |A_{k-1}(x) - A_k(x)| \le 10K^2m^2.$$

For $x \in \mathcal{C}$ we define $F(x) = \langle w, x\rangle - \Phi(x)$. Assume $i, j, k, l \in [K]$ are four different indices. Then $\mathcal{C}_{i,j} \cap \mathcal{C}_{k,l} = \emptyset$ and $\mathcal{C}_{i,j} \cap \mathcal{C}_{i,l} = u_i$. Hence F is a continuous piecewise affine function on $\mathcal{C}$.

For every $x \in \mathcal{C}$ there is an $i_x \in [K]$ so that $|\Phi(x) - \langle w_{i_x}, x\rangle| \le 1/10$. Hence

$$|F(x)| \le |w - w_{i_x}| + |\Phi(x) - \langle w_{i_x}, x\rangle| < 1/5.$$

Since $F' = w - \Phi'$,

$$|\langle v, w - F'(x)\rangle| = |\langle v, \Phi'(x) - w_i\rangle| \le |\Phi'(x) - w_i| \le C\mathrm{dist}\,(x, p_i)$$

for all $x \in \mathcal{C}_{i,j}$ and $v \in S^{d-1}$ orthogonal to p_i. □

In order to use Lemma 2.1 in the proof of Theorem 2.3, we need the following reduction to lines as in the Setting.

Lemma 2.2. *Suppose $K, d \in \mathbb{N}$ are such that $K_0 \le K$ and $4K < d$. Suppose that for every $\varepsilon > 0$, there is a family $\mathcal{L}_\varepsilon$ of K lines in $\mathbb{R}^d$ and a sequence $z_1, \dots, z_{m_\varepsilon}$ of projections on these lines so that*

$$|z_1 - z_{m_\varepsilon}| = 1 \text{ and } \sum_{i=1}^{m_\varepsilon - 1} |z_{i+1} - z_i|^2 < \varepsilon. \tag{9}$$

Let $w \in S^{d-1}$ and $0 < \delta < 1/10^4$ be given. Then for every $\varepsilon > 0$, there exists a family $\mathcal{L}_\varepsilon$ as above which also satisfies the conditions in the Setting. Moreover, $\langle z_{m_\varepsilon} - z_1, w\rangle = 1$.

Proof. We denote the lines in $\mathcal{L}_\varepsilon$ by $p_1^\varepsilon, \dots, p_K^\varepsilon$ and call γ_ε the piecewise linear curve $z_1, \dots, z_{m_\varepsilon}$. To construct a "better" γ_ε, that is, one defined by a family $\mathcal{L}_\varepsilon$ of almost parallel lines, we pick a somewhat better sub-curve of γ_ε, then we choose a still better sub-curve of the new curve, and so on. We always truncate γ_ε at one of the points z_i. At the end we blow the resulting curve up to diameter one.

First we make sure that all curves γ_ε are uniformly bounded. Translating the whole picture by $-z_1$ we can assume that each γ_ε starts at the origin. We truncate γ_ε the first time it gets out of the unit ball, to get $|z_i| \le 1$ for all i. Then all lines in $\mathcal{L}_\varepsilon$ that are really in use (and from now on, we include in $\mathcal{L}_\varepsilon$ only such lines, if needed repetitiously) are contained in the compact set of lines which intersect $B(0,1)$. We can therefore also assume that $\lim_{\varepsilon\to 0} p_i^\varepsilon = q_i$ for $i \in [K]$. Not all of the K lines q_i are necessarily different. By passing to sub-curves we will actually arrange that all p_i's are close to one line q.

The reason why this is possible is intuitively obvious. If a curve γ of diameter one is contained in the union of K lines, then a "long" sub-curve of γ is contained in one of the lines. We make this more precise.

Let I consist of all possible intersections of the lines in $Q = \{q_1, \dots, q_K\}$; then $|I| < K^2/2$. We thicken up the lines in Q to pipes of radius r so that the pipes intersect only within $B(I, 1/K^2)$. In particular, we choose $0 < r < \delta/(19K^6)$ so that if $q_i \neq q_j$ then

$$B(q_i, r) \cap B(q_j, r) \subset B(q_i \cap q_j, 1/K^2). \tag{10}$$

We choose $0 < \sqrt{\varepsilon} < r/2$ so small that all p_i^ε's are already close to the q_i's:

$$p_i^\varepsilon \cap B(0,1) \subset B(q_i, r/2). \tag{11}$$

Hence

$$\gamma_\varepsilon \subset B(\mathcal{L}_\varepsilon, \sqrt{\varepsilon}) \cap B(0,1) \subset B(Q, r).$$

Since $\operatorname{diam} \gamma_\varepsilon \geq 1 - \sqrt{\varepsilon}$, and $|I| < K^2/2$, there exist k and a point z of γ_ε so that $z \in B(q_k, r) \setminus B(I, 2/K^2)$. By watching where γ_ε leaves the ball $B(z, 1/K^2)$, we get a sub-curve $\tilde{\gamma}_1$ so that

$$1/(2K^2) < \operatorname{diam} \tilde{\gamma}_1 < 2/K^2.$$

The curve $\tilde{\gamma}_1 \subset B(Q, r) \setminus B(I, 1/K^2)$ can be by (10) contained only in one component of the latter set. Hence $\tilde{\gamma}_1 \subset B(q_k, r)$. Moreover, $\operatorname{dist}(\tilde{\gamma}_1, p_j) > r/2 > \sqrt{\varepsilon}$ always when $q_j \neq q_k$ by (11), hence

$$p \cap B(0,1) \subset B(q_k, r/2) \text{ if } p \in \tilde{\mathcal{L}}, \tag{12}$$

where $\tilde{\mathcal{L}} \subset \mathcal{L}_\varepsilon$ is the set of the lines really in use in $\tilde{\gamma}_1$. We call $p_1, \dots, p_k$ the lines in $\tilde{\mathcal{L}}$; if needed, we use repetition to achieve $|\tilde{\mathcal{L}}| = K$.

Let $e_1, \dots, e_{2K}$ be orthonormal vectors in the orthogonal complement of $\operatorname{span} \tilde{\mathcal{L}}$ and $\alpha > 0$ be very small. To ensure that the lines in $\tilde{\mathcal{L}}$ are skew, we replace the original lines $p_i = x_i + \operatorname{span} w_i$ by their small perturbations

$$(x_i + \alpha e_i) + \operatorname{span}(w_i + \alpha e_{2i}),$$

so that (12) is still satisfied, and call them p_i's again. Notice, that (12) ensures that the directional vectors of the lines are at distance at most δ from w.

Since the projections are Lipschitz mappings, for the perturbed lines we obtain a curve $\tilde\gamma_2$ very close to $\tilde\gamma_1$ so that the corresponding sequence $z_1, \dots, z_m$ of projections satisfies

$$1/(2K^2) < \operatorname{diam} \tilde\gamma_2 < 2/K^2 \text{ and } \sum_{i=1}^{m-1} |z_{i+1} - z_i|^2 < \varepsilon.$$

To ensure (ii) of the Setting we need $\tilde\gamma_2$ to avoid the K^2 points of $\mathcal{Y}$. Since $\operatorname{diam} \tilde\gamma_2 > 1/(2K^2)$, there exists $z \in \tilde\gamma_2$ so that $|z - y| > 1/(2K^4)$ for all $y \in \mathcal{Y}$. By watching where $\tilde\gamma_2$ leaves the ball $B(z, 1/(8K^5))$, we obtain a part $\tilde\gamma_3$ of $\tilde\gamma_2$ such that

$$1/(9K^5) \le \operatorname{diam} \tilde\gamma_3 < 1/(4K^5) \text{ and } \operatorname{dist}(\tilde\gamma_3, \mathcal{Y}) \ge 1/(4K^4). \tag{13}$$

By translation, we can assume that $\tilde\gamma_3$ starts at zero on, say, the line p_1. If we blow the whole picture up by $c = 1/\operatorname{diam} \tilde\gamma_3$, we get a curve γ and a corresponding sequence $0 = z_1, \dots, z_m$ of projections on the lines in $\mathcal{L} = c\tilde{\mathcal{L}}$. Since $4K^5 < c \le 9K^5$,

$$|z_1 - z_m| = |z_m| = 1 \text{ and } \sum_{i=1}^{m-1} |z_{i+1} - z_i|^2 < 9K^5 \varepsilon.$$

Up to an isometry we can assume that $w = z_m$. Then $\langle z_m - z_1, w\rangle = 1$. Since $z_m \in p_2$, say, $z_m \in B(p_1, cr)$ by (12). Since p_1 contains the origin, we also have $-w = -z_m \in B(p_1, cr)$, and again by (12),

$$u \subset B(p, 2cr) \subset B(p, \delta/K)$$

for $p \in \tilde{\mathcal{L}}$ and (i) of the Setting is satisfied. By (13),

$$|y_{i,j}| \ge \operatorname{dist}(\mathcal{Y}, 0) \ge \operatorname{dist}(\mathcal{Y}, \gamma) \ge c/(4K^4) > K,$$

and (ii) of the Setting is satisfied as well. □

Next comes our main result on the rate of convergence of projections on lines. The case where all of the lines intersect at one point, and the sequence of projections necessarily converges, appears in [DR]. Somewhat surprisingly, our proof for general lines seems to be conceptually simpler than the one in [DR].

Theorem 2.3. *For every $K \in \mathbb{N}$ there is a constant $c(K)$ depending only on K with the following property. If $\mathcal{L}$ is a family of K lines in $\mathbb{R}^d$ and $z_1, z_2, \dots$ is a sequence of orthogonal projections on the lines in $\mathcal{L}$ then*

$$\operatorname{diam}^2 \{z_i\}_{i=1}^\infty \le c(K) \sum_{i=1}^\infty |z_{i+1} - z_i|^2.$$

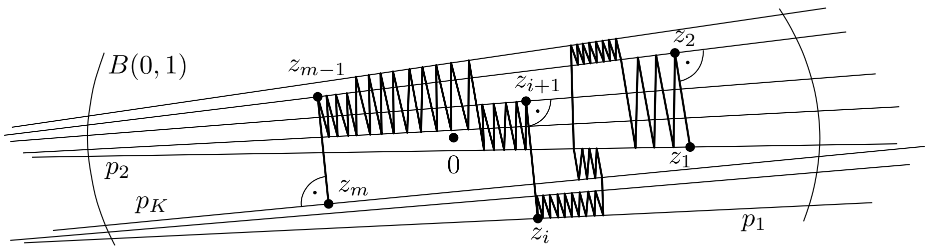

FIGURE 4. Sequence of projections on the lines in $\mathcal{L}$.

Proof. Assume the statement of the theorem is false for some K. We can assume that K is larger than a fixed constant K_0 and that $d > 4K$. By scaling the whole picture we then get, for every $\varepsilon > 0$, a collection $\mathcal{L}_\varepsilon = \{p_1, \ldots, p_K\}$ of K lines, a sequence $k_1, \ldots, k_{m_\varepsilon} \in [K]$, and $z_1 \in p_{k_1}$ with the following property. If we denote by P_k the projection onto p_k, and define $z_{i+1} = P_{k_{i+1}} z_i$, then

$$|z_1 - z_{m_\varepsilon}| = 1 \text{ and } \sum_{i=1}^{m_\varepsilon - 1} |z_{i+1} - z_i|^2 < \varepsilon. \tag{14}$$

Let $C > 0$ be the constant from Lemma 2.1. We fix some $0 < \varepsilon < 1/(5C)$ and $0 < \delta < 1/99$, and from now on we drop the indices ε. By Lemma 2.2 we can assume that $\mathcal{L}$ is as in the Setting for some fixed $w \in S^{d-1}$ and the piecewise linear curve $\gamma = (z_1, \ldots, z_m)$ is contained in $\mathcal{C}$ (see Fig. 4). We use an arc-length parametrization $\gamma : [0, s] \to \mathcal{C}$. Let $0 = s_1 < s_2 < \cdots < s_m = s$ satisfy $\gamma(s_i) = z_i$. We denote

$$v_i = \frac{z_{i+1} - z_i}{|z_{i+1} - z_i|} \in S^{d-1}.$$

Then v_i is orthogonal to $p_{k_{i+1}}$. Moreover, $\gamma(t) \in \mathcal{C}_{k_i, k_{i+1}}$ and $\gamma'(t) = v_i$ for $t \in (s_i, s_{i+1})$. By Lemma 2.2,

$$\begin{aligned} 1 = \langle z_m - z_1, w \rangle &= \sum_{i=1}^{m-1} \langle z_{i+1} - z_i, w \rangle = \sum_{i=1}^{m-1} \int_{s_i}^{s_{i+1}} \langle \gamma'(t), w \rangle dt \\ &= \sum_{i=1}^{m-1} \int_{s_i}^{s_{i+1}} \langle v_i, w \rangle dt. \end{aligned} \tag{15}$$

Let F be the continuous piecewise affine function from Lemma 2.1. Then

$$\langle v_i, w \rangle \le C \operatorname{dist}(\gamma(t), p_{k_{i+1}}) + \langle F'(\gamma(t)), v_i \rangle$$

for all $t \in (s_i, s_{i+1})$, and we can continue with (15) as follows:

$$\begin{aligned}
&\leq C \sum_{i=1}^{m-1} \int_{s_i}^{s_{i+1}} \operatorname{dist}(\gamma(t), p_{k_{i+1}})dt + \sum_{i=1}^{m-1} \int_{s_i}^{s_{i+1}} \langle F'(\gamma(t)), v_i \rangle dt \\
&\leq C \sum_{i=1}^{m-1} \int_{0}^{s_{i+1}-s_i} t dt + \sum_{i=1}^{m-1} \int_{s_i}^{s_{i+1}} \langle F'(\gamma(t)), \gamma'(t) \rangle dt \\
&= C/2 \sum_{i=1}^{m-1} (s_{i+1} - s_i)^2 + \sum_{i=1}^{m-1} F(z_{i+1}) - F(z_i) \\
&= C/2 \sum_{i=1}^{m-1} |z_{i+1} - z_i|^2 + F(z_m) - F(z_1) \\
&\leq C\varepsilon/2 + 2/5 < 1/2,
\end{aligned}$$

and this is a contradiction. □

3. Projections on subspaces

Let $\mathcal{L}$ be a family of K closed linear subspaces of ℓ_2. Any sequence $\{z_i\}$ of orthogonal projections on the spaces in $\mathcal{L}$ converges weakly according to [AA]. If $K = 2$ the sequence of projections even converges in norm [vN]. If $K \geq 3$, this is known only under additional assumptions, for example, if the sequence $\{k_i\}$ is (quasi) periodic [H, S]. In this section we give a necessary and sufficient condition ensuring norm convergence.

The following observation is well known. In Theorem 2.3 we verified that its assumptions are satisfied for finite families of one-dimensional affine subspaces of ℓ_2. For one-dimensional linear subspaces this was done already in [DR].

Proposition 3.1. *Suppose that for some $K \in \mathbb{N}$ there is a constant $c(K)$ with the following property. If $z_1, z_2, \ldots$ is a sequence of orthogonal projections on K finite dimensional subspaces ℓ_2, then*

$$|z_1 - z_m|^2 \leq c(K) \sum_{i=1}^{m-1} |z_{i+1} - z_i|^2.$$

Then if $\mathcal{L}$ is a family of K closed linear subspaces of ℓ_2, then any sequence of orthogonal projections on the subspaces in $\mathcal{L}$ converges in norm.

Proof. Suppose $z_1, z_2, \ldots$ are successive projections on K closed subspaces of ℓ_2. By Pythagoras' theorem, $|z_i|^2 = |z_{i+1}|^2 + |z_{i+1} - z_i|^2$.

Hence

$$|z_j - z_k|^2 \le c(K) \sum_{i=j}^{k-1} |z_{i+1} - z_i|^2 = c(K)(|z_j|^2 - |z_k|^2). \tag{16}$$

Since the sequence $\{|z_i|^2\}$ is nonincreasing and hence convergent, the sequence $\{z_i\}$ is Cauchy. □

A slightly weaker assumption than the one of Proposition 3.1 already causes random projections to converge. Conversely, its lack implies the existence of a sequence of random projections that converges only weakly but not in norm as we will show in Theorem 3.2.

Let $K \in \mathbb{N}$, and let $\boldsymbol{\delta}_K : (0,1] \to (0,2]$ be defined by

$$\boldsymbol{\delta}_K(\varepsilon) = \sup |z_1 - z_m|,$$

where the supremum is taken over all sequences $\{z_i\}_1^{m_\varepsilon} \subset B_{\ell_2}$ of projections on some K finite dimensional subspaces of ℓ_2, for which

$$|z_1|^2 - |z_m|^2 = \sum_{i=1}^{m-1} |z_{i+1} - z_i|^2 \le \varepsilon.$$

Clearly, $\boldsymbol{\delta}_K$ is an increasing positive function of ε, hence $\lim_{\varepsilon \to 0} \boldsymbol{\delta}_K(\varepsilon)$ always exists. Proposition 3.1 above deals with the hypothetical situation when $\boldsymbol{\delta}_K(\varepsilon) \le c(K)\sqrt{\varepsilon}$ for all $\varepsilon > 0$ and some $c(K) > 0$ depending on K only.

Theorem 3.2. *Let $K \in \mathbb{N}$.*

(i) *Suppose $\lim_{\varepsilon \to 0} \boldsymbol{\delta}_K(\varepsilon) = 0$. If $\mathcal{L}$ is a family of K closed linear subspaces of ℓ_2, then any sequence of orthogonal projections on the subspaces in $\mathcal{L}$ converges in norm.*

(ii) *Suppose $\lim_{\varepsilon \to 0} \boldsymbol{\delta}_K(\varepsilon) = r > 0$. Then for every $\tilde{K} > 9K/r$, there is a family $\mathcal{L}$ of $\tilde{K}$ closed linear subspaces of ℓ_2 and a sequence of orthogonal projections on the subspaces in $\mathcal{L}$ that does not converge in norm.*

Proof. To show (i) we proceed exactly as in Proposition 3.1. Suppose $z_1, z_2, \dots$ are successive projections on K closed subspaces of ℓ_2. By Pythagoras' theorem,

$$|z_j - z_k| \le \boldsymbol{\delta}_K\Big(\sum_{i=j}^{k-1} |z_{i+1} - z_i|^2\Big) = \boldsymbol{\delta}_K(|z_j|^2 - |z_k|^2). \tag{17}$$

Since the sequence $\{|z_i|^2\}$ is nonincreasing, the sequence $\{z_i\}$ is Cauchy.

To verify (ii), let $u, v \in S_{\ell_2}$ so that $|u - v| \le r$ and $1 \ge s > t > 1/2$ be given. By the assumptions, there exist $K+1$ finite dimensional subspaces of ℓ_2 and a sequence $z_1, \dots, z_m$ of projections on these subspaces so that $z_1 = su$ and $z_m = t'v$, where $s > t' \geq t$. Indeed, for $\varepsilon > 0$ small enough it suffices to choose K subspaces and a sequence $x_1, \dots, x_n \in B_{\ell_2}$ of projections on these spaces so that $|x_1|^2 - |x_n|^2 \leq \varepsilon$ and $|x_1 - x_n|$ is nearly equal to r. We truncate the sequence so that the angle $x_1, 0, x_n$ nearly corresponds to the angle $u, 0, v$. Then we scale the sequence so that $|x_1| = s$. We use an isometry to achieve $x_1 = su$ and that x_n nearly lies on the line $p = \operatorname{span} v$. Finally, we project on p.

Let $k = \lceil \pi/(2r) \rceil$. We will inductively construct an orthonormal sequence $\{e_n\}$ in ℓ_2 and finite dimensional subspaces $F^n_{i,j}$, $i \in [K+1]$, $j \in [k]$, and $n \in \mathbb{N}$, so that if $|m - n| \geq 2$, then $F^m_{i,j}$ is orthogonal to $F^n_{i,j}$. For $i \in [K+1]$ and $j \in [k]$, we define

$$p_{i,j} = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} F^{2n-1}_{i,j},$$
$$q_{i,j} = \overline{\operatorname{span}} \bigcup_{n=1}^{\infty} F^{2n}_{i,j}.$$

This is a family $\mathcal{L}$ of $2k(K+1) \leq 9K/r$ closed linear subspaces of ℓ_2. In the construction we, moreover, arrange that there is as sequence of projections on the spaces in $\mathcal{L}$, which contains $t_n e_n$ for some $1 = t_1 > t_2 > \cdots > 1/2$ as a subsequence. Such a sequence does not converge in norm.

To start the induction we choose two orthonormal vectors e_1 and e_2. We divide the quarter-circle connecting e_1 and e_2 into k sectors of equal length; we call the division points $e_1 = u_0, u_1, \dots, u_k = e_2$. We choose some $1 = s_0 > s_1 > \cdots > s_k > 1/2 + 1/3$. We choose finite dimensional spaces $F^1_{i,1}$, $i \in [K+1]$, and a sequence of projections on these spaces, so that the first point is $e_1 = u_0$ and the last point is $s'_1 u_1$ for some $s'_1 \geq s_1$. Next we choose the spaces $F^1_{i,2}$ and a sequence of projections on them starting at $s'_1 u_1$ and finishing at $s'_2 u_2$ for some $s'_2 \geq s_2$. We continue in this manner, till we reach via the spaces $F^1_{i,k}$ the point $t_2 e_2$ for some $t_2 > 1/2 + 1/3$.

Suppose orthonormal vectors $e_1, \dots, e_{n-1}$ and spaces $F^m_{i,k}$ for $m \leq n-1$ with a sequence of points finishing at $t_{n-1} e_{n-1}$ with $t_{n-1} > 1/2 + 1/n$ have already been constructed. We choose $e_n \in S_{\ell_2}$ orthogonal to all vectors $e_1, \dots, e_{n-1}$, and to all spaces $F^m_{i,k}$, $m \leq n-1$. We again divide the quarter-circle connecting e_{n-1} and e_n into k sectors of equal length and construct in k steps the spaces $F^n_{i,1}, \dots, F^n_{i,k}$, $i \in [K+1]$, and sequences of projections connecting $t_{n-1} e_{n-1}$ to $t_n e_n$ for some $t_n > 1/2 + 1/(n+1)$. Moreover, in each step we make sure that $F^n_{i,j}$ is orthogonal to all vectors $e_1, \dots, e_{n-2}$, and to all spaces $F^m_{i,k}$, $m \le n-2$. □

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