MONOUNARY ALGEBRAS WITH TWO DIRECT LIMITS

Emília Halušková, Košice

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Abstract. In this paper we describe all algebras A with one unary operation such that by a direct limit construction exactly two nonisomorphic algebras can be obtained from A.

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For an algebra A we denote by $\mathbf{L}[A]$ the class of all isomorphic copies of algebras which can be obtained by a direct limit construction from A. We investigate classes $\mathbf{L}[A]$ in the case when A is a monounary algebra.

Every algebra A such that every endomorphism of A is an isomorphism has the property that whenever $B \in \mathbf{L}[A]$, then B is isomorphic to A. In [4] monounary algebras A such that $\mathbf{L}[A]$ consists of isomorphic copies of A were characterized. The natural question arises whether there exists a monounary algebra A such that the class $\mathbf{L}[A]$ contains exactly two nonisomorphic types of algebras.

In the present paper we construct a countable system of nonisomorphic types of monounary algebras with the mentioned property and we show that there are no other types of monounary algebras with this property.

1. Preliminaries

As usual, by a monounary algebra we understand an algebra with a single unary operation; cf. e.g. [9], [10]. For monounary algebras we will use the terminology as in [9].

The class of all monounary algebras will be denoted by $\mathcal{U}$. The class of all connected monounary algebras will be denoted by $\mathcal{U}^c$.

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We will use the symbol f for the operation in algebras of $\mathcal{U}$.

The symbol $\mathbb{N}$ denotes the set of all positive integers.

If $k \in \mathbb{N}$ and $A_1, \ldots, A_k$ are algebras, then by $[A_1, \ldots, A_k]$ we will understand the class of all isomorphic copies of algebras $A_1, \ldots, A_k$.

Let I be a nonempty set. For each $i \in I$ let A_i be a monounary algebra. We denote by $\sum\limits_{i \in I} A_i$ a monounary algebra which is a disjoint union of monounary algebras A_i, $i \in I$. If the set I is finite, $I = \{1, \ldots, n\}$, then instead of $\sum\limits_{i \in I} A_i$ we write $A_1 + \ldots + A_n$.

We recall the notion of a direct limit, cf. [2].

Let $\langle P, \leqslant \rangle$ be an upward directed partially ordered set, $P \neq \emptyset$. For each $p \in P$ let A_p be a monounary algebra and assume that if $p, q \in P$, $p \neq q$, then $A_p \cap A_q = \emptyset$. Suppose that for each pair of elements p and q in P with $p < q$ a homomorphism φ_{pq} of A_p into A_q is defined and that $p < q < s$ implies that $\varphi_{ps} = \varphi_{pq} \circ \varphi_{qs}$. Let φ_{pp} be the identity on A_p for each $p \in P$. We say that $\{P, A_p, \varphi_{pq}\}$ is a *direct family.*

Assume that $p, q \in P$ and $x \in A_p$, $y \in A_q$. Put $x \equiv y$ if there exists $s \in P$ with $p \leqslant s$, $q \leqslant s$ such that $\varphi_{ps}(x) = \varphi_{qs}(y)$. For each $z \in \bigcup\limits_{p \in P} A_p$ put $\overline{z} = \Big\{t \in \bigcup\limits_{p \in P} A_p \colon z \equiv t\Big\}$. Denote $\overline{A} = \Big\{\overline{z} \colon z \in \bigcup\limits_{p \in P} A_p\Big\}$.

If z_1, z_2 are elements of $\bigcup\limits_{p \in P} A_p$ such that $\overline{z}_1 = \overline{z}_2$, then clearly $\overline{f(z_1)} = \overline{f(z_2)}$. Hence if we put $f(\overline{z}_1) = \overline{f(z_1)}$, then the operation f on $\overline{A}$ is correctly defined and with respect to this operation $\overline{A}$ is a monounary algebra. It is said to be the *direct limit* of the direct family $\{P, A_p, \varphi_{pq}\}$. We will express this situation by writing

$$\{P, A_p, \varphi_{pq}\} \longrightarrow \overline{A}. \tag{1}$$

The autor is aware of the fact that the term 'direct limit' is rather out-of-date, and that the term 'directed colimit' (cf. [1]) would be more up-to-date.

Nevertheless, since the present paper can be considered as a continuation of the articles [4] and [3] where the term 'direct limit' was used, the author prefers the application of this therm also in this paper.

Let $A \in \mathcal{U}$ and (1) be valid. If $A_p \cong A$ for every $p \in P$, then we will write

$$\{P, A, \varphi_{pq}\} \longrightarrow \overline{A} \tag{2}$$

and say that $\overline{A}$ is a direct limit of A. We denote by $\mathbf{L}[A]$ the class of all monounary algebras which are isomorphic to some of the direct limits of A.

The next lemma is an immediate consequence of the definition of the relation (2).

Lemma 1. *Let $A \in \mathcal{U}$ and let (2) be valid.*

(i) *If φ_{pq} is an isomorphism for every $p, q \in P$ such that $p \leqslant q$, then $\overline{A} \cong A$.*

(ii) *The algebra $\overline{A}$ has a cycle if and only if A has a cycle.*

(iii) *Let $k \in \mathbb{N}$. If $\overline{A}$ contains a cycle of length k, then A contains a cycle of length k.*

(iv) *If A is connected, then $\overline{A}$ is connected.*

(v) *If the operation of A is injective, then the operation of $\overline{A}$ is injective.*

Lemma 2. *Let $A, B, D \in \mathcal{U}$. Suppose that $A = B \cup D$, $A = B + D$ and that (2) is valid.*

Let ψ_p be an isomorphism from A into A_p for every $p \in P$. Denote $B_p = \psi_p(B)$, $D_p = \psi_p(D)$ for every $p \in P$. Further, let $\varphi_{pq}(B_p) \subseteq B_q, \varphi_{pq}(D_p) \subseteq D_q$ for every $p, q \in P$, $p \leqslant q$.

Then $\{P, B_p, \varphi_{pq}\}$, $\{P, D_p, \varphi_{pq}\}$ are direct families (where φ_{pq} are the corresponding restrictions) and if $\{P, B_p, \varphi_{pq}\} \to \overline{B}$, $\{P, D_p, \varphi_{pq}\} \to \overline{D}$, then $\overline{A} = \overline{B} \cup \overline{D}$ and $\overline{A} = \overline{B} + \overline{D}$.

Proof. It follows from the fact that direct limits commute with sums. $\square$

Let us denote by N the monounary algebra defined on the set $\mathbb{N}$ with the operation of successor. Further, let Z be the monounary algebra defined on the set of all integers with the operation of successor.

Let A be a monounary algebra and let $\{B_j, j \in J\}$ be the set of all components of A. If $j \in J$ and $k \in \mathbb{N}$ are such that B_j contains a cycle of the length k, then let C_j be a cycle of the length k. If $j \in J$ is such that B_j contains no cycle, then put $C_j \cong Z$. We denote $A^\diamond = \sum\limits_{j \in J} C_j$.

The following result is proved in [3]:

Lemma 3. *Let $A \in \mathcal{U}$. Then $A^\diamond \in \mathbf{L}[A]$.*

Lemma 4.

$$\mathbf{L}[N] = [N, Z].$$

Proof. Since $N^\diamond = Z$ we have $\{N, Z\} \subseteq \mathbf{L}[N]$. Let (1) be valid and $A_p \cong N$ for every $p \in P$. In view of Lemma 1 (iv) and (v) the algebra $\overline{A}$ is connected and the operation of $\overline{A}$ is injective. Therefore $\overline{A} \cong Z$ or $\overline{A} \cong N$. $\square$

Let us denote

$$\begin{aligned}\mathcal{T} = \{A \in \mathcal{U}\colon\ & \text{every component of } A \text{ is a cycle and there are no}\\ & \text{components } C_1, C_2 \text{ of } A \text{ such that } C_1 \neq C_2 \text{ and the length of } C_1\\ & \text{divides the length of } C_2\}.\end{aligned}$$

In view of Theorem 1 of [4] we have

Lemma 5. $\mathbf{L}[A] = [A]$ *if and only if* $A \in \mathcal{T} \cup [Z]$.

Let $A \in \mathcal{U}$. Let B be a subalgebra of A. Assume that there exists a homomorphism φ of A onto B such that $\varphi(b) = b$ for each $b \in B$. Then B is said to be a *retract* of A and φ is called a *retract mapping* corresponding to B.

This definition yields

Lemma 6. *Let* $A \in \mathcal{U}$. *Let* J *be a set and let* B_j *be a component of* A *for every* $j \in J$. *If* B' *is a retract of the algebra* $\sum\limits_{j \in J} B_j$, *then the algebra*

$$\Big(A - \bigcup_{j \in J} B_j\Big) \cup B'$$

is a retract of A.

Retracts of monounary algebras were thoroughly studied by D. Jakubíková-Studenovská [5], [6]. The following lemma we obtain from Theorem 1.3 of [5]

Lemma 7. *Let* $A \in \mathcal{U}$. *If* A *contains a cycle, then there exists a retract* T *of* A *such that* $T \in \mathcal{T}$.

We will often use the following well-known property of direct limits; cf. [1] 2.4 and 1.5.

Lemma 8. *Let* $A \in \mathcal{U}$ *and let* B *be a retract of* A. *Then* $B \in \mathbf{L}[A]$.

Let $A \in \mathcal{U}$ and $R \subset A$. The set R is said to be a *chain* of the algebra A, if one of the following conditions is satisfied:

1. $R = \{a_0, \dots, a_n\}, n \in \mathbb{N} \cup \{0\}$, $a_i \neq a_j$ for $i \neq j$ and $f(a_i) = a_{i-1}$ for $i = 1, 2, \dots, n$;
2. $R = \big\{a_i, i \in \mathbb{N} \cup \{0\}\big\}$, $a_i \neq a_j$ for $i \neq j$ and $f(a_i) = a_{i-1}$ for each $i \in \mathbb{N}$.

We denote

$$\begin{aligned}\mathcal{T}_1 = \{A \in \mathcal{U}\colon\ &\text{there exists a chain } R \text{ of } A \text{ such that}\\ &A - R \in \mathcal{T} \text{ and } R \text{ fails to be a subalgebra of } A\}.\end{aligned}$$

If $A \in \mathcal{T}_1$ and R is a chain of A from the definition of $\mathcal{T}_1$, then we put $A^* = A - R$. It is easy to see that $\{A, A^*\} \subseteq \mathbf{L}[A]$.

In this section we will prove that if $A \in \mathcal{T}_1$, then $\mathbf{L}[A] = [A, A^*]$, i. e., if (2) is valid, then either $\overline{A} \cong A$ or $\overline{A} \cong A^*$.

The definition of $\mathcal{T}_1$ yields the following lemma.

Lemma 9. *Let $A \in \mathcal{T}_1$. If A is not connected, then there exist monounary algebras B, D such that $B \in \mathcal{T}_1 \cap \mathcal{U}^c$, $D \in \mathcal{T}$, $A = B \cup D$ and $A = B + D$.*

Theorem 1. *Let $A \in \mathcal{T}_1$. Then $\mathbf{L}[A] = [A, A^*]$.*

Proof. Suppose that A is connected.

Since $A^* \cong A^\diamond$, we have $A^* \in \mathbf{L}[A]$ according to Lemma 3. Thus $[A, A^*] \subseteq \mathbf{L}[A]$.

The algebra A is connected and thus A^* is a cycle of A. If $a \in A - A^*$, then there exists $k \in \mathbb{N}$ such that $f^k(a) \in A^*$ and $f^{k-1}(a) \notin A^*$. Let (2) be valid. Suppose that for every $p \in P$ a mapping ψ_p is an isomorphism from A onto A_p. For every $p \in P$ and $a \in A$ we denote $a_p = \psi_p(a)$ and $A_p^* = \psi_p(A^*)$.

The algebra $\overline{A}$ is connected and $\overline{A}$ has a cycle of the same length as A^*. If $p \in P$ and $x \in A_p^*$, then $\overline{x}$ belongs to the cycle of $\overline{A}$.

Suppose that $\overline{A}$ is not isomorphic to A. We need to prove that $\overline{A}$ is a cycle. Let $p \in P$. We need to prove that for every $a \in A$ we can find $s \in P$ such that $p \leqslant s$ and $\varphi_{ps}(a_p) \in A_s^*$.

By induction on k we show:

If $a \in A$ is such that $f^k(a_p) \in A_p^*$ and $f^{k-1}(a_p) \notin A_p^*$, then there exists $s \in P$ such that $p \leqslant s$ and $\varphi_{ps}(a_p) \in A_s^*$.

Let $a \in A$ and $k \in \mathbb{N}$. It is obvious that the following three assertions are equivalent:

(i) $f^k(a) \in A^*$ and $f^{k-1}(a) \notin A^*$;

(ii) there exists $p \in P$ such that $f^k(a_p) \in A_p^*$ and $f^{k-1}(a_p) \notin A_p^*$;

(iii) for every $q \in P$ we have $f^k(a_q) \in A_q^*$ and $f^{k-1}(a_q) \notin A_q^*$.

Let $k = 1$. Put $Q = \{q \in P\colon\ p \leqslant q\}$. Then $\{Q, A, \varphi_{q,q'}\} \to \overline{A}$. Assume that the equality $\varphi_{pq}(a_p) = a_q$ is satisfied for every $q \in Q$. Let $q, q' \in Q$ be such that $q \leqslant q'$. Then $\varphi_{qq'}(a_q) = \varphi_{qq'}(\varphi_{pq}(a_p)) = \varphi_{pq'}(a_p) = a_{q'}$. That means $\varphi_{qq'}$ is an isomorphism. Therefore $\overline{A} \cong A$, a contradiction. We conclude that there exists $q \in Q$ such that $\varphi_{pq}(a_p) \neq a_q$. This implies that there exists $i \in \mathbb{N}$ such that $\varphi_{pq}(a_p) = f^i(a_q)$ and so $\varphi_{pq}(a_p) \in A_q^*$.

Let $k \in \mathbb{N}$, $k > 1$ and let the claim hold for every natural number less than k. Analogously as in the first step there exist $q \in P$ such that $p \leqslant q$ and $\varphi_{pq}(a_p) = f^i(a_q)$ for some $i \in \mathbb{N}$. If $i \geqslant k$, then $\varphi_{pq}(a_p) \in A_q^*$. If $i < k$, then there exists $s \in P$ such that $q \leqslant s$ and $\varphi_{qs}(f^i(a_q)) \in A_s^*$ by the induction hypothesis (for $q \in P$ and $f^i(a_q)$). Thus $\varphi_{ps}(a_p) = \varphi_{qs}(\varphi_{pq}(a_p)) = \varphi_{qs}(f^i(a_q)) \in A_s^*$.

We conclude $\mathbf{L}[A] = [A, A^*]$.

Now suppose that A is not connected. Take B and D from Lemma 9. Then $A^* = B^* + D$.

Let (2) be valid. According to Lemma 2 we have that $\{P, B, \varphi_{pq}\}$, $\{P, D, \varphi_{pq}\}$, where φ_{pq} are the corresponding restrictions, are direct families. If $\{P, B, \varphi_{pq}\} \to \overline{B}$, then $\overline{B} \in [B, B^*]$ since B is a connected algebra from $\mathcal{T}_1$. If $\{P, D, \varphi_{pq}\} \to \overline{D}$, then $\overline{D} \cong D$ according to 5. In view of Lemma 2 we obtain

$$\overline{A} = \overline{B} + \overline{D} \in [B + D, B^* + D] = [A, A^*].$$

$\square$

3. Classes $\mathcal{T}_2, \mathcal{T}_3$

We denote

$$\begin{aligned}\mathcal{T}_2 = \{A \in \mathcal{U}\colon\ &\text{there exist } B \in \mathcal{T} \text{ and } k, l \in \mathbb{N} \text{ such that}\\ &A = B + C, \text{ where } C \text{ is a cycle of length } l, B \text{ contains a cycle of}\\ &\text{length } k \text{ and } l \text{ is a multiple of } k\}.\end{aligned}$$

If $A \in \mathcal{T}_2$, then we denote by A^* a subalgebra of A which is isomorphic to the algebra B from the definition of $\mathcal{T}_2$.

Further, we denote

$$\mathcal{T}_3 = \{A \in \mathcal{U}\colon \text{there exists } B \in \mathcal{T} \text{ such that } A = B + Z\}.$$

If $A \in \mathcal{T}_3$, then we denote by A^* a subalgebra of A such that $A^* \in \mathcal{T}$ and $A - A^*$ is an algebra isomorphic to Z.

If $A \cong Z + Z$, then we denote by A^* a subalgebra of A which is isomorphic to Z.

Theorem 2. *Let* $A \in \mathcal{T}_2 \cup \mathcal{T}_3 \cup [Z + Z]$. *Then* $\mathbf{L}[A] = [A, A^*]$.

P r o o f. Let us remark that if φ is an endomorphism of A, then φ has the following tree properties:

1. $\varphi(A^*) \cong A^*$;
2. $\varphi(A) = A$ or $\varphi(A) \cong A^*$;
3. if φ is onto A, then φ is an isomorphism.

It is obvious that $\{A, A^*\} \subseteq \mathbf{L}[A]$. Let (2) be valid.

Suppose that there exists $p \in P$ such that for every $q \in P$ the following conditon is valid: if $p \leqslant q$ then $\varphi_{pq}(A_p) = A_q$. Denote $Q = \{q \in P\colon\ p \leqslant q\}$. Let $q, s \in Q$ be such that $q \leqslant s$. In view of $\varphi_{ps} = \varphi_{pq} \circ \varphi_{qs}$ we have $\varphi_{qs}(A_q) = \varphi_{qs}(\varphi_{pq}(A_p)) = \varphi_{ps}(A_p) = A_s$. Thus φ_{qs} is an isomorphism between A_q and A_s. The set Q is cofinal with P and thus the direct limit of the family $\{Q, A, \varphi_{qs}\}$ is isomorphic to $\overline{A}$. We obtain $\overline{A} \cong A$ according to Lemma 1 (i).

Suppose that for every $p \in P$ there exists $q \in P$ such that $p \leqslant q$ and $\varphi_{pq}(A_p) \neq A_q$. Thus for every $p \in P$ there exists $q \in P,\ p \leqslant q$ such that $\varphi_{pq}(A_p) \cong A^*$. Choose $p \in P$. Let B_p be a subset of A_p such that $B_p \cong A^*$. Denote $R = \{r \in P\colon\ p \leqslant r\}$ and $B_r = \varphi_{pr}(B_p)$ for every $r \in R$. Then $B_r \cong A^*$ for every $r \in R$ and $\{R, B_r, \varphi_{rs}\}$ is a direct family. Assume that $\{R, B_r, \varphi_{rs}\} \to B$. Since $A^* \in \mathcal{T}$ or $A^* \cong Z$ we have $B \cong A^*$ according to Lemma 5.

Let $q \in R$. Take $s \in P$ such that $q \leqslant s$ and $\varphi_{qs}(A_q) \cong A^*$. Then $s \in R$ and $\varphi_{qs}(A_q) = B_s$ according to $B_s \cong A^*$. We obtain that $B \cong \overline{A}$; an isomorphism is $\psi(b) = \overline{a}$, where $a \in b$. $\square$

4. Class $\mathcal{T}_4$

Let us denote

$$\mathcal{T}_4 = \{A \in \mathcal{U}^c\colon \text{ there exists a chain } R \text{ of } A \text{ such that } A - R \cong Z\}.$$

In this section we will prove that if $A \in \mathcal{T}_4$, then $\mathbf{L}[A] = [Z, A]$.

Lemma 10. *Let $A \in \mathcal{T}_4$ and let R be a subset of A such that $A - R \cong Z$. If R is finite, then $\mathbf{L}[A] = [A, Z]$.*

P r o o f. Obviously $\{A, Z\} \subseteq \mathbf{L}[A]$.

Let C be a subalgebra of A which is isomorphic to Z. Since R is finite there exists exactly one element $a \in A$ such that $f(x) \neq a$ for every $x \in A$. Suppose that $n \in \mathbb{N}$ is such that $f^n(a) \in C$ and $f^{n-1}(a) \notin C$. Assume that $\{P, A, \varphi_{pq}\} \to \overline{A}$. Let ψ_p be an isomorphism from A into A_p for every $p \in P$. For every $p \in P$ and $x \in A$ denote $x_p = \psi_p(x)$ and $C_p = \psi_p(C)$.

Let $p, q \in P,\ p \leqslant q$. We remark that

1. $\varphi_{pq}(A_p) \cong A$ if and only if $\varphi_{pq}(a_p) = a_q$;
2. $\varphi_{pq}(A_p) \cong Z$ if and only if $\varphi_{pq}(a_p) \in C_q$;
3. if φ_{pq} is onto A_q, then φ_{pq} is an isomorphism.

Denote

$(*)$ There exists $p \in P$ such that $\varphi_{pq}(A_p) \cong A$ whenever $q \in P$ and $p \leqslant q$.

$(**)$ For every $p \in P$ there exists $q \in P$ such that $p \leqslant q$ and $\varphi_{pq}(A_p) \cong Z$.

We will prove that

a) $(*)$ is not fulfilled if and only if $(**)$ is fulfilled;

b) $(*)$ implies $\overline{A} \cong A$;

c) $(**)$ implies $\overline{A} \cong Z$.

a) Clearly if $(**)$ is fulfilled then $(*)$ is not fulfilled.

Suppose that $(*)$ is not fulfilled, i.e., for every $p \in P$ there exists $q \in P$ such that $p \leqslant q$ and $\varphi_{pq}(A_p)$ is not isomorphic to A.

Let $p_0 \in P$. Choose $p_1, \dots, p_n \in P$ such that $\varphi_{p_i p_{i+1}}(A_{p_i})$ is not isomorphic to A for $i \in \{0, \dots, n-1\}$. We get $\varphi_{p_i p_{i+1}}(a_{p_i}) \neq a_{p_{i+1}}$ for every $i \in \{0, \dots, n-1\}$. Therefore $\varphi_{p_0 p_n}(a_{p_0}) \in C_{p_n}$ and thus $\varphi_{p_0 p_n}(A_{p_0}) \cong Z$.

b) Let $(*)$ hold. If $\varphi_{pq}(A_p) \cong A$, then $\varphi_{pq}(A_p) = A_q$. Denote $Q = \{q \in P\colon\ p \leqslant q\}$. Let $q, s \in Q$ be such that $q \leqslant s$. We have $\varphi_{qs}(A_q) = \varphi_{qs}(\varphi_{pq}(A_p)) = \varphi_{ps}(A_p) = A_s$. Therefore φ_{qs} is an isomorphism and $\overline{A} \cong A$.

c) Assume that $(**)$ is valid. The algebra $\overline{A}$ is connected and contains no cycle according to Lemma 1. We need to show that $\overline{A}$ has a bijective operation.

Let $p, q \in P,\ x \in A_p,\ y \in A_q$ and $f(\overline{x}) = f(\overline{y})$. Then $\overline{f(x)} = \overline{f(y)}$ and thus there exists $s \in P$ such that $p, q \leqslant s$ and $\varphi_{ps}(f(x)) = \varphi_{qs}(f(y))$. The validity of $(**)$ yields that there exists $t \in P$ such that $s \leqslant t$ and $\varphi_{st}(A_s) \cong Z$. We have

$$f(\varphi_{st}(\varphi_{ps}(x))) = \varphi_{st}(\varphi_{ps}(f(x))) = \varphi_{st}(\varphi_{qs}(f(y))) = f(\varphi_{st}(\varphi_{qs}(y))).$$

Therefore $\varphi_{st}(\varphi_{ps}(x)) = \varphi_{st}(\varphi_{qs}(y))$ according to the injectivity of the operation of the algebra $\varphi_{st}(A_s)$. We conclude that $\varphi_{pt}(x) = \varphi_{qt}(y)$ and $\overline{x} = \overline{y}$.

Let $p \in P$ and $x \in A_p$. Choose $q \in P$ such that $\varphi_{pq}(A_p) \cong Z$. Then there is $y \in A_q$ such that $f(y) = \varphi_{pq}(x)$. Hence $f(\overline{y}) = \overline{x}$. $\square$

Lemma 11. *Let $A \in \mathcal{T}_4$ and let R be a subset of A such that $A - R \cong Z$. If R is infinite, then $\mathbf{L}[A] = [A, Z]$.*

P r o o f. According to Lemma 3 we have $Z \in \mathbf{L}[A]$.

Suppose that $\{P, A, \varphi_{pq}\} \to \overline{A}$. The algebra $\overline{A}$ is connected and Z is isomorphic to a subalgebra of $\overline{A}$ according to Lemma 1.

Let $p \in P$ and $x \in A_p$. Take $y \in A_p$ such that $f(y) = x$. Then $f(\overline{y}) = \overline{x}$.

Let $a, b, u, v \in \overline{A}$ be such that $a \neq b$, $u \neq v$, $a \neq u$, $f(a) = f(b)$ and $f(u) = f(v)$. It is easy to verify that then there exists $s \in P$ such that the set A_s contains elements a', b', u', v' which satisfy $a' \neq b'$, $u' \neq v'$, $a' \neq u'$, $f(a') = f(b')$ and $f(u') = f(v')$. This is a contradiction since $A_s \cong A$.

We conclude that $\overline{A} \cong Z$ or $\overline{A} \cong A$. $\square$

Theorem 3. *Let $A \in \mathcal{T}_4$. Then $\mathbf{L}[A] = [A, Z]$.*

P r o o f. It follows from Lemmas 10 and 11. $\square$

5. Main result

In this section we will characterize all monounary algebras A such that the class $\mathbf{L}[A]$ contains exactly two nonisomorphic types of monounary algebras.

Lemmas 3, 7 and 8 will be often used. Further, we will apply some results of D. Jakubíková-Studenovská from [7], [8].

Let $A \in \mathcal{U}$ and $k \in \mathbb{N}$. If $\mathbf{L}[A]$ contains at least k nonisomorphic types of algebras, then we will write $|\mathbf{L}[A]| \geqslant k$. If $\mathbf{L}[A]$ contains exactly k nonisomorphic types of algebras, then we will write $|\mathbf{L}[A]| = k$. If $\mathbf{L}[A]$ contains at most k nonisomorphic types of algebras, then we will write $|\mathbf{L}[A]| \leqslant k$.

Lemma 12. *Let A be an algebra without a cycle and let A be not isomorphic to N. If A does not contain a subalgebra isomorphic to Z, then $|\mathbf{L}[A]| \geqslant 3$.*

P r o o f. Let K be a component of A. We have $K^\diamond \cong Z$, because A is an algebra without a cycle. Further, $K^\diamond$ is not isomorphic to K, because A does not contain a subalgebra isomorphic to Z.

Suppose that $M = \{K_i, i \in I\}$ is the set of all components of A which are isomorphic to N.

First let $M \neq \emptyset$. Let $K \in M$. If M possesses only one component of A, then $A - K$ is a retract of A and the algebras A, $A^\diamond$, $A - K$ are nonisomorphic algebras from $\mathbf{L}[A]$. If $M - \{K\} \neq \emptyset$, then K is a retract of the algebra $\bigcup\limits_{i \in I} K_i$. In view of Lemma 6 we have that $\Bigl(A - \bigcup\limits_{i \in I} K_i\Bigr) \cup K$ is a retract of A. Thus A, $\Bigl(A - \bigcup\limits_{i \in I} K_i\Bigr) \cup K$, $A^\diamond$ are nonisomorphic algebras from $\mathbf{L}[A]$.

Now let $M = \emptyset$. Let K be a component of A. Then K contains at least two nonisomorphic retracts according to Lemma 3.1 of [8]. Assume that K' is a retract of K such that $K' \ncong K$. Let $L = \{K'_j, j \in J\}$ be the set of all components of A which are isomorphic to K. Since K' is a retract of the algebra $\bigcup\limits_{j \in J} K'_j$, we obtain that the algebra $\Big(A - \bigcup\limits_{j\in J} K_j'\Big) \cup K'$ is a retract of A according to Lemma 6. Moreover, $\Big(A - \bigcup\limits_{j\in J} K_j'\Big) \cup K' \ncong A$ because the algebra $\Big(A - \bigcup\limits_{j\in J} K_j'\Big) \cup K'$ contains no component isomorphic to K. Thus A, $\Big(A - \bigcup\limits_{j\in J} K_j'\Big) \cup K$, $A^\diamond$ are nonisomorphic algebras from the class $\mathbf{L}[A]$. $\square$

Lemma 13. *Let $A^\diamond \in \mathcal{T} \cup [Z]$. If $|\mathbf{L}[A]| \leqslant 2$, then*

$$A \in \mathcal{T} \cup \mathcal{T}_1 \cup \mathcal{T}_4 \cup [N, Z].$$

P r o o f. Let $A \notin \mathcal{T} \cup \mathcal{T}_1 \cup \mathcal{T}_4 \cup [N, Z]$.

If $A^\diamond \in \mathcal{T}$, then in view of $A \notin \mathcal{T} \cup \mathcal{T}_1$ there exists a component K of A such that K satisfies the assumptions of Lemmas 1.1, 1.2, 1.5, 1.6 or 2.3 from the paper [7]. It is proved in these lemmas that the algebra K has a retract K' such that $K' \ncong K$ and K' is not a cycle. Let $L = \{K_j', j \in J\}$ be the set of all components of A which are isomorphic to K. Since K' is a retract of A, Lemma 6 yields that $\Big(A - \bigcup\limits_{j\in J} K_j'\Big) \cup K'$ is a retract of A. Further, $A \ncong \Big(A - \bigcup\limits_{j\in J} K_j'\Big) \cup K'$ because the algebra $\Big(A - \bigcup\limits_{j\in J} K_j'\Big) \cup K'$ does not contain a component isomorphic to K. Thus A, $\Big(A - \bigcup\limits_{j\in J} K_j'\Big) \cup K'$, $A^\diamond$ are nonisomorphic algebras from the class $\mathbf{L}[A]$.

If A contains a subalgebra isomorphic to Z, then A is connected. In view of $A \notin \mathcal{T}_4 \cup [Z]$ the algebra A satisfies the assumptions of Lemma 2.3 or of Lemma 3.1 from the paper [8]. It is proved there that A has a retract B such that $B \ncong A$ and $B \ncong Z$. We have $A, B, Z \in \mathbf{L}[A]$.

If A does not contain a subalgebra isomorphic to Z and $A^\diamond \cong Z$, then $|\mathbf{L}[A]| \geqslant 3$ according to Lemma 12. $\square$

Lemma 14. *Let $A^\diamond \notin \mathcal{T} \cup [Z]$. If $|\mathbf{L}[A]| \leqslant 2$, then*

$$A \in \mathcal{T}_2 \cup \mathcal{T}_3 \cup [Z + Z].$$

P r o o f. The algebra A is not connected and $A \notin \mathcal{T}_1 \cup \mathcal{T}_4$. Suppose that $A \notin \mathcal{T}_2 \cup \mathcal{T}_3$ and A is not isomorphic to $Z + Z$.

Assume that A has no cycle. If A does not contain a subalgebra isomorphic to Z, then $|\mathbf{L}[A]| \geqslant 3$ according to Lemma 12. If A contains a subalgebra isomorphic to Z and A is not isomorphic to $A^\diamond$, then $A, A^\diamond, Z$ are nonisomorphic algebras of

$\mathbf{L}[A]$. If A contains a subalgebra isomorphic to Z and $A \cong A^\diamond$, then $A, Z+Z, Z$ are nonisomorphic algebras of $\mathbf{L}[A]$.

Assume that A has a cycle. Let $T \in \mathcal{T}$ be a retract of A. If $A^\diamond$ is not isomorphic to A, then $A, A^\diamond, T$ are mutually nonisomorphic and are in $\mathbf{L}[A]$.

Let $A^\diamond \cong A$. Then f is a bijective operation on A.

Let A contain a component K such that $A - K \in \mathcal{T}$. In view of $A \notin \mathcal{T}_3$ the algebra K is a cycle. Further, in view of $A \notin \mathcal{T}_2 \cup \mathcal{T}$ there exists a component K_1 of $A - K$ such that the number of elements of K_1 is a multiple of the number of elements of K. Hence $A - K_1$ is a retract of A. We obtain $A - K_1 \notin \mathcal{T}$ according to $A \notin \mathcal{T}_2$. We conclude that $A, T, A - K_1$ are nonisomorphic algebras of $\mathbf{L}[A]$.

Now let $A - K \notin \mathcal{T}$ for every component K of A. Then the algebra $A - T$ has at least two components and so A has at least three nonisomorphic retracts. $\square$

Theorem 4. *Let $A \in \mathcal{U}$. Then $|\mathbf{L}[A]| = 2$ if and only if*

$$A \in \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4 \cup [Z+Z, N].$$

P r o o f. Let $|\mathbf{L}[A]| = 2$. Then $A \in \mathcal{T} \cup \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3 \cup \mathcal{T}_4 \cup [Z, Z+Z, N]$ according to Lemmas 13 and 14. In view of Lemma 5 we have $A \notin \mathcal{T} \cup [Z]$.

Theorems 1, 2, 3 and Lemma 4 yield the opposite implication. $\square$

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Author's address: *Emília Halušková*, Mathematical Institute, Slovak Academy of Sciences, Grešákova 6, 040 01 Košice, Slovakia, e-mail: `ehaluska@mail.saske.sk`.