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Remarks on Bishop - type operators

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INTRODUCTION

Let $\alpha \in (0,1)$ be an irrational number and $T_\alpha : L^2[0,1) \to L^2[0,1)$ be the operator acting on the space $L^2[0,1)$ consisting of all classes of square integrable functions on the real interval $[0,1)$ by

$$(T_\alpha h)(x) = xh(\{x+\alpha\}), \qquad x \in [0,1)$$

where for any real number y,

$$\{y\} := \text{ the fractional part of } y,$$

namely we write $y = n + s$ with $n \in \mathbb{Z}$, $s \in [0,1)$ and set $\{y\} := s$; we note $[y] := n$ for the integer part of y.

Equivalently, if we identify the interval $[0,1)$ and the unit circle $\mathbb{T} = \{e^{2\pi ix} : x \in [0,1)\}$ endowed with the normalized Lebesgue measure, then

$$(T_\alpha h)(e^{2\pi ix}) = xh(e^{2\pi i(x+\alpha)})$$

on the space $H := L^2(\mathbb{T})$.

The operator T_α was suggested in the 50's by E. Bishop as a candidate for an operator without closed linear invariant subspaces $\neq \{0\}$, H.

The main answer to the this question concerning the operators of the form T_α was given in the 70's by A. Davie [D]:

1 Theorem [D] *For almost all α the operator T_α does have a hyperinvariant (in particular, invariant) subspace.*

We call a subspace *hyperinvariant* for T_α if it is invariant under all bounded linear maps $S : H \to H$ such that $ST_\alpha = T_\alpha S$ (in particular, under T_α).

Let us note that if $\alpha \in \mathbb{Q}$, one easily shows that T_α does have invariant subspaces; while if $\alpha \notin \mathbb{Q}$, then T_α has no eigenvectors [D]. More general operators like

$$T_{\alpha,\varphi} h\,(x) = \varphi(x) h(e^{2\pi i(x+\alpha)})$$

with $\varphi \in L^\infty(\mathbb{T})$ were then considered by MacDonald, Blecher, Flattot, Chalendar, Partington etc, on various spaces $L^p(\mathbb{T})$ etc. Theorem 1 was subsequently extended to wider classes of numbers α as well as multipliers φ, including for instance the real analytic ones in neighborhoods of $[0,1]$.

We remind below a few known number theoretic topics.

Definition The *index* $\operatorname{ind} \alpha$ of an irrational number α is the supremum of all $l > 0$ such that for any $k > 0$ there exist p, q with

$$|\alpha - \frac{p}{q}| < \frac{k}{q^l}.$$

As it is well known by Liouville's theorem, if $\operatorname{ind} \alpha = \infty$ then α is transcendent. Recall also that by Dirichlet's theorem, for all irrational α we have $\operatorname{ind} \alpha \geq 2$. It's been proved by Roth that if α is algebraic irrational then $\operatorname{ind} \alpha = 2$. Also, Jarnik has shown that almost all numbers $\alpha \in (0,1)$ have finite index.

Theorem 1 from above [D] thus holds for all irrational $\alpha \in (0,1)$ with $\operatorname{ind} \alpha < \infty$, and $\varphi(x) = x$ on L^2. The result was generalized in [M] to the case of those multipliers φ with $\ln |\varphi|$ well-approximable by step functions of intervals; also, for those φ with $\ln |\varphi| \in L^p$ piecewise monotone and $p > \operatorname{ind} \alpha$; in particular, for φ analytic in a neighbourhood of $[0,1]$ on spaces L^p with $1 < p < \infty$. The case $\varphi(x) = x^s$ was considered in [F] on L^2 for a larger class of α's including some non–Liouville numbers. Then a slight generalization was stated in [CP] for products of two such Bishop type operators. In [M2] the existence of joint invariant subspaces was proved for finitely many commuting Bishop operators, too.

However, even for the simplest case $\varphi(x) = x$ the question of the existence of invariant subspaces remains open in general, that is, the answer is still unknown for highly transcendent numbers α.

We give in this note some versions of the various existing results of this type.

Following [D], let us note $T := eT_\alpha$. Hence the spectral radius $r(T)$ of T is 1. More generally, we have

$$r(T_{\varphi,\alpha}) = e^{\int_0^1 \ln|\varphi(x)|\,dx}$$

for a wide class of φ's including the continuous ones [M]; then we let $T := r(T_\alpha)^{-1}T_\alpha$ so that $r(T) = 1$. Using the well known formula for the spectral radius $r(T) = \lim_{n\to\infty} \|T^n\|$, one can derive, briefly speaking, good estimates of $\|T^{\pm n}\|$ for large n, which leads to the existence of invariant subspaces by known techniques of Wermer, Atzmon etc. A first obstacle to this aim is that generally T is not invertible. Moreover, using the technique mentioned from above requires to deal with operators T having a rich functional calculus - almost unitaries, in some sense. For these (and other) reasons, a renorming of the space under consideration will be necessary, so that T extends to a more suitable (invertible etc) operator, say $\tilde{T} : \tilde{H} \to \tilde{H}$ on some Hilbert space $\tilde{H}$ that contains H densely.

Then the main tool in obtaining the existence of a rich functional calculus for $\tilde{T}$ is Denjoy-Carleman's theorem on quasi-analytic functions. We remind below some known facts in this sense.

Given a sequence of weights $\rho_n \geq 1$ where $n \in \mathbb{Z}$ such that $\rho_{n+m} \leq \rho_n\rho_m$ for all n, m and $\lim_{|n|\to\infty} \rho_n^{1/|n|} = 1$, the space of all continuous functions $f(e^{it}) = \sum_n c_n e^{int}$ on the unit circle such that

$$\|f\| := \sum_{n\in\mathbb{Z}} |c_n|\rho_n < \infty$$

becomes a Banach algebra $A_{(\rho_n)}$.

If Beurling's condition

$$\sum_{n\in\mathbb{Z}} \frac{\ln \rho_n}{n^2+1} < \infty$$

is verified (for example, if $\rho_n := |n|^{|n|^\rho}$, where $0 < \rho < 1$ is fixed), the algebra $A_{(\rho_n)}$ is regular. In particular, $A_{(\rho_n)}$ contains functions f, $g \not\equiv 0$ such that $fg \equiv 0$.

For an arbitrary complex Hilbert space H, let $B(H)$ denote the algebra of all bounded linear maps on H.

Definition (see [CF]) Let $T \in B(H)$ be invertible. Set $\rho_n = \|T^n\|$ and $A_T := A_{(\rho_n)_n}$. We call T *A_T-unitary* if A_T is regular and there exists a continuous morphism of algebras

$$A_T \ni f \mapsto f(T) \in B(H)$$

taking any polynomial $\sum_n c_n z^n$ into $\sum_n c_n T^n$.

2 Theorem (Wermer) *If T is A_T-unitary, it has invariant subspaces.*

Proof: One applies, roughly speaking, the multiplicativity property of the functional calculus of T, namely write that $f(T)g(T) = (fg)(T) = 0$, while $f(T) \neq 0$ and $g(T) \neq 0$. Then set $H_0 := \ker f(T)$. We have $TH_0 \subset H_0$, since $f(T)h = 0$ implies that $f(T)Th = Tf(T)h = 0$, too. We omit the details, that are known [W].

We need certain topics on diophantine approximation. Remind that every irrational number $x \in (0,1)$ has a *continuous fraction* representation

$$x = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} \ (a_1, a_2, a_3, \ldots \in \mathbb{N}).$$

That is, we write $\frac{1}{x} = a_1 + t_1$ with a_1 integer and $0 < t_1 < 1$, namely $a_1 = [\frac{1}{x}]$ and $t_1 = \{\frac{1}{x}\}$, then $\frac{1}{t_1} = a_2 + t_2$ with $a_2 \in \mathbb{N}$ and $t_2 \in (0,1)$, namely $a_2 = [\frac{1}{t_1}]$ etc. By the formula $t_{n+1} = \{\frac{1}{t_n}\}$ for $n \geq 1$, it follows inductively that all $t_n = t_n(x)$ (and hence, all *partial quotients* $a_n = a_n(x)$) are measurable functions of $x \in (0,1) \setminus \mathbb{Q}$. Truncating the continued fraction of x at the n-th partial quotient a_n for each $n \geq 1$ provides the *convergents* $\frac{p_n}{q_n}$ of x

$$\frac{p_n}{q_n} := \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots \frac{1}{a_n}}} \ (n \geq 1),$$

namely $\frac{p_1}{q_1} = \frac{1}{a_1}$, $\frac{p_2}{q_2} = \frac{1}{a_1 + \frac{1}{a_2}} = \frac{a_2}{a_1 a_2 + 1}$ etc where $p_1 = 1$ and $q_1 = a_1$, $p_2 = a_2$ and $q_2 = a_1 a_2 + 1$ etc. Then $p_n = p_n(x) \geq 1$ and $q_n = q_n(x) \geq 1$ also are (integer-valued) measurable functions of x. For these topics we refer for instance to [EN]

3 Theorem (see [EN]) *For every irrational $x \in (0,1)$ we have $\lim_{n\to\infty} \frac{p_n}{q_n} = x$, and for every $n \geq 1$ the numbers p_n and q_n are relatively prime such that*

$$|x - \frac{p_n}{q_n}| < \frac{1}{q_n^2}$$

and

$$\frac{p_{2n}}{q_{2n}} < x < \frac{p_{2n-1}}{q_{2n-1}}.$$

4 Theorem (see [EN]) *For almost all irrational $x \in (0,1)$, we have $\lim_{n\to\infty} \frac{1}{n} \ln q_n(x) = \frac{\pi^2}{12 \ln 2}$.*

5 Corollary *For every ϵ_1, ϵ_2, $\mu \in (0,1)$ with $\epsilon_1 < \epsilon_2$, there exist a number $m_0 \geq 1$ and a measurable subset $M \subset (0,1)$ with $\lambda(M) > \mu$, such that for each natural number $m \geq m_0$ and every point $x \in M$ there are relatively prime integers $p \geq 1$ and $q \geq 1$ with*

$$0 < x - \frac{p}{q} < \frac{1}{q^2}$$

and

$$m^{1-\epsilon_2} \leq q \leq m^{1-\epsilon_1}.$$

Moreover, for m fixed we can select $p = p(x)$ and $q = q(x)$ ($x \in M$) such that $p(\cdot)$ and $q(\cdot)$ are measurable functions.

Proof. Let ϵ_1, ϵ_2, $\mu \in (0,1)$ with $\epsilon_1 < \epsilon_2$. Set $c = \frac{\pi^2}{12 \ln 2}$. Fix a positive $\epsilon = \epsilon(\epsilon_1, \epsilon_2)$ sufficiently small such that

$$\frac{1-\epsilon_1}{c+\epsilon_1} - \frac{1-\epsilon_2}{c-\epsilon_2} > \frac{1}{2}\frac{\epsilon_2 - \epsilon_1}{c}. \quad (1)$$

By Levy's theorem from above, the sequence of almost everywhere defined measurable functions $\frac{1}{n} \ln q_n$ is almost everywhere convergent to the constant function c. By Egorov's theorem, there exists a measurable set $M \subset (0,1)$ with $\lambda(M) > \mu$ such that $\frac{1}{n} \ln q_n \to c$ uniformly on M as $n \to \infty$. Let $n_0 \geq 1$ such that $\frac{1}{n} \ln q_n(x) \in (c - \varepsilon, c + \varepsilon)$ for all $n \geq n_0$ and almost all $x \in M$. Take $m_0 = \max(e^{\frac{(c+\epsilon)(n_0+1)}{1-\epsilon_1}}, e^{\frac{4c}{\epsilon_2 - \epsilon_1}})$. Now let $m \geq m_0$ be arbitrary. Set $\nu = [\frac{\ln m^{1-\epsilon_1}}{c+\epsilon}]$. Since $m \geq m_0 \geq e^{\frac{(c+\epsilon)(n_0+1)}{1-\epsilon_1}}$, $\frac{\ln m^{1-\epsilon_1}}{c+\epsilon_1} - 1 \geq n_0$ and so

$\nu - 1 \geq n_0$. If ν is even, let $n = \nu$; if ν is odd, let $n = \nu - 1$. In any case n is even and $n \geq n_0$. For every irrational $x \in M$, we may let $\frac{p}{q}$ be the n-th convergent of x, namely define $p := p_n(x)$ and $q = q_n(x)$. By Dirichlet's theorem on rational approximation from above, $\frac{p}{q} < x$ and $x - \frac{p}{q} < \frac{1}{q^2}$. Using that $y - 1 \leq [y]$ for $y = \frac{\ln m^{1-\epsilon_1}}{c+\epsilon}$, we obtain

$$\frac{\ln m^{1-\epsilon_1}}{c+\epsilon} - 2 \leq \nu - 1 \leq n \leq \nu \leq \frac{\ln m^{1-\epsilon_1}}{c+\epsilon}. \tag{2}$$

Since $m \geq e^{\frac{4c}{\epsilon_2 - \epsilon_1}}$, $\ln m \geq \frac{4c}{\epsilon_2 - \epsilon_1}$. By (1), this gives

$$(\frac{1-\epsilon_1}{c-\epsilon} - \frac{1-\epsilon_2}{c+\epsilon}) \ln m \geq 2$$

and so

$$\frac{\ln m^{1-\epsilon_2}}{c-\epsilon} \leq \frac{\ln m^{1-\epsilon_1}}{+\epsilon} - 2 \tag{3}$$

From (2) and (3) we derive

$$\frac{\ln m^{1-\epsilon_2}}{c-\epsilon} \leq n \leq \frac{\ln m^{1-\epsilon_1}}{+\epsilon}.$$

Hence

$$m^{1-\epsilon_2} \leq e^{n(c-\epsilon)} \ ; \ e^{n(c+\epsilon)} \leq m^{1-\epsilon_1}.$$

Since $n \geq n_0$, we have $c - \epsilon \leq \frac{1}{n} \ln q_n(x) \leq c + \epsilon$ for almost all $x \in M$, that is, $e^{n(c-\epsilon)} \leq q \leq e^{n(c+\epsilon)}$ almost everywhere. Then $m^{1-\epsilon_2} \leq q \leq m^{1-\epsilon_1}$.

6 Lemma *Fix a real $p > 1$, a positive $\omega < 1 - \frac{1}{p}$ and a decreasing sequence of numbers $t_k > 0$ ($k \geq 1$) with $\lim_{k\to\infty} t_k = 0$. Then for any $f \in L^p[0,1]$ nonnegative almost everywhere, the sequence of sets*

$$E_k = \{x \in [0,1) \ : \ f(\{x - n\alpha\}) \geq \frac{n^{1-\omega}}{t_k} \ \textit{for all } n \geq 1\} \quad (k \geq 1)$$

is increasing and satisfy $\lambda(\cup_k E_k) = 1$.

Proof For every $k \geq 1$, $[0,1) \setminus E_k = \cup_{n\geq 1} M_{kn}$ where $M_{kn} = \{x \in (0,1) : f(x - n\alpha) > \frac{n^{1-\omega}}{t_k}$. For every $n \geq 1$, define the function σ_n on the real line by $\sigma_n(y) = y - n\alpha$. Set also $\sigma(y) = y$. Now σ_n is a translation of σ

by $n\alpha$. Then a brief look at the graph of τ shows that the restriction $\tau|_I$ is measure-preserving on any interval $I \subset \mathbb{R}$ of length one. In particular, $\tau_n|_{[0,1)}$ is measure-preserving. Also, $M_{kn} = \tau_n(N_n)$ where $N_n = \{s \in (0,1) : f(s) > \frac{n^{1-\omega}}{t_k}\}$. Then

$$\lambda(M_{kn}) = \lambda(\tau_n(N_n)) = \lambda(N_n).$$

It follows that

$$\lambda([0,1) \setminus E_k) = \lambda(\cup_n M_{kn}) \le \sum_n \lambda(M_{kn}) = \sum_n \lambda(N_n).$$

Now

$$(\frac{n^{1-\omega}}{t_k})^p \lambda(N_n) \le \int_{N_n} f^p \, d\lambda \le \|f\|_p^p.$$

Hence

$$\lambda([0,1) \setminus E_k) \le \sum_n \|f\|_p^p (\frac{t_k}{n^{1-\omega}})^p = \|f\|_p^p (\sum_n \frac{1}{n^{(1-\omega)p}} = c \cdot t_k^p$$

for a constant c since $(1-\omega)p > 1$. Hence $\lambda([0,1) \setminus E_k) \to 0$ as $k \to \infty$.

A TECHNIQUE OF INVARIANT SUBSPACES FOR T_α

We summarize in what follows, in a unified way, the main six steps of the various proofs known so far to have provided invariant subspaces for Bishop type operators.

(1) We extend T to a space L of (classes of) Lebesgue measurable functions defined almost everywhere on $[0,1)$, so that T^{-n} also exists for $n = 1, 2, \ldots$, given by the formula

$$T^{-n} f(x) = F_n(x) f(\{x - \alpha n\})$$

where

$$F_n(x) = \frac{e^{-n}}{\{x-\alpha\} \cdots \{x - n\alpha\}}.$$

(2) We write $[0,1] = \cup_t E_t$ with $E_t = \{x : \{x - n\alpha\}$ are bounded from below$\}$, see Lemma 6.

(3) We make use of Dirichlet's theorem (see Corollary 5) stating that for every integer n there are p, q relatively prime such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$, $q \leq \sqrt{n}$;

moreover, for almost all α, we can take $q \geq n^{1/4}$.

To this aim, a Levy's theorem is used, providing us with approximations: $\alpha \approx \frac{p_n(\alpha)}{q_n(\alpha)}$ by continuous fractions, so that $\lim_n \frac{1}{n} \ln q_n(\alpha) =$ a universal constant. The proof of this fact is a nice application of the ergodic theory of numbers.

(4) Since we have good approximations of the form $\{x - n\alpha\} \approx \{x - n\frac{p}{q}\}$, then we can derive suitable estimates for the multipliers $F_n(x)$ on each of the sets E_t (by Stirling's fomula, in the case $\varphi(x) = x$).

Therefore, we obtain estimates of the form

$$\|T^{\pm n}\|_{L^2(E_t)} \leq n^{n^\rho}.$$

(5) We take, for example, $\tilde{H} := \{f \,:\, \|f\|_{\tilde{H}} < \infty\}$ where for suitable constants c_t (with countably many positive $t = t_k \to 0$ as $k \to +\infty$),

$$\|f\|_{\tilde{H}} := \sum_t c_t \int_{E_t} \sum_{n \in \mathbb{Z}} |T^n f\,(x) e^{-|n|^\rho}|^2 dx$$

Using (4) we obtain

$$\|f\|_{\tilde{H}} \leq ct.\, \|f\|_{L^2}$$

Let $\tilde{T}$ denote the operator T acting on $\tilde{H}$. Then by Beurling's condition, $\tilde{T}$ is $A_{\tilde{T}}$-unitary; moreover, one proves that its spectrum $\sigma(\tilde{T}) =$ the unit circle.

Let $A : L^2 \to \tilde{H}$ denote the inclusion $L^2 \subset \tilde{H}$. Then A is bounded with dense range.
Also, $AT = \tilde{T}A$, that is, $T = A^{-1}\tilde{T}A$.

Definition (see [CF]) We call T from above as a *quasiaffine transformation* of $\tilde{T}$, and write

$$T < \tilde{T}$$

7 Proposition (see [CF]) *If $B < C$, then $C^* < B^*$.*

If B is A_B-unitary, then B^* is A_{B^*}-unitary.

Now, as one can esaily check, T^* also is a Bishop-type operator. Then similarly one obtains that

$$T^* < U$$

for an A_U-unitary operator U. Hence

$$U^* < T^{**} = T$$

with $U^* = A_{U^*}$-unitary. Thus $U^* < T < \tilde{T}$.

(6) The existence of a hyperinvariant subspace of T follows then from the theorem:

7 Theorem (see [CF]) *If $V < T < W$ with $V = A_V$-unitary and $W = A_W$-unitary such that $\sigma(V) \neq$ a single point, then T has hyperinvariant subspaces.*

CONCLUSIONS

The general idea behind all known proofs seems to be the following: once the multiplier F_n that appears in the formula $T^{-n}f(x) = F_n(x)f(\{x - \alpha n\})$ has a rather concrete form (in the case $\varphi(x) = x$ for instance), then estimating $F_n(x)$ is equivalent to find estimates for $|\,\frac{1}{n}\ln F_n(x)\,| =$

$$= |\,\frac{1}{n}[F(x) + F(\tau(x)) + \cdots + F(\tau^{n-1}(x))] - \int F\,dx\,| \,\leq\, a_n$$

where $F(x) := \ln x$ and $\tau(z) := ze^{2\pi i\alpha}$ acts on the unit circle by a rotation of angle $2\pi\alpha$ (Weyl automorphism), with a suitable sequence of constants $a_n \to 0$ when $n \to \infty$, for example of the form $a_n := \frac{1}{n^\varepsilon}$ with $\varepsilon > 0$ for very good φ's (this is not the case for $\varphi(x) = x$, by the way). That is, good hypotheses on α and φ should lead to uniform estimates of the speed of convergence in this case of Birkhoff's ergodic theorem. There are few concrete results on the speed of convergence $(a_n)_n$, for example let $F := \ln|\varphi|$ and write $F(e^{2\pi ix}) = \sum_k c_k e^{2\pi i\cdot kx}$; if $F \in L^1$ with $|c_k| \leq \frac{ct}{k^{ind\,\alpha+1+\varepsilon}}$, we may take $a_n = O(1/n)$ [Ko]; if $|c_k| \leq \frac{ct}{k^{2+\varepsilon}}$, we may take $a_n = O(1/\,n^\epsilon)$ using results in [K]; if α is well approximable by rationals then τ is well approximable by periodic automorphisms, which also leads to an ergodic behaviour [SC]. As it is known however, there cannot be a universal estimate of the speed of convergence in the ergodic theorem, even for a continuous function F.

Moreover, there exist examples [M2], [N] of Bishop-type operators with a bad behaviour of the sequence of norms $\|T^n\|$ so that the known techniques presented here can not lead to significant improvements. New ideas are then necessary in order to deal with the general case, more precisely with the case of the highly transcendent parameters α.

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