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BEM and the Neumann problem for the Poisson equation on Lipschitz domains

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Abstract: The weak Neumann problem for the Poisson eqution is studied on Lipschitz domain with compact boundary using the direct integral equation method. The domain is bounded or unbounded, the boundary might be disconnected. The problem leads to a uniquely solvable integral equation in $H^{1/2}(\partial\Omega)$. It is proved that we can get the solution of this equation using the successive approximation method.

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1 Introduction

The theory of integral equations is an important tool in the theory of boundary value problems and in a numerical application. It is standard to look for a classical solution of the Neumann problem for the Laplace equation in a form of a single layer potential with density B. If a corresponding domain Ω is bounded and has smooth boundary, then the original problem is reduced to the integral equation $TB = F$ on the boundary $\partial\Omega$, where F is a boundary condition. It is a classical result that for Ω convex we can express a solution of the equation $TB = F$ in the form of a Neumann series

$$B = 2\sum_{j=0}^{\infty}(I - 2T)^j F, \tag{1}$$

where I is the identity operator. (For the history of the problem see [3].) Weaker solutions of the Neumann problem for the Laplace equation have been studied on nonsmooth domains by the integral equation method for the last fifty years (see [14], [17]). A solution is again looked for in the form of a single layer potential, which leads to the integral equation $TB = F$. J. Král and I. Netuka studied in 1977 this problem on general bounded convex domains with a boundary condition given by a real measure on $\partial\Omega$ (see [15]) and proved the representability of a solution of the equation $TB = F$ in the form (1). They proved that the spectral radius of the operator $I - 2T$ is smaller than 1 in the space of real measures μ on $\partial\Omega$ with $\mu(\partial\Omega) = 0$. E. Fabes, M. Sand, J. K. Seo studied the Neumann problem with a boundary condition from the space $L^2(\partial\Omega)$ on bounded convex domains in 1992 (see [6]) and they proved that a solution of the equation

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$TB = F$ can be given by (1). D. Medková studied in 1998 the representability of a solution of the equation $TB = F$ in the form (1) for domains with smooth compact boundaries and boundary conditions given by real measures (see [18]). It was shown that a solution of the equation $TB = F$ can be express by (1) if and only if Ω is a bounded domain with connected boundary. For a domain Ω with arbitrary topology she proved that a solution of the equation $TB = F$ can be calculating by

$$B = \sum_{j=0}^{\infty} \alpha^{-1}(I - \alpha^{-1}T)^j F, \tag{2}$$

where α is an arbitrary constant greater than 1/2 (see [19]). (For $\alpha = 1/2$ we get an expression (1).) O. Steinbach and W. L. Wendland studied in 2001 the weak Neumann problem for the Laplace equation in $W^{1,2}(\Omega)$ with boundary conditions from $H^{-1/2}(\partial\Omega)$ on bounded Lipschitz domains with connected boundary in R^2 and R^3 (see [27]). They proved that the operator $I - T$ is a contractive operator in the space $\{F \in H^{-1/2}(\partial\Omega); F(1) = 0\}$ and the integral equation $TB = F$ is solvable by the Neumann series

$$B = \sum_{j=0}^{\infty} (I - T)^j F. \tag{3}$$

(It is a series (2) with $\alpha = 1$.) The same result was proved by a totally different method by M. Constanda in 2007 (see [3]). He proved that T is a nonnegative operator for which $\sigma(T) \subset \langle 0, 1\rangle$. (Here $\sigma(T)$ is the spectrum of the operator T.) Since T is invertible on $\{F \in H^{-1/2}(\partial\Omega); F(1) = 0\}$ the representation (3) holds true. T. Chang and K. Lee studied the spectral properties of the operator T in 2008 (see [1]). They proved using properties of single layer potentials and double layer potentials and the interpolation that $\sigma(T) \subset \langle 0, 1\rangle$. So, the representation (3) can be shown by a further method.

D. Medková studied in 2007 the weak Neumann problem for the Laplace equation and later the weak Neumann problem for the Poissson equation on a general open set $\Omega \subset R^m$, $m > 2$ (see [20], [21])). It was shown that each solution of this problem is a Newton potential, which density B is a distribution with finite energy supported on the closure of Ω. The distribution B is supported on the boundary of Ω in the case of the Laplace equation. If we look for a solution in the form of a Newton potential then the original problem reduces to the integral equation $TB = F$, where F is the corresponding right side. It was proved for a general open set Ω that a solution of the equation $TB = F$ can be expressed by the series (3) provided the Neumann problem is solvable. It is a generalization of the result of Steinbach and Wendland because $H^{-1/2}(\partial\Omega)$ is the space of distributions with finite energy supported on $\partial\Omega$ for a bounded domain Ω with Lipschitz boundary.

So, we are able to solve the Neumann problem for the Poisson equation using the indirect integral equation method. But some experts from numerical analysis prefer the direct integral equation method, i.e. the method using the fact that $u = V^\Omega(-\Delta u) + \mathcal{S}^\Omega(\partial u/\partial n) - \mathcal{D}^\Omega u$, where $V^\Omega \varphi$ is a volume potential, $\mathcal{S}^\Omega \varphi$ is a single layer potential and $\mathcal{D}^\Omega \varphi$ is a double layer potential. It is usually supposed that $-\Delta u = 0$ in Ω. If we study the Neumann problem for the Laplace equation with the boundary condition f we get the integral equation $(1/2)u + Ku = \mathcal{S}^\Omega f$ on $\partial\Omega$. (See for example [2], [26], [9].) We shall study a weak solution of the Neumann problem for the Poisson equation on a domain $\Omega \subset R^m$, $m > 2$, with compact locally Lipschitz boundary. We do not suppose either that Ω is bounded or that the boundary $\partial\Omega$ is connected. We study a weak solution in the space $\hat{H}^{1,2}(\Omega)$ with the right side $\mathcal{F} \in [\hat{H}^{1,2}(\Omega)]'$. If Ω is bounded then $\hat{H}^{1,2}(\Omega) = W^{1,2}(\Omega)$ is a common Sobolev space. If Ω is unbounded then $\hat{H}^{1,2}(\Omega) = \{u \in L^{2m/(m-2)}(\Omega); \nabla u \in [L^2(\Omega)]^m\}$. If u is a solution of the problem $-\Delta u = f$ in Ω, $\partial u/\partial n = g$ on $\partial\Omega$ with $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, then the right side $\mathcal{F} = f(\mathcal{H}_m|\Omega) + g(\mathcal{H}_{m-1}|\partial\Omega)$. If we solve the Neumann problem for the Laplace equation then $\mathcal{F} \in H^{-1/2}(\partial\Omega) \subset [H^{1,2}(\Omega)]'$. We shall study the Neumann problem for the Poisson equation with the right side $\mathcal{F} \in [\hat{H}^{1,2}(\Omega)]'$ and we shall get the integral equation $(1/2)u + Ku = \mathcal{S}^\Omega \mathcal{F}$ on $\partial\Omega$. We shall solve this integral equation in the space $H^{1/2}(\partial\Omega)$.

Suppose first that Ω is unbounded. Then $\frac{1}{2}I + K$ is an invertible operator in $H^{1/2}(\Omega)$. We shall show that

$$\left(\frac{1}{2}I + K\right)^{-1} = \sum_{j=0}^{\infty} \left(\frac{1}{2}I - K\right)^j$$

and the successive approximation method converges. Moreover, we shall show that the spectral radius of the operator $K - \frac{1}{2}I$ is smaller than 1. That means that there is a norm $\|\cdot\|$ on $H^{1/2}(\partial\Omega)$ equivalent to the original norm such that $\|K - \frac{1}{2}I\| < 1$ (see [8]). Now we can approximate $\mathcal{S}^\Omega \mathcal{F}$ by f_j and the operator K by the operator K_j. If $f_j \to \mathcal{S}^\Omega \mathcal{F}$ and $K_j \to K$ then for sufficiently large j the equation $\frac{1}{2}u_j + K_j u_j = f_j$ is uniquely solvable and $u_j \to u$. Moreover $\|\frac{1}{2}I - K_j\| < 1$,

$$\left(\frac{1}{2}I + K_j\right)^{-1} = \sum_{j=0}^{\infty} \left(\frac{1}{2}I - K_j\right)^j$$

and the successive approximation method converges.

Let now Ω be bounded. Denote by $\tilde{K}$ the restriction of K onto the range $(\frac{1}{2}I + K)(H^{1/2}(\partial\Omega))$. We shall show that $\frac{1}{2}I + \tilde{K}$ is invertible and

$$\left(\frac{1}{2}I + \tilde{K}\right)^{-1} = \sum_{j=0}^{\infty} \left(\frac{1}{2}I - \tilde{K}\right)^j.$$

If $g \in (\frac{1}{2}I + K)(H^{1/2}(\partial\Omega))$, $f_0 \in H^{1/2}(\partial\Omega)$,

$$f_{j+1} = \left(\frac{1}{2}I - K\right) f_j + g,$$

then $f_j \to f$, where f is some solution of the equation $(\frac{1}{2}I+K)f = g$. Since this equation is not uniquely solvable, different choices of f_0 give different solutions of the equation $(\frac{1}{2}I+K)f = g$. If we solve the equation $(\frac{1}{2}I+K)\tilde{f} = \tilde{g}$, where $\tilde{g}$ is close to g then this equation will stop to be solvable. For the numerical practice we need to solve the integral equation which is stable under small perturbation. So we introduce a new operator

$$Mf = \frac{1}{2}f + Kf + \frac{1}{\mathcal{H}_{m-1}(\partial\Omega)} \int\limits_{\partial\Omega} f \, d\mathcal{H}_{m-1}.$$

We shall show that $Mf = g$ is uniquely solvable. Moreover, if $Mf = g$ and there is a solution of the integral equation $(\frac{1}{2}I+K)\varphi = g$ then $(\frac{1}{2}I+K)f = g$. If there is a solution of the Neumann problem for the Poisson equation with the right side $\mathcal{F}$ then the solution f of the integral equation $Mf = \mathcal{U}\mathcal{F}$ is the trace of a solution of the Neumann problem for the Poisson equation with the right side $\mathcal{F}$. We shall show that the spectral radius of the operator $M - I$ is smaller than 1. That means that there is a norm $\| \cdot \|$ on $H^{1/2}(\partial\Omega)$ equivalent to the original norm such that $\|M - I\| < 1$ (see [8]). Thus

$$M^{-1} = \sum_{j=0}^{\infty} (I - M)^j$$

and the successive approximation method converges. If $f_j \to \mathcal{S}^{\Omega}\mathcal{F}$ and $M_j \to M$ then for sufficiently large j the equation $M_j u_j = f_j$ is uniquely solvable and $u_j \to u$. Moreover $\|M_j - I\| < 1$,

$$M_j^{-1} = \sum_{j=0}^{\infty} (I - M_j)^j$$

and the successive approximation method converges.

2 Neumann problem

We study the Neumann problem for the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \qquad \frac{\partial \mathrm{u}}{\partial \mathrm{n}} = \mathrm{g} \quad \text{on } \partial\Omega \tag{4}$$

for a domain $\Omega \subset R^m$, $m > 2$, with compact Lipschitz boundary $\partial\Omega$, where n is the outward unit normal of Ω. We say that Ω has Lipschitz boundary if for each $x \in \partial\Omega$ there is a coordinate system centered at x and a Lipschitz function Φ on R^{m-1} such that $\Phi(0, \ldots, 0) = 0$ and in some neighborhood of x the set Ω lies under the graph of Φ and $R^m \setminus \overline{\Omega}$ lies above the graph of Φ. (Here $\overline{\Omega}$ denotes the closure of Ω.) We do not suppose that $\partial\Omega$ is connected.

Suppose first that Ω is a bounded domain with smooth boundary and $u \in \mathcal{C}^2(\overline{\Omega})$ is a classical solution of the problem. Denote by $\mathcal{H}_k$ the k-dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesgue measure in R^k. Then Green's formula yields

$$\int_{\Omega} \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = \int_{\Omega} f\varphi \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} g\varphi \, \mathrm{d}\mathcal{H}_{m-1} \tag{5}$$

for $\varphi \in \mathcal{D}(R^m)$, where $\mathcal{D}(G)$ is the space of all compactly supported infinitely differentiable real functions in G. This motivates a formulation of a weak solution in the Sobolev space $W^{1,2}(\Omega)$ of the Neumann problem for the Poisson equation on bounded domains with Lipschitz boundary.

If $\Omega \subset R^m$ is a nonempty open set denote by $L^2_{loc}(\Omega)$ the class of all measurable functions in Ω that are in $L^2(K)$ for every compact subset K of Ω and by $L^{1,2}(\Omega)$ the space of all functions in $L^2_{loc}(\Omega)$ for which all generalized derivatives of order 1 are in $L^2(\Omega)$. The Sobolev space $W^{1,2}(\Omega) = L^{1,2}(\Omega) \cap L^2(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,2}(\Omega)} = \sqrt{\int_{\Omega} (|u|^2 + |\nabla u|^2) \, \mathrm{d}\mathcal{H}_m} \tag{6}$$

and $(W^{1,2}(\Omega))'$ is the dual space of $W^{1,2}(\Omega)$. Remark, that for Ω bounded with Lipschitz boundary $\partial\Omega$ we have $W^{1,2}(\Omega) = L^{1,2}(\Omega) = \{u|\Omega; u \in L^{1,2}(R^m\}$.

If Ω is a bounded domain with Lipschitz boundary then $u \in W^{1,2}(\Omega)$ is a weak solution in $W^{1,2}(\Omega)$ of the Neumann problem for the Poisson equation in Ω with the right side $F \in (W^{1,2}(\Omega))'$ if

$$\int_{\Omega} \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = F(\varphi) \quad \forall \varphi \in W^{1,2}(\Omega)$$

(see [23], Exemple 2.8). Since $\mathcal{D}(R^m)$ is a dense subset of $W^{1,2}(\Omega)$ (see [23], Chapitre 2, Théorem 3.1) we can consider only test functions $\varphi \in \mathcal{D}(R^m)$. If u is a weak solution of the Neumann problem for the Poisson equation in Ω then u can be extended onto the whole R^m such that $u \in L^{1,2}(R^m)$ (see for example [23], Théorème 3.9). Using these facts we define a weak solution of the Neumann problem for the Poisson equation on general open subset of R^m.

Let Ω be a nonempty open subset of R^m, $\mathcal{F} \in \mathcal{D}'(R^m)$ supported in $\overline{\Omega}$, where $\mathcal{D}'(R^m)$ is the space of distributions on R^m. We say that a function u is a weak solution of the Neumann problem for the Poisson equation in Ω with the right side $\mathcal{F}$ if u can be extended onto R^m as a function from $L^{1,2}(R^m)$ and

$$\int_{\Omega} \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = \mathcal{F}(\varphi) \quad \forall \varphi \in \mathcal{D}(R^m)$$

for each $\varphi \in \mathcal{D}(R^m)$.

This problem was studied in [21] using the indirect integral equation method. In the rest of the paragraph we briefly sketch the results. (See also [18] and [4].)

For $x, y \in R^m$ denote

$$h_x(y) = \frac{1}{(m-2)\mathcal{H}_{m-1}(\partial B(0;1))} |x-y|^{2-m}$$

where $B(x;r)$ is the open ball with the center x and the radius r. For a closed set F denote by $\mathcal{C}'(F)$ the space of all finite real Borel measures with support in F.

Denote $\mathcal{E}(R^m) = \{\Delta u; u \in L^{1,2}(R^m)\}$ the space of distributions with finite energy. If $\mu \in \mathcal{C}'(R^m)$, then $\mu \in \mathcal{E}(R^m)$ if and only if

$$\int\limits_{R^m} \int\limits_{R^m} h_x(y) \, \mathrm{d}|\mu|(x) \, \mathrm{d}|\mu|(y) < \infty,$$

where $|\mu|$ denotes the total variation of μ. If $\mu \in \mathcal{E}(R^m) \cap \mathcal{C}'(R^m)$ then the Newton potential corresponding to μ

$$\mathcal{U}\mu(x) = \int\limits_{R^m} h_x(y) \, \mathrm{d}\mu(y) \tag{7}$$

is defined for almost all $x \in R^m$ and $\mathcal{U}\mu = h_0 * \mu \in L^{1,2}(R^m)$. (Here $*$ denotes the convolution.) Define now the Newton potential for $\mathcal{F} \in \mathcal{E}(R^m)$. Fix $\mathcal{F} \in \mathcal{E}(R^m)$. Then $\mathcal{F}$ is a tempered distribution and the Fourier transform $\hat{\mathcal{F}}$ of $\mathcal{F}$ is defined. Moreover, $\hat{\mathcal{F}} \in L^1_{loc}(R^m)$ and there is only tempered distribution $\mathcal{U}\mathcal{F}$ such that for its Fourier transform we have $\widehat{\mathcal{U}\mathcal{F}}(x) = \hat{\mathcal{F}}(x)|x|^{-2}$. Remark that $\mathcal{U}\mathcal{F} \in L^{1,2}(R^m)$ and $\Delta\mathcal{U}\mathcal{F} = -\mathcal{F}$. If $\mathcal{F} \in \mathcal{C}'(R^m)$ then $\mathcal{U}\mathcal{F}$ is given by (7). If $\mathcal{F}$ has compact support, then $\mathcal{U}\mathcal{F} = h_0 * \mathcal{F}$. It is chosen such representation of $\mathcal{U}\mathcal{F}$ that

$$\mathcal{U}\mathcal{F}(x) = \lim_{r \searrow 0} (\mathcal{H}_m(B(x;r)))^{-1} \int\limits_{B(x;r)} \mathcal{U}\mathcal{F} \, \mathrm{d}\mathcal{H}_m \tag{8}$$

at each $x \in R^m$ for which the limit on the right side exists. The function $\mathcal{U}\mathcal{F}$ is called the Newton potential of $\mathcal{F}$.

The space $\mathcal{E}(R^m)$ is a Hilbert space with inner product

$$(\mathcal{F}, \mathcal{G})_{\mathcal{E}} = \int\limits_{R^m} \nabla\mathcal{U}\mathcal{F} \cdot \nabla\mathcal{U}\mathcal{G} \, \mathrm{d}\mathcal{H}_m. \tag{9}$$

If $\mathcal{F} \in \mathcal{E}(R^m)$ and $\nu \in \mathcal{E}(R^m) \cap \mathcal{C}'(R^m)$, then

$$(\mathcal{F}, \nu)_{\mathcal{E}} = \int \mathcal{U}\mathcal{F} \, \mathrm{d}\nu. \tag{10}$$

For the closed set K denote by $\mathcal{E}(K)$ the space of all distribution from $\mathcal{E}(R^m)$ supported on K with the energy $\| \, \|_{\mathcal{E}}$ as a norm. Then $\mathcal{E}(K)$ is a Hilbert space. If Ω is a bounded domain with Lipschitz boundary then $\mathcal{E}(\partial\Omega) = H^{-1/2}(\partial\Omega)$, $\mathcal{E}(\overline{\Omega}) = (W^{1,2}(\Omega))'$ and the corresponding norms are equivalent. (Here $H^{1/2}(\partial\Omega)$ is the space of traces of functions from $W^{1,2}(\Omega)$ and $H^{-1/2}(\partial\Omega)$ is its dual space.)

Fix a nonempty open subset Ω of R^m, $m > 2$. If u is a solution of the weak Neumann problem for the Poisson equation in Ω then there are $\mathcal{G} \in \mathcal{E}(\overline{\Omega})$ and a real number a such that $u = \mathcal{U}\mathcal{G} + a$ in Ω. So, we can look for a solution in the form of a Newton potential $\mathcal{U}\mathcal{G}$ with $\mathcal{G} \in \mathcal{E}(\overline{G})$. If $\mathcal{G} \in \mathcal{E}(\overline{G})$ then $\mathcal{U}\mathcal{G}$ is a weak solution of the Neumann problem for the Poisson equation in Ω with the right side $\mathcal{F}$ if and only if $\mathcal{F} \in \mathcal{E}(\overline{\Omega})$ and $J_\Omega \mathcal{G} = \mathcal{F}$, where $J_\Omega \mathcal{G}$ is the unique element of $\mathcal{E}(R^m)$ such that

$$\int_\Omega \nabla \mathcal{U}\mathcal{B} \cdot \nabla \mathcal{U}\mathcal{G} \, \mathrm{d}\mathcal{H}_m = (\mathcal{B}, J_\Omega \mathcal{G})_{\mathcal{E}}$$

for each $\mathcal{B} \in \mathcal{E}(R^m)$. The operator $J_\Omega : \mathcal{G} \mapsto J_\Omega \mathcal{G}$ is a bounded linear nonnegative operator on $\mathcal{E}(R^m)$ with $\|J_\Omega\| \le 1$. Moreover, $J_\Omega(\mathcal{E}(R^m)) = J_\Omega(\mathcal{E}(\overline{\Omega})) \subset \mathcal{E}(\overline{\Omega})$, $J_\Omega(\mathcal{E}(\partial\Omega)) \subset \mathcal{E}(\partial\Omega)$. Fix $\mathcal{F} \in \mathcal{E}(\overline{\Omega})$ such that the Neumann problem for the Poisson equation in Ω with the right side $\mathcal{F}$ is solvable. Then the series

$$\tilde{\mathcal{G}} = \sum_{j=0}^{\infty} (I - J_\Omega)^j \mathcal{F} \tag{11}$$

is convergent in $\mathcal{E}(\overline{\Omega})$ and $\mathcal{U}\tilde{\mathcal{G}}$ is a weak solution of the Neumann problem for the Poisson equation in Ω with the right side $\mathcal{F}$. Fix $\mathcal{G}_0 \in \mathcal{E}$. Put

$$\mathcal{G}_n = (I - J_\Omega)\mathcal{G}_{n-1} + \mathcal{F} \tag{12}$$

for positive integer n. Then there is $\mathcal{G} \in \mathcal{E}$ such that $\mathcal{G}_n \to \mathcal{G}$ as $n \to \infty$ and $\mathcal{U}\mathcal{G}$ is a weak solution of the Neumann problem for the Poisson equation in Ω with the right side $\mathcal{F}$. If $\mathcal{G}_0 \in \mathcal{E}(\overline{\Omega})$ then $\mathcal{G}_n, \mathcal{G} \in \mathcal{E}(\overline{\Omega})$. If $\mathcal{F}, \mathcal{G}_0 \in \mathcal{E}(\partial\Omega)$ then $\mathcal{G}_n, \mathcal{G} \in \mathcal{E}(\partial\Omega)$. If we choose $\mathcal{G}_0 = 0$ then $\mathcal{G} = \tilde{\mathcal{G}}$. By different choice of $\mathcal{G}_0$ we obtain different solutions of $J_\Omega \mathcal{G} = \mathcal{F}$, because this equation is not uniquely solvable for each Ω. But we do not get fundamentally different solutions $\mathcal{U}\mathcal{G}$ because for two solutions u, v of the Neumann problem for the Poisson equation in Ω with the right side $\mathcal{F}$ the function $u - v$ is constant on each component of Ω.

For general Ω we are not able to say how quickly converge $\mathcal{G}_n$ to $\mathcal{G}$ and this convergence is not uniform on bounded sets of $\mathcal{E}(\overline{\Omega})$. For domains with compact boundary and $W^{1,2}$-extension property we can say more. We restrict ourselves to domains with compact Lipschitz boundary. We shall suppose from now that Ω is a domain with compact Lipschitz boundary. (We do not suppose that

$\partial\Omega$ is connected.) If the Neumann problem with the right side $\mathcal{F}$ is solvable, $\mathcal{G}_0 \in \mathcal{E}(R^m)$, $\mathcal{G}_n$ is given by (12) and $\mathcal{G}_n \to \mathcal{G}$, then

$$\|\mathcal{G} - \mathcal{G}_n\|_{\mathcal{E}} \leq q^n [(1-q)^{-1} \|\mathcal{F}\|_{\mathcal{E}} + \|\mathcal{G}_0\|_{\mathcal{E}}]$$

where the constant $q \in (0,1)$ depends only on Ω. If Ω is unbounded then there is a weak solution of the Neumann problem for the Poisson equation in Ω with the right side $\mathcal{F}$ if and only if $\mathcal{F} \in \mathcal{E}(\overline{\Omega})$. If Ω is bounded, then there is a weak solution of the Neumann problem for the Poisson equation in Ω with the right side $\mathcal{F}$ if and only if $\mathcal{F} \in \mathcal{E}(\overline{\Omega})$ and $\mathcal{F}(1) = 0$.

We can ask how to calculate $J_G\mathcal{F}$. It is well-known for some special cases. Denote by $\mathcal{H}$ the restriction of $\mathcal{H}_{m-1}$ onto $\partial\Omega$. Then there is the exterior unit normal $n^\Omega(x)$ of Ω at $\mathcal{H}$-a.a. $x \in \partial\Omega$. Let $\mathcal{F} = f\mathcal{H} \in \mathcal{E}(\partial\Omega)$, where $f \in L^p(\mathcal{H})$, $1 < p < \infty$. Then $\mathcal{UF}$ is the single layer potential with density f and for $\mathcal{H}$-a.a. $x \in \partial\Omega$ there exists the limit

$$K^* f(x) = \lim_{\epsilon \searrow 0} \int\limits_{\partial\Omega \setminus B(x;\epsilon)} n^\Omega(x) \cdot \nabla h_y(x) f(y) \, \mathrm{d}\mathcal{H}(y) \tag{13}$$

and $J_\Omega \mathcal{F} = (\frac{1}{2} f + K^* f)\mathcal{H}$.

3 Representation by potentials

We shall use the following convention: If $X(G)$ is a Hilbert space of functions or distributions on an open set G with an inner product (f,g), denote by $X(G,C)$ its complexification, i.e. $X(G,C) = \{f_1 + i f_2; f_1, f_2 \in X(G)\}$, $(f_1 + f_2, g_1 + i g_2) = (f_1, g_1) - (f_2, g_2) + i(f_1, g_2) + i(f_2, g_1)$.

Let $\Omega \subset R^m$ be an open set with compact Lipschitz boundary, $g \in L^2(\partial\Omega)$. Define

$$\mathcal{S}^\Omega g(x) = \int\limits_{\partial\Omega} g(y) h_x(y) \, \mathrm{d}\mathcal{H}_{m-1}(y) \tag{14}$$

the single layer potential corresponding to g and

$$\mathcal{D}^\Omega g(x) = \int\limits_{\partial\Omega} g(y) \frac{\partial h_x(y)}{\partial n} \, \mathrm{d}\mathcal{H}_{m-1}(y) = \int\limits_{\partial\Omega} \frac{n^\Omega(y) \cdot (x-y)}{\mathcal{H}_{m-1}(\partial B(0;1)) |x-y|^m} \, \mathrm{d}\mathcal{H}_{m-1}(y) \tag{15}$$

the double layer potential corresponding to g whenever these integrals make sense. Here $n(y)$ denotes the outward unit normal of Ω at y. (Remark that $\mathcal{S}^\Omega g = \mathcal{U}(g\mathcal{H}_{m-1}|\partial\Omega)$. Then $\mathcal{S}^\Omega g$, $\mathcal{D}^\Omega g$ are harmonic functions in the complement of $\partial\Omega$. If $g \in L^2(\Omega)$ with compact support, denote $V^\Omega f = \mathcal{U}(f\mathcal{H}_m|\Omega)$ the volume potential.

Suppose for a while that Ω is bounded. If $u \in \mathcal{C}^2(\overline{\Omega})$ then $u = V^\Omega(-\Delta u) + \mathcal{S}^\Omega(\partial u/\partial n) - \mathcal{D}^\Omega u$. Since u is a classical solution of the problem (4) with

$f = -\Delta u$ and $g = \partial u/\partial n$, the function u is a weak solution of Neumann problem for the Poisson equation in Ω with the right side $\mathcal{F} = f\mathcal{H}_m|\Omega + g\mathcal{H}_{m-1}|\partial\Omega$. We can rewrite the formula for the representation of u by $u = \mathcal{U}\mathcal{F} - \mathcal{D}^\Omega u$ in Ω.

We would like to prove the same representation for arbitrary weak solution of the Neumann problem. But it does not hold in general for unbounded domains. (If $u \equiv 1$ then u is a weak solution of the Neumann problem for the Poisson equation with right side $\mathcal{F} = 0$. If Ω is an exterior domain then $\mathcal{D}^\Omega u = 0$ in Ω and $\mathcal{U}\mathcal{F} + \mathcal{D}^\Omega u \equiv 0 \neq 1 \equiv u$ in Ω.) In the case of exterior domains we restrict ourselves to solutions in the form of a Newton potential $\mathcal{U}\mathcal{G}$ with $\mathcal{G} \in \mathcal{E}(R^m)$.

If $G \subset R^m$ is an open set, define on $\mathcal{D}(G)$ an inner product by

$$(u, v)_{\hat{H}_0^{1,2}(G)} = \int_G \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m. \tag{16}$$

Denote by $\hat{H}_0^{1,2}(G)$ the completion of $\mathcal{D}(G)$ with respect to the corresponding norm. This space is used when the Dirichlet problem with the homogeneous boundary condition is studied in an exterior domain (see for example [7], [11], [13]). According to [12], Lemma 2.2 we have $\hat{H}_0^{1,2}(R^m) = \{u \in L^{2m/(m-2)}(R^m); \nabla u \in L^2(R^m; R^m)\}$.

Remark that $\hat{H}_0^{1,2}(R^m) = \{\mathcal{U}\mathcal{G}; \mathcal{G} \in \mathcal{E}(R^m)\}$. If $\varphi \in \mathcal{D}(R^m)$ then $\Delta\varphi \in \mathcal{E}(R^m)$ and $\varphi = \mathcal{U}(-\Delta\varphi)$. This gives that $\hat{H}_0^{1,2}(R^m) \subset \{\mathcal{U}\mathcal{G}; \mathcal{G} \in \mathcal{E}(R^m)\}$. Since $\mathcal{D}(R^m)$ is dense in $\{\mathcal{U}\mathcal{G}; \mathcal{G} \in \mathcal{E}(R^m)\}$ (see [16], Theorem 1.17 and [28], Lemma 6.5), we infer that $\hat{H}_0^{1,2}(R^m) = \{\mathcal{U}\mathcal{G}; \mathcal{G} \in \mathcal{E}(R^m)\}$. The relations (9), (16) give that $\mathcal{F} \mapsto -\mathcal{U}\mathcal{F}$ is an isometric isomorphism of the Hilbert space $\mathcal{E}(R^m)$ onto the Hilbert space $\hat{H}_0^{1,2}(R^m)$. The inverse operator $u \mapsto -\Delta u$ is an isometric isomorphism of the Hilbert space $\hat{H}_0^{1,2}(R^m)$ onto the Hilbert space $\mathcal{E}(R^m)$.

If $G \subset R^m$ is an open set denote $\hat{H}^{1,2}(G) = \{u|G; u \in \hat{H}_0^{1,2}(R^m)\}$. Clearly, $\hat{H}^{1,2}(R^m) = \hat{H}_0^{1,2}(R^m)$ and we use the same norm on both spaces. As usually, the norm on $\hat{H}^{1,2}(G)$ is given by

$$\|u\|_{\hat{H}^{1,2}(G)} = \inf\{\|v\|_{\hat{H}^{1,2}(R^m)}; v \in \hat{H}^{1,2}(R^m), u = v|G\}. \tag{17}$$

Fix $u \in \hat{H}^{1,2}(G)$. If $v \in \hat{H}^{1,2}(R^m)$, $v = u$ in G, then there is $\mathcal{F} \in \mathcal{E}(R^m)$ such that $v = \mathcal{U}\mathcal{F}$. (Remark that $\mathcal{F} = -\Delta v$.) Denote by $\mathcal{G}_u$ the orthogonal projection of $\mathcal{F}$ onto $\mathcal{E}(\overline{G})$. Since $\mathcal{U}\mathcal{G}_u = \mathcal{U}\mathcal{F}$ in G (see [16], Chapter VI, Theorem 6.3), we have $u = \mathcal{U}\mathcal{G}_u$ in G. Since

$$\|\mathcal{U}\mathcal{G}_u\|_{\hat{H}^{1,2}(R^m)} = \|\mathcal{G}_u\|_{\mathcal{E}} \leq \|\mathcal{F}\|_{\mathcal{E}} = \|v\|_{\hat{H}^{1,2}(R^m)},$$

we deduce that

$$\|u\|_{\hat{H}^{1,2}(G)} = \|\mathcal{U}\mathcal{G}_u\|_{\hat{H}^{1,2}(R^m)} = \|\mathcal{G}_u\|_{\mathcal{E}}. \tag{18}$$

So, (17) gives a norm on $\hat{H}^{1,2}(G)$ and $\mathcal{U} : \mathcal{F} \mapsto \mathcal{U}\mathcal{F}$ represents an isometric isomorphism from the Hilbert space $\mathcal{E}(\overline{G})$ onto the Hilbert space $\hat{H}^{1,2}(G)$.

We now show that $(\hat{H}^{1,2}(G))' = \mathcal{E}(\overline{G})$ and the corresponding norms are equivalent. Fix $\mu \in \mathcal{E}(\overline{G}) \cap \mathcal{C}'(\overline{G})$. If $\varphi \in \mathcal{D}(R^m)$ then

$$\mu(\varphi) = \int \varphi \, \mathrm{d}\mu = \int \mathcal{U}\mathcal{G}_\varphi \, \mathrm{d}\mu = (\mathcal{G}_\varphi, \mu)_\mathcal{E}$$

by (10). If $\mathcal{F} \in \mathcal{E}(\overline{G})$ then there is a sequence μ_n in $\mathcal{E}(\overline{G}) \cap \mathcal{C}'(\overline{G})$ such that $\mu_n \to \mathcal{F}$ in $\mathcal{E}(\overline{G})$ (see [4], p. 143). Since μ_n converges to $\mathcal{F}$ in distributional sense we have

$$\mathcal{F}(\varphi) = (\mathcal{G}_\varphi, \mathcal{F})_\mathcal{E}$$

for each $\varphi \in \mathcal{D}(R^m)$. Thus

$$|\mathcal{F}(\varphi)| \leq \|\mathcal{G}_\varphi\|_\mathcal{E} \cdot \|\mathcal{F}\|_\mathcal{E} = \|\varphi\|_{\hat{H}^{1,2}(G)} \cdot \|\mathcal{F}\|_\mathcal{E}.$$

Since $\mathcal{D}(R^m)$ is a dense subset of $\hat{H}^{1,2}(G)$, there is unique extension of $\mathcal{F}$ as a bounded linear functional on $\hat{H}^{1,2}(G)$

$$\mathcal{F}(\varphi) = (\mathcal{G}_\varphi, \mathcal{F})_\mathcal{E}, \quad \|\mathcal{F}\|_{\hat{H}^{1,2}(G))'} \leq \|\mathcal{F}\|_\mathcal{E}. \tag{19}$$

Since

$$\mathcal{F}(\mathcal{U}\mathcal{F}) = \|\mathcal{F}\|^2 = \|\mathcal{F}\| \cdot \|\mathcal{U}\mathcal{F}\|_{\hat{H}^{1,2}(G)},$$

we infer

$$\|\mathcal{F}\|_{\hat{H}^{1,2}(G))'} = \|\mathcal{F}\|_\mathcal{E}. \tag{20}$$

Thus $\mathcal{E}(\overline{G})$ is a closed subspace of $\hat{H}^{1,2}(G))'$. Let now $F \in (\hat{H}^{1,2}(G))'$. Since $\hat{H}^{1,2}(G)$ is a Hilbert space, Riesz representation theorem gives that there is $u \in \hat{H}^{1,2}(G)$ such that

$$F(v) = (v, u)_{\hat{H}^{1,2}(G)} = (\mathcal{G}_u, \mathcal{G}_v)_\mathcal{E} \qquad \forall v \in \hat{H}^{1,2}(G).$$

Thus $F = \mathcal{G}_u \in \mathcal{E}(R^m)$.

Let now $\Omega \subset R^m$, $m > 2$, be an unbounded domain with compact Lipschitz boundary. We shall show that the inner product (16) gives a norm on $\hat{H}^{1,2}(\Omega)$ which is equivalent to the norm given by (17). According to the definition we have

$$\int_\Omega |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m \leq \|u\|^2_{\hat{H}^{1,2}(\Omega)}$$

for each $u \in \hat{H}^{1,2}(\Omega)$. It was shown in [20] and [21] that J_Ω is a positive invertible operator on $\mathcal{E}(\overline{\Omega}, C)$. Moreover, $0 \leq J_\Omega \leq I$, where I is the identity operator. Therefore there is $\alpha > 0$ such that $\sigma(J_\Omega) \subset (\alpha, 1\rangle$, where $\sigma(J_\Omega)$ is the spectrum of J_Ω on $\mathcal{E}(\overline{\Omega}, C)$ (see [5], Proposition 4.15). Put $T = I - J_\Omega$. Then $\sigma(T) \subset \langle 0, 1-\alpha)$. According to [30], Chapter VII, §3, Theorem 3 and [25], Theorem 1 we have $1 - \alpha > \sup\{\lambda \in \sigma(T)\} = \|T\| = \sup\{(T\mathcal{F}, \mathcal{F})_\mathcal{E}; \|\mathcal{F}\|_\mathcal{E} =$

$1\} = 1 - \inf\{(J_\Omega \mathcal{F}, \mathcal{F})_\mathcal{E}; \|\mathcal{F}\|_\mathcal{E} = 1\}$. This gives $(J_\Omega \mathcal{F}, \mathcal{F})_\mathcal{E} \geq \alpha(\mathcal{F}, \mathcal{F})_\mathcal{E}$ for each $\mathcal{F} \in \mathcal{E}(\overline{\Omega})$. Fix $u \in \hat{H}^{1,2}(\Omega)$. We have shown that there is $\mathcal{G} \in \mathcal{E}(\overline{\Omega})$ such that $u = \mathcal{U}\mathcal{G}$ on Ω and (18) holds. Then

$$\|u\|^2_{\hat{H}^{1,2}(\Omega)} = \|\mathcal{G}\|_\mathcal{E} \leq \alpha^{-1}(J_\Omega \mathcal{G}, \mathcal{G})_\mathcal{E} = \alpha^{-1} \int_\Omega |\nabla \mathcal{U}\mathcal{G}|^2 \, \mathrm{d}\mathcal{H}_m = \alpha^{-1} \int_\Omega |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m.$$

Therefore the norm (17) is equivalent to the norm

$$\sqrt{\int_\Omega |\nabla u|^2 \, \mathrm{d}\mathcal{H}_m} \tag{21}$$

given by the inner product (16). Consequently, $\hat{H}^{1,2}(\Omega)$ is the completion of the inner product space $\{u|\Omega; u \in \mathcal{D}(R^m)\}$ endowed with the inner product (16).

Let now Ω be a bounded open set with Lipschitz boundary. Then $\hat{H}^{1,2}(\Omega) = W^{1,2}(\Omega)$. Clearly, (21) is not a norm on $\hat{H}^{1,2}(\Omega)$. We shall show that the norm given by (17) is equivalent to the usual norm (6) in $W^{1,2}(\Omega)$. According to [20], Lemma 8.4 there is a positive constant C such that

$$C^{-1}\|\mathcal{F}\|_\mathcal{E} \leq \|\mathcal{U}\mathcal{F}\|_{W^{1,2}(\Omega)} \leq C\|\mathcal{F}\|_\mathcal{E} \tag{22}$$

for each $\mathcal{F} \in \mathcal{E}(\overline{\Omega})$. Fix $u \in \hat{H}^{1,2}(\Omega)$. We have proved that there is $\mathcal{G} \in \mathcal{E}(\overline{\Omega})$ such that $u = \mathcal{U}\mathcal{G}$ on Ω and (18) holds. Using (18) and (22) we obtain

$$C^{-1}\|u\|_{\hat{H}^{1,2}(\Omega)} = C^{-1}\|\mathcal{G}\|_\mathcal{E} \leq \|u\|_{W^{1,2}(\Omega)} \leq C\|\mathcal{G}\|_\mathcal{E} = C\|u\|_{\hat{H}^{1,2}(\Omega)}.$$

We can reformulate the Neumannn problem for the Poisson equation as follows: *The function $u \in \hat{H}^{1,2}(\Omega)$ is a weak solution of the Neumann problem for the Poisson equation with the right side $\mathcal{F} \in (\hat{H}^{1,2}(\Omega))'$ if*

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}\mathcal{H}_m = \mathcal{F}(v) \qquad \forall v \in \hat{H}^{1,2}(\Omega).$$

Now we express a solution of the Neumann problem for an open set Ω with compact Lipschitz boundary in the form of potentials. We can find this result in [26] for $\Omega \subset R^3$ a bounded domain with connected Lipschitz boundary.

Proposition 3.1. *Let $\Omega \subset R^m$, $m > 2$, be an open set with compact Lipschitz boundary. Let $u \in \hat{H}^{1,2}(\Omega)$ be a weak solution of the Neumann problem for the Poisson equation in Ω with the right side $\mathcal{F}$. Then*

$$u = \mathcal{U}\mathcal{F} - \mathcal{D}^\Omega u \quad \text{in } \Omega, \tag{23}$$

where u on $\partial\Omega$ is the trace of u.

Proof. We can suppose that $u \in \hat{H}^{1,2}_0(R^m)$. Suppose first that Ω is bounded. Fix $u_k \in \mathcal{D}(R^m)$ such that $u_k \to u$ in $\hat{H}^{1,2}_0(R^m)$ as $k \to \infty$. Put $\mathcal{G} = -\Delta u$,

$\mathcal{G}_k = -\Delta u_k$. Then $u = \mathcal{U}\mathcal{G}$, $u_k = \mathcal{U}\mathcal{G}_k$ and $\mathcal{G}_k \to \mathcal{G}$ in $\mathcal{E}(R^m)$. Put $\mathcal{F}_k = J_\Omega \mathcal{G}$. Then $\mathcal{F}_k = J_\Omega \mathcal{G} \to J_\Omega \mathcal{G} = \mathcal{F}$ in $\mathcal{E}(R^m)$. Since $\mathcal{F}_k = (-\Delta u_k)(\mathcal{H}_m|\Omega) + (\partial u_k/\partial n)(\mathcal{H}_{m-1}|\partial\Omega)$, we have

$$u_k = \mathcal{U}\mathcal{F}_k - \mathcal{D}^\Omega u_k \tag{24}$$

in Ω. Since $u_k \to u$ in $\hat{H}_0^{1,2}(R^m)$, we infer that $u_k \to u$ in $W^{1,2}(\Omega)$. Since $\mathcal{F}_k \to \mathcal{F}$ in $\mathcal{E}(R^m)$ we obtain $\mathcal{U}\mathcal{F}_k \to \mathcal{U}\mathcal{F}$ in $\hat{H}_0^{1,2}(R^m)$ and hence also in $W^{1,2}(\Omega)$. Since $u_k \to u$ in $W^{1,2}(\Omega)$ we have $u_k \to u$ in $H^{1/2}(\partial\Omega)$. This gives that $\mathcal{D}^\Omega u_k \to \mathcal{D}^\Omega u$ locally uniformly in Ω. Letting $k \to \infty$ in (24) we get (23).

Let now Ω be unbounded. Fix $R > 0$ such that $R^m \setminus \Omega \subset B(0;R)$. Choose $\psi \in \mathcal{D}(R^m)$ supported in $B(0,R)$ such that $\psi = 1$ on a neighborhood of $R^m \setminus \Omega$. Denote $u_1 = u\psi$, $u_2 = u - u_1$. Then u_j is a weak solution of the Neumann problem for the Poisson equation in Ω with the right side $\mathcal{F}_j$ for $j = 1, 2$ and $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$. Since $u_2 \in \hat{H}_0^{1,2}(R^m)$, there is $\mathcal{G} \in \mathcal{E}(R^m)$ such that $u_2 = \mathcal{U}\mathcal{G}$ in R^m. Since $u_2 = 0$ on the neighborhood of $R^m \setminus \Omega$ we have for $\varphi \in \mathcal{D}(R^m)$

$$\mathcal{F}_2(\varphi) = \int_\Omega \nabla u_2 \cdot \nabla \varphi \, d\mathcal{H}_m = \int_{R^m} \nabla \mathcal{U}\mathcal{G} \cdot \nabla \varphi \, d\mathcal{H}_m = \mathcal{G}(\varphi).$$

Thus $\mathcal{F}_2 = \mathcal{G}$ and $u_2 = \mathcal{U}\mathcal{F}_2$. Since u_2 vanishes on $\partial\Omega$ we have obtained (23) for u_2.

Let $R < \rho < \infty$. Since $u_1 = 0$ in $R^m \setminus B(0;R)$, we deduce that $\mathcal{F}_1$ is supported in $\overline{\Omega \cap B(0;R)}$ and u_1 is a weak solution of the Neumann problem for the Poisson equation in $\Omega \cap B(0;\rho)$ with the right side $\mathcal{F}_2$. Since $\Omega \cap B(0;\rho)$ is bounded we have $u_1 = \mathcal{U}\mathcal{F}_1 - \mathcal{D}^{\Omega \cap B(0;\rho)} u_1$ in $\Omega \cap B(0;\rho)$. Since u_1 vanishes on $\partial B(0;\rho)$ we obtain (23) for u_1.

4 Integral equation

We shall suppose that Ω is a domain with compact Lipschitz boundary. If $x \in \partial\Omega$, $\alpha > 0$, denote the non-tangential approach region of opening α at the point x

$$\Gamma_\alpha(x) = \{y \in \Omega; |x - y| < (1 + \alpha)\operatorname{dist}(y, \partial\Omega)\},$$

where $\operatorname{dist}(y, \partial\Omega)$ is the distance of y from $\partial\Omega$. If u is a function on Ω and

$$c = \lim_{y \to x, y \in \Gamma_\alpha(x)} u(y)$$

for each $\alpha > \alpha_0$, we say that c is the nontangential limit of u at x.

Since Ω is a Lipschitz domain there is $\alpha_0 > 0$ such that $x \in \overline{\Gamma}_\alpha(x)$ for each $x \in \partial\Omega$, $\alpha > \alpha_0$.

If $f \in L^2(\partial\Omega)$ then the single layer potential $\mathcal{S}^\Omega f(x)$ is defined for almost all $x \in \partial\Omega$ and it is the nontangential limit of $\mathcal{S}^\Omega f$. Moreover, since $\mathcal{S}^\Omega f \in L^2(\partial\Omega)$, we have $f(\mathcal{H}_{m-1}|\partial\Omega) \in \mathcal{E}(\partial\Omega)$ and $\mathcal{S}^\Omega f \in \hat{H}^{1,2}(R^m)$. The nontangential limit of $\mathcal{S}^\Omega f$ on $\partial\Omega$ is also the trace of $\mathcal{S}^\Omega f$ and thus $\mathcal{S}^\Omega f$ is the trace of $\mathcal{S}^\Omega f$ on $\partial\Omega$. If K^* is given by (13), i.e.

$$K^* f(x) = \lim_{\epsilon \searrow 0} \int\limits_{\partial\Omega\setminus B(x;\epsilon)} \frac{n^\Omega(x)\cdot(y-x)}{\mathcal{H}_{m-1}(\partial B(0;1))|x-y|^m} f(y)\, \mathrm{d}\mathcal{H}_{m-1}(y),$$

then K^* is a bounded linear operator on $L^2(\partial\Omega)$. If $f \in L^2(\partial\Omega)$ then there is the nontangential limit of $\nabla\mathcal{S}^\Omega f$ at almost all points of boundary $\partial\Omega$ and $n^\Omega \cdot \nabla\mathcal{S}^\Omega f = \frac{1}{2} f + K^* f$ in the sense $L^2(\partial\Omega)$ (see [10], Theorem 2.2.13).

If $f \in L^2(\partial\Omega)$ then for almost all $x \in \partial\Omega$ there is

$$Kf(x) = \lim_{\epsilon\to 0_+} \frac{1}{\mathcal{H}_{m-1}(\partial B(0;1))} \int\limits_{\partial\Omega\setminus B(x;\epsilon)} \frac{n^\Omega(y)\cdot(x-y)}{|y-x|^m} f(y)\, \mathrm{d}\mathcal{H}_{m-1}(y).$$

Moreover, $-\frac{1}{2}f(x)+Kf(x)$ is the nontangential limit of $\mathcal{D}^\Omega f$ at x for almost all $x \in \partial\Omega$ (see [10], Theorem 2.2.13). The operator K is a bounded linear operator in $L^2(\partial\Omega)$ and K^* is the adjoint operator of K (see [10], Theorem 2.2.13 and [29], Lemma 2.18).

If $f \in L^2(\partial\Omega)$ we can identify f with $f(\mathcal{H}_{m-1}|\partial\Omega)$. Since $K^* f = J_\Omega f - \frac{1}{2} f$ for $f \in L^2(\partial\Omega)$, the operator K^* can be extended as a bounded linear operator on $H^{-1/2}(\partial\Omega) = \mathcal{E}(\partial\Omega)$ by the formula $K^* f = J_\Omega f - \frac{1}{2} f$. Similarly we can put $\mathcal{S}^\Omega f = \mathcal{U} f$ for $f \in H^{-1/2}(\partial\Omega) = \mathcal{E}(\partial\Omega)$. Then $\mathcal{S}^\Omega f \in \hat{H}^{1,2}(\Omega)$ is harmonic and its normal derivative is $J_\Omega f = \frac{1}{2} f + K^* f$. Since K^* is a bounded operator on $H^{-1/2}(\partial\Omega)$, the adjoint operator of K^* is a bounded operator on $H^{1/2}(\partial\Omega)$. Since K and K^* are adjoint operators on $L^2(\partial\Omega)$, we deduce that K is a bounded operator on $H^{-1/2}(\partial\Omega)$ which is the adjoint operator of K^*. (For $\Omega \subset R^3$ compare [26].)

If $g \in H^{1/2}(\partial\Omega)$ then there is $u \in \hat{H}^{1,2}(\Omega)$ such that the trace of u on $\partial\Omega$ is equal to g. The function u is a solution of the Neumann problem for the Poisson equation with a right side $\mathcal{F} \in (\hat{H}^{1,2}(\Omega))' = \mathcal{E}(\overline{\Omega})$. Proposition 3.1 gives that $\mathcal{D}^\Omega g = \mathcal{U}\mathcal{F} - u \in \hat{H}^{1,2}(\Omega)$. The trace of $\mathcal{D}^\Omega g$ is the nontangetial limit of $\mathcal{D}^\Omega g$, i.e. $-\frac{1}{2} g + Kg$.

Let now $u \in \hat{H}^{1,2}(\Omega)$ be a weak solution of the Neumann problem for the Poisson equation with the right side $\mathcal{F} \in (\hat{H}^{1,2}(\Omega))' = \mathcal{E}(\overline{\Omega})$. Then $u = \mathcal{U}\mathcal{F} - \mathcal{D}^\Omega u$ in Ω by Proposition 3.1. Since $\mathcal{U}\mathcal{F}$ is the trace of $\mathcal{U}\mathcal{F}$, we obtain $u = \mathcal{U}\mathcal{F} - (-\frac{1}{2} u + Ku)$ on $\partial\Omega$. So, we get the integral equation

$$\frac{1}{2} u + Ku = \mathcal{U}\mathcal{F} \tag{25}$$

on $\partial\Omega$. If we find a solution of this equation then we reconstruct a solution of the Neumann problem using Proposition 3.1.

5 Unbounded domain

Suppose first that Ω is an unbounded domain with compact Lipschitz boundary. We know that the Neumann problem is solvable for each $\mathcal{F} \in (\hat{H}^{1,2}(\Omega))'$. Since two solutions of the problem differ by a constant and $v \equiv 1$ is not an element of $\hat{H}^{1,2}(\Omega)$, a solution of the Neumann problem is unique. We now show how to solve the equation $\frac{1}{2}u + Ku = g(= \mathcal{U}\mathcal{F})$.

We shall use the following Proposition 5.1 proved in [8].

Proposition 5.1. *Let X be a complex Banach space. Denote by $\mathcal{N}$ the set of all norms on X equivalent to the original norm. If T is a bounded linear operator in X denote by $\sigma(T)$ the spectrum of T and*

$$r(T) = \sup\{|\lambda|; \lambda \in \sigma(T)\}$$

the spectral radius of T. Then

$$r(T) = \inf_{\|\cdot\| \in \mathcal{N}} \|T\|.$$

Theorem 5.2. *Let $\Omega \subset R^m$, $m > 2$, be an unbounded domain with Lipschitz boundary. Then $\frac{1}{2}I + K$ is a continuously invertible operator in $H^{1/2}(\partial\Omega)$ and*

$$\left(\frac{1}{2}I + K\right)^{-1} = \sum_{j=0}^{\infty} \left(\frac{1}{2}I - K\right)^j.$$

Moreover, there are constants $d \geq 1$ and $q \in (0, 1)$ such that

$$\left\| \left(\frac{1}{2}I - K\right)^j \right\|_{H^{1/2}(\partial\Omega)} \leq dq^j$$

for each nonnegative integer j. Fix now $g \in H^{1/2}(\partial\Omega)$, $f_0 \in H^{1/2}(\partial\Omega)$. Put

$$f_{j+1} = \left(\frac{1}{2}I - K\right) f_j + g \tag{26}$$

for a nonnegative integer j. Then there exists

$$f = \lim_{j \to \infty} f_j,$$

f is the unique solution of the equation $\frac{1}{2}f + Kf = g$ and

$$\|f - f_j\|_{H^{1/2}(\partial\Omega)} \leq Cq^j (\|g\|_{H^{1/2}(\partial\Omega)} + \|f_0\|_{H^{1/2}(\partial\Omega)})$$

for arbitrary j, where C is a constant dependent only on Ω.

Proof. Put $T = I - (\frac{1}{2}I + K) = \frac{1}{2}I - K$. Then $(I-T)' = \frac{1}{2}I + K' = J_\Omega$ is a continuously invertible operator in $H^{-1/2}(\partial\Omega)$ and $r(T') < 1$ (see [21]). By a duality we obtain that $I - T = \frac{1}{2}I + K$ is a continuously invertible operator in $H^{1/2}(\partial\Omega)$ and $r(T) < 1$. According to Proposition 5.1 there is an equivalent norm $\|\cdot\|$ on $H^{1/2}(\partial\Omega)$ such that $\|T\| < 1$. The rest is a classical result. (Compare also Proposition 6.1 bellow.)

6 Bounded domain

Suppose now that Ω is a bounded domain with Lipschitz boundary. Then the Neumann problem for the Poisson equation with the right side $\mathcal{F} \in (\hat{H}^{1,2}(\Omega)' = \mathcal{E}(\overline{\Omega})$ is solvable if and only if $\mathcal{F}(1) = 0$. We would like to solve the equation $\frac{1}{2}u + Ku = \mathcal{U}\mathcal{F}$. We know that there is a solution of this equation and thus $g = \mathcal{U}\mathcal{F} \in (\frac{1}{2}I + K)(H^{1/2}(\partial\Omega))$. If v is constant then v is a solution of the Neumann problem with the zero right side. Thus v is a solution of the homogeneous equation $(\frac{1}{2}I + K)v = 0$. Lemma 6.2 will show that each solution of the homogeneous equation is constant. Since a solution of the Neumann problem is given up to an additive constant, every solution u of the integral equation $(\frac{1}{2}I + K)u = \mathcal{U}\mathcal{F}$ gives the trace of some solution of the Neumann problem with the right side $\mathcal{F}$.

We should like to express a solution of the equation (25) by a Neumann series. We shall need the following Proposition 6.1 proved in [22] (Proposition 3):

Proposition 6.1. *Let X be a Banach space, T be a bounded linear operator on X. Suppose that X is the direct sum of* $\mathrm{Ker}(I-T)$ *and $(I-T)(X)$. Denote by $\tilde{T}$ the restriction of T onto $(I-T)(X)$. Suppose that $r(\tilde{T}) < 1$. Then there are constants $d \geq 1$ and $q \in (0,1)$ such that*

$$\|\tilde{T}^j\| \leq dq^j$$

for each nonnegative integer j. Moreover,

$$(I - \tilde{T})^{-1} = \sum_{j=0}^{\infty} \tilde{T}^j.$$

Fix now $y \in (I-T)(X)$, $x_0 \in X$. Put

$$x_{j+1} = Tx_j + y$$

for a nonnegative integer j. Then there exists

$$x = \lim_{j\to\infty} x_j$$

and

$$\|x - x_j\| \le Cq^j(\|y\| + \|x_0\|)$$

for arbitrary j, where C is a constant dependent only on T.

Lemma 6.2. *Let T be a bounded linear operator on a Banach space X such that $T(X)$ is closed and X is the direct sum of $T(X)$ and $\mathrm{Ker}\,T = \{x \in X; Tx = 0\}$. Denote by T' the adjoint operator of T defined on X', the dual space of X. Then X' is the direct sum of $T'(X')$ and $\mathrm{Ker}\,T'$.*

Proof.

If $A \subset X$ denote $A^0 = \{S \in X'; Sx = 0\ \forall x \in A\}$ the annihilator of A. According to [24], Theorem 3.16 we have $T'(X') = [\mathrm{Ker}\,T]^0$ and $T'(X')$ is closed. Moreover, $\mathrm{Ker}\,T' = [T(X)]^0$ by [24], Theorem 3.7 and [24], Lemma 3.6. Denote by P the projection of X onto $T(X)$ along $\mathrm{Ker}\,T$. If S in X' then $S = SP + S(I-P)$, $SP \in [\mathrm{Ker}\,T]^0 = T'(X')$, $S(I-P) \in [T(X)]^0 = \mathrm{Ker}\,T'$. If $S \in T'(X') \cap \mathrm{Ker}\,T' = [T(X)]^0 \cap [\mathrm{Ker}\,T]^0$, $x \in X$ then $Sx = SPx + S(I-P)x = 0$ and thus $S = 0$. So, X' is the direct sum of $T'(X')$ and $\mathrm{Ker}\,T'$.

Lemma 6.3. *Let $\Omega \subset R^m$, $m > 2$, be a bounded domain with Lipschitz boundary. Then the space $H^{1/2}(\partial\Omega, C)$ is the direct sum of $(\frac{1}{2}I + K)(H^{1/2}(\partial\Omega, C))$ and $\mathrm{Ker}(\frac{1}{2}I + K)$. The kernel $\mathrm{Ker}(\frac{1}{2}I + K)$ is the space of all constant functions on $\partial\Omega$, $\sigma(\frac{1}{2}I + K) \subset \langle 0, 1\rangle$. Denote by $\tilde{K}$ the restriction of K onto $(\frac{1}{2}I + K)(H^{1/2}(\partial\Omega, C))$. Then $r(\tilde{K} - \frac{1}{2}I) < 1$.*

Proof. $(\frac{1}{2}I + K')(H^{-1/2}(\partial\Omega, C)) = \{\mathcal{F} \in H^{-1/2}(\partial\Omega, C); \mathcal{F}(1) = 0\}$ by [20], [21]. Since $\frac{1}{2}I + K$ is an adjoint operator of $\frac{1}{2}I + K'$, $\mathrm{Ker}(\frac{1}{2}I + K) = \{u \in H^{1/2}(\partial\Omega, C); \mathcal{F}(u) = 0\ \forall \mathcal{F} \in (\frac{1}{2}I + K')((H^{-1/2}(\partial\Omega, C))\}$ by [24], Theorem 3.7 and [24], Lemma 3.6, we infer that $\mathrm{Ker}(\frac{1}{2}I + K)$ is the space of all constant functions on $\partial\Omega$. Since $H^{-1/2}(\partial\Omega, C)$ is the direct sum of $\mathrm{Ker}(\frac{1}{2}I + K')$ and $(\frac{1}{2}I + K')(H^{-1/2}(\partial\Omega, C))$ (see [20], Proposition 7.5 and [20], Theorem 8.8), Lemma 6.2 gives that $H^{1/2}(\partial\Omega, C)$ is the direct sum of $\mathrm{Ker}(\frac{1}{2}I + K)$ and $(\frac{1}{2}I + K)(H^{1/2}(\partial\Omega, C))$. The spectrum $\sigma(\frac{1}{2}I + K') \subset \langle 0, 1\rangle$ by [20], Proposition 5.4, [5], Proposition 4.15 and [30], Chapter VII, §3, Theorem 3. The duality argument gives that $\sigma(\frac{1}{2}I + K) \subset \langle 0, 1\rangle$ (see [24], Theorem 6.24). Since $H^{1/2}(\partial\Omega, C)$ is the direct sum of $\mathrm{Ker}(\frac{1}{2}I + K)$ and $(\frac{1}{2}I + K)(H^{1/2}(\partial\Omega, C))$, we deduce that $\sigma(\frac{1}{2}I + \tilde{K}) \subset (0, 1\rangle$. Since $\sigma(-\frac{1}{2}I + \tilde{K})$ is a closed subset of the interval $(-1, 0\rangle$ (see [24], Theorem 6.3), we have $r(\tilde{K} - \frac{1}{2}I) < 1$.

Theorem 6.4. *Let $\Omega \subset R^m$, $m > 2$, be a bounded domain with Lipschitz boundary. Then $\frac{1}{2}I + \tilde{K}$ is a continuously invertible operator in the subspace $(\frac{1}{2}I + K)(H^{1/2}(\partial\Omega))$ and*

$$\left(\frac{1}{2}I + \tilde{K}\right)^{-1} = \sum_{j=0}^{\infty} \left(\frac{1}{2}I - \tilde{K}\right)^j.$$

Moreover, there are constants $d \geq 1$ and $q \in (0,1)$ such that

$$\left\| \left(\frac{1}{2} I - \tilde{K} \right)^j \right\| \leq d q^j$$

for each nonnegative integer j. Fix $g \in (\frac{1}{2}I + K)(H^{1/2}(\partial\Omega))$, $f_0 \in H^{1/2}(\partial\Omega)$. Put

$$f_{j+1} = \left(\frac{1}{2} I - K \right) f_j + g \tag{27}$$

for a nonnegative integer j. Then there exists

$$f = \lim_{j \to \infty} f_j,$$

f is a solution of the equation $\frac{1}{2} f + Kf = g$ and

$$\| f - f_j \|_{H^{1/2}(\partial\Omega)} \leq C q^j (\| g \|_{H^{1/2}(\partial\Omega)} + \| f_0 \|_{H^{1/2}(\partial\Omega)})$$

for arbitrary j, where C is a constant dependent only on Ω.

Proof. If f_j is given by (26) and $f_j \to f$ then $f = (\frac{1}{2} I - K) f + g$ by a limit procedure.

Put $T = I - (\frac{1}{2} I + K) = \frac{1}{2} I - K$. Now we use Lemma 6.3 and Proposition 4.1.

7 Uniquely solvable integral equation

In this paragraph we shall suppose that Ω is a bounded domain with Lipschitz boundary, $\mathcal{F} \in (\hat{H}^{1,2}(\Omega)' = \mathcal{E}(\overline{\Omega})$, $\mathcal{F}(1) = 0$. We would like to solve the equation $\frac{1}{2} u + Ku = \mathcal{U}\mathcal{F}$. Theorem 6.4 shows how we can approximate a solution of this problem. In the numerical practice we approximate $\mathcal{U}\mathcal{F}$, i.e. instead of the equation $\frac{1}{2} u + Ku = \mathcal{U}\mathcal{F}$ we solve the equation $\frac{1}{2} u + Ku = g$ where g is close to $\mathcal{U}\mathcal{F}$. If the given data $g \notin (\frac{1}{2} I + K)(H^{1/2}(\partial\Omega))$ then the sequence f_j given by (26) does converge.

Instead of the equation $\frac{1}{2} u + Ku = g$ we shall solve a uniquely solvable equation $Mu = g$. For $f \in H^{1/2}(\partial\Omega)$ put

$$Mf = \frac{1}{2} f + Kf + \frac{1}{\mathcal{H}_{m-1}(\partial\Omega)} \int_{\partial\Omega} f \, d\mathcal{H}_{m-1}.$$

We shall show that if $g \in (\frac{1}{2} I + K)(H^{1/2}(\partial\Omega))$ and $Mf = g$ then $(\frac{1}{2} I + K) f = g$. So, if there is a solution of the Neumann problem for the Poisson equation with the right side $\mathcal{F}$ and f is a solution of the equation $Mf = \mathcal{U}\mathcal{F}$ then f is the trace of some solution of the Neumann problem for the Poisson equation with the right side $\mathcal{F}$. We shall show that $r(I - M) < 1$ and we can approximate a solution of the equation $Mf = g$ by the successive approximation method. But in the numerical application we approximate also the operator M. So, instead of the integral equation $Mf = g$ we solve an equation $\tilde{M}f = g$ where $\tilde{M}$ is close to M. Since $r(I - M) < 1$ there is a norm $\| \cdot \|$ on $H^{1/2}(\partial\Omega)$ equivalent to the original norm such that $\|I - M\| < 1$ (see Proposition 5.1). Thus if $M_k \to M$, then $M_k^{-1}g \to M^{-1}g$, $\|I - M_k\| < 1$ for sufficiently large k and we can use the successive approximation method for solving the equation $M_k f = g$.

Lemma 7.1. *The operator M is a continuously invertible operator in the complex Hilbert space $H^{1/2}(\partial\Omega, C)$ with $\sigma(M) \subset (0, 1\rangle$.*

Proof. Let $Mf = 0$. Since $H^{1/2}(\partial\Omega, C)$ is the direct sum of the space of constants and $(\frac{1}{2}I + K)(H^{1/2}(\partial\Omega, C))$ (see Lemma 6.3), we deduce that

$$\int\limits_{\partial\Omega} f \, d\mathcal{H}_{m-1} = 0, \quad \left(\frac{1}{2}I + K\right) f = 0. \tag{28}$$

Since $f \in \mathrm{Ker}(\frac{1}{2}I + K)$, Lemma 6.3 gives that f is constant. According to (28) we infer that $f = 0$.

The operator $\frac{1}{2}I + K$ is a Fredholm operator with index 0 by Lemma 6.3 (i.e. $\dim \mathrm{Ker}(\frac{1}{2}I + K) = \mathrm{codim}(\frac{1}{2}I + K)(H^{1/2}(\partial\Omega, C)) < \infty$). Since the operator $M - (\frac{1}{2}I + K)$ is an operator of finite rank and therefore compact, the operator M is a Fredholm operator with index 0, too (see [24], Theorem 5.10). Since M is injective, it is surjective. According to [5], Theorem 1.42 the operator M is continuously invertible.

Fix $\lambda \in C \setminus \langle 0, 1\rangle$. According to Lemma 6.3 we have $\sigma(\frac{1}{2}I + K) \subset \langle 0, 1\rangle$. Since the operator $M - (\frac{1}{2}I + K)$ is compact, the operator $\lambda I - M$ is a Fredholm operator with index 0 (see [24], Theorem 5.10). If $\lambda \in \sigma(M)$ then λ is an eigenvalue of M. Suppose that $Mf = \lambda f$. Since $H^{1/2}(\partial\Omega, C)$ is the direct sum of the space of constants and $(\frac{1}{2}I + K)(H^{1/2}(\partial\Omega, C))$ (see Lemma 6.3), there are $\tilde{f} \in (\frac{1}{2}I + K)(H^{1/2}(\partial\Omega, C))$ and a constant c such that $f = \tilde{f} + c$. Since $c \in \mathrm{Ker}(\frac{1}{2}I + K)$ we have

$$\lambda\tilde{f} + \lambda c = \left(\frac{1}{2}I + K\right)\tilde{f} + \frac{1}{\mathcal{H}_{m-1}(\partial\Omega)} \int\limits_{\partial\Omega} f \, d\mathcal{H}_{m-1}.$$

Since $H^{1/2}(\partial\Omega, C)$ is the direct sum of $(\frac{1}{2}I + K)(H^{1/2}(\partial\Omega, C))$ and the space of constants, we deduce that $\lambda\tilde{f} = (\frac{1}{2}I + K)\tilde{f}$. Since λ is not an eigenvalue of $\frac{1}{2}I + K$, we infer that $\tilde{f} = 0$. Since $f = c$ we have $\lambda c = Mc = c$. Since $\lambda \neq 1$ we obtain $c = 0$. This forces that $\lambda \notin \sigma(M)$.

Theorem 7.2. *Let $\Omega \subset R^m$, $m > 2$, be a bounded domain with Lipschitz boundary. Then M is a continuously invertible operator in $H^{1/2}(\partial\Omega)$ and*

$$M^{-1} = \sum_{j=0}^{\infty} (I - M)^j.$$

Moreover, there are constants $d \geq 1$ and $q \in (0,1)$ such that

$$\|(I-M)^j\|_{H^{1/2}(\partial\Omega)} \leq dq^j$$

for each nonnegative integer j. Fix now $g \in H^{1/2}(\partial\Omega)$, $f_0 \in H^{1/2}(\partial\Omega)$. Put

$$f_{j+1} = (I-M)f_j + g \tag{29}$$

for a nonnegative integer j. Then there exists

$$f = \lim_{j\to\infty} f_j,$$

f is the unique solution of the equation $Mf = g$ and

$$\|f - f_j\|_{H^{1/2}(\partial\Omega)} \leq Cq^j(\|g\|_{H^{1/2}(\partial\Omega)} + \|f_0\|_{H^{1/2}(\partial\Omega)})$$

for arbitrary j, where C is a constant dependent only on Ω. If moreover $g \in (\frac{1}{2}I + K)(H^{1/2}(\partial\Omega))$ then $\frac{1}{2}f + Kf = g$.

Proof. Let $g \in (\frac{1}{2}I + K)(H^{1/2}(\partial\Omega))$ and $Mf = g$. Since $H^{1/2}(\partial\Omega, C)$ is the direct sum of the space of constants and $(\frac{1}{2}I + K)(H^{1/2}(\partial\Omega, C))$ (see Lemma 6.3), $g = Mf = (\frac{1}{2}I + K)f + c$ with some constant c, we infer that $c = 0$ and thus $g = (\frac{1}{2}I + K)f$.

Lemma 7.1 gives that $\sigma(M) \subset (0,1\rangle$. Since $\sigma(I-M) \subset (-1,0\rangle$ is a closed set (see [24], Theorem 6.3), we have $r(I-M) < 1$. According to Proposition 5.1 there is a norm $\|\cdot\|$ on $H^{1/2}(\partial\Omega, C)$ equivalent to the original norm such that $\|I - M\| < 1$. The rest is a classical result. (See also Proposition 6.1 for $T = I - M$.)

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