REMARKS ON THE SHERMAN-MORRISON-WOODBURY FORMULAE

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Abstract. We present some results on generalized inverses and their application to generalizations of the Sherman-Morrison-Woodbury-type formulae.

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1. Introduction

As the final goal, we are interested in extending the well known Sherman-Morrison formula [6]

$$(A+uv^T)^{-1} = A^{-1} - \frac{1}{1+v^TA^{-1}u}A^{-1}uv^TA^{-1}$$

(A is a nonsingular matrix, u, v column vectors) to the case that A is singular.

We recall first the notion of quasidirect sum of two matrices ([2], [3]), or, rank-additivity in the terminology of [5].

If A, B are matrices of the same order, then the sum $A+B$ is *quasidirect* if for the ranks,

$$\operatorname{rank}(A+B) = \operatorname{rank} A + \operatorname{rank} B.$$

Equivalent statements are:

1. The column space of $A+B$ is the direct sum of the column space of A and the column space of B; or, similarly, for the row spaces.

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2. There exist nonsingular matrices P and Q such that

$$PAQ = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad PBQ = \begin{pmatrix} 0 & 0 \\ 0 & B_0 \end{pmatrix},$$

where the partitionings on the right-hand sides are identical.

We will also be using the notion [1] of the $(1,2)$-generalized inverse to an $m \times n$ matrix A, and that of the Moore-Penrose inverse of such a matrix. A $(1,2)$-inverse of A is an $n \times m$ matrix X which satisfies

$$AXA = A, \tag{1}$$

$$XAX = X. \tag{2}$$

Such a matrix X is well known to always exist—even over a general field—and to have the same rank as A. It is, however, in general not uniquely determined.

The Moore-Penrose inverse A^+, usually in the case of the complex field, is the unique matrix which satisfies, in addition to (1) and (2), the relations

$$(AA^+)^* = AA^+, \tag{3}$$

$$(A^+A)^* = A^+A. \tag{4}$$

Here, as usual, the operation X^* means the conjugate transpose (in the real case, of course, just the transpose).

In Theorem 2.1, we will add a property to the theory of $(1,2)$-inverses which is formulated analogously to [4]. As usual, we call a square matrix P a *projector* if it satisfies $P^2 = P$, and for completeness, prove a simple lemma.

Lemma 1.1. *Let A be an $m \times n$ matrix of rank r, $A = RS$ its rank decomposition, i.e. R is $m \times r$, S is $r \times n$, where $r = \operatorname{rank} A$. If P is a projector of rank r for which $PA = A$, then $P = RU$ for some $r \times m$ matrix U satisfying $UR = I$.*

Proof. If a projector P satisfies $PA = A$, then, of course, $\operatorname{rank} P \geqslant \operatorname{rank} A$. Suppose now that $A = RS$ is a rank decomposition of A. Then for any row vector x with m coordinates, $xP = 0 \to xA = 0 \to xR = 0$. Thus, $P = RU$ for some $r \times m$ matrix U. Since $RURU = RU$, it follows that the nonsingular matrix UR satisfies $(UR)^3 = (UR)^2$, i.e. $UR = I$. □

We also need the following known results:

Theorem 1.2 ([1], Ch. 5, Theorem 8). *Let A be an (in general, complex) $m \times n$ matrix of rank r. Let V be an $n \times (n-r)$ matrix of rank $n-r$ for which $AV = 0$, let U be an $m \times (m-r)$ matrix of rank $m-r$ for which $U^*A = 0$. Then the matrix*

$$\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix} \tag{5}$$

is nonsingular and its inverse is

$$\begin{pmatrix} X & Y \\ Z & 0 \end{pmatrix}, \tag{6}$$

where X is the Moore-Penrose inverse A^+ of A and $Y = V(V^*V)^{-1}$, $Z = (U^*U)^{-1}U^*$.

In addition, $A^+U = 0$ and $V^*A^+ = 0$.

Remark 1.3. If the annihilating matrices U and V in Theorem 1.2 are "normalized", i.e. if we replace U by $U(U^*U)^{-\frac{1}{2}}$ and V by $V(V^*V)^{-\frac{1}{2}}$, then $Y = V$ and $Z = U^*$.

Theorem 1.4 (Woodbury's formula [7]). *Let A be a nonsingular $n \times n$ matrix, let U, V be $n \times r$ matrices of rank r, X a nonsingular $r \times r$ matrix.*

Then the matrix

$$A + UXV^T$$

is nonsingular if and only if the $r \times r$ matrix

$$X^{-1} + V^TA^{-1}U$$

is nonsingular. In that case,

$$(A + UXV^T)^{-1} = A^{-1} - A^{-1}U(X^{-1} + V^TA^{-1}U)^{-1}V^TA^{-1}. \tag{7}$$

2. Results

All results in this section—unless specified otherwise—hold for matrices over an arbitrary field.

Theorem 2.1. *Let A be an $m \times n$ matrix. Then:*

1. *If X is a $(1,2)$-inverse of A, then there exist projectors P, Q such that*

$$PA = A, \qquad AQ = A, \tag{8}$$

for which

$$\operatorname{rank}\begin{pmatrix} A & P \\ Q & X \end{pmatrix} = \operatorname{rank} A. \tag{9}$$

2. *If P, Q are projectors satisfying* (8)*, both with the same rank as A, then there exists a matrix X satisfying* (9)*. This matrix is uniquely determined and satisfies $AX = P$, $XA = Q$.*
3. *If for projectors P, Q satisfying* (8) *and for some matrix X* (9) *holds, then the matrix X is a $(1,2)$-inverse of A.*

Proof. To prove 1, choose $P = AX$, $Q = XA$. These are indeed projectors and

$$\operatorname{rank}\begin{pmatrix} A & P \\ Q & X \end{pmatrix} \leqslant \operatorname{rank} A$$

since, if r is the rank of A,

$$\begin{pmatrix} A & AX \\ XA & X \end{pmatrix}\begin{pmatrix} X & U \\ -I & 0 \end{pmatrix} = 0$$

for U of rank $n-r$ for which $AU = 0$, and the second matrix has rank $m+n-r$. Thus (9) holds.

To prove 2, observe first that by (9) the matrix X is uniquely determined. Indeed, every entry of X is contained in an $(r+1)\times(r+1)$ singular matrix which extends some nonsingular submatrix of A of order r. Now, by Lemma 1.1, if $A = RS$ is a rank decomposition of A, then $P = RU$ and $UR = I$, and analogously $Q = VS$ and $SV = I$. Choosing $X = VU$, (9) is then the product

$$\begin{pmatrix} R \\ V \end{pmatrix}(S\ U),$$

and thus has rank r.

To prove 3, let (9) be satisfied for projectors P and Q for which (8) holds. Multiply

$$\begin{pmatrix} A & P \\ Q & X \end{pmatrix}\begin{pmatrix} I & 0 \\ -A & I \end{pmatrix} = \begin{pmatrix} 0 & P \\ Q - XA & X \end{pmatrix}.$$

We have thus for the ranks

$$r = \operatorname{rank}(Q - XA) + \operatorname{rank} P.$$

Since $\operatorname{rank} P \geqslant r$, $Q = XA$. Analogously, premultiplication by

$$\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix}$$

yields $P = AX$. Further, observe that in the matrix

$$\begin{pmatrix} A & AX \\ XA & Y \end{pmatrix}$$

with rank equal to $\operatorname{rank} A$ the matrix Y is uniquely determined.

Now,

$$\begin{pmatrix} A & AX \\ XA & XAX \end{pmatrix} = \begin{pmatrix} I \\ X \end{pmatrix} A \begin{pmatrix} I & X \end{pmatrix},$$

so that $X = XAX$. Since $A = PA$, we have $A = AXA$ and X is indeed a $(1,2)$-inverse of A. $\square$

R e m a r k 2.2. If in 2 of Theorem 2.1 both projectors P and Q are Hermitian (or, symmetric in the real case), then X is the Moore-Penrose inverse of A.

Theorem 2.3. *Let A be an $n \times n$ matrix of rank $r < n$. Let $AP = 0$ and $Q^T A = 0$, where P and Q are $n \times (n-r)$ matrices of rank $n - r$. Let X be a nonsingular $(n-r) \times (n-r)$ matrix and let U, V be $n \times (n-r)$ matrices such that both the matrices $V^T P$ and $Q^T U$ are nonsingular.*

If α, β are numbers, then the matrix

$$\alpha A + \beta U X V^T$$

is nonsingular if and only if $\alpha\beta \neq 0$. In this case,

$$(\alpha A + \beta U X V^T)^{-1} = \alpha^{-1} B + \beta^{-1} P (V^T P)^{-1} X^{-1} (Q^T U)^{-1} Q^T, \tag{10}$$

where B is the (unique) matrix which satisfies one of the following four equivalent conditions:

$$AB = I - U(Q^T U)^{-1} Q^T, \quad V^T B = 0, \tag{11}$$

$$BA = I - P(V^T P)^{-1} V^T, \quad BU = 0, \tag{12}$$

$$\begin{pmatrix} A & U \\ V^T & 0 \end{pmatrix} \begin{pmatrix} B & P(V^T P)^{-1} \\ (Q^T U)^{-1} Q^T & 0 \end{pmatrix} = I_{2n-r}, \tag{13}$$

$$\operatorname{rank} \begin{pmatrix} A & I - U(Q^T U)^{-1} Q^T \\ I - P(V^T P)^{-1} V^T & B \end{pmatrix} = r. \tag{14}$$

In addition, both sums in (10) *are quasidirect.*

P r o o f. Observe first that (11) and (13) as well as (12) and (13) are equivalent. Let us show that also (14) is equivalent to (12). Let first (12) hold. The matrix

$$\begin{pmatrix} 0 & I \\ U(Q^TU)^{-1}Q^T & -A \end{pmatrix}$$

has rank $2n-r$ and annihilates the matrix Z on the left-hand side of (14). Consequently, the rank of Z is at most r. Since $\operatorname{rank} A = r$, equality in (14) holds.

Conversely, let (14) hold. Postmultiply Z by $\begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}$. The resulting matrix

$$\begin{pmatrix} A & 0 \\ I - P(V^TP)^{-1}V^T & BU \end{pmatrix}$$

has rank at most r, which implies $BU = 0$. Analogously, premultiplying Z by $\begin{pmatrix} I & 0 \\ 0 & V^T \end{pmatrix}$ yields $V^TB = 0$.

Postmultiply now Z by $\begin{pmatrix} B & I \\ -I & 0 \end{pmatrix}$. The resulting matrix

$$\begin{pmatrix} AB - I + U(Q^TU)^{-1}Q^T & A \\ 0 & I - P(V^TP)^{-1}V^T \end{pmatrix}$$

has then rank r so that, since $I - P(V^TP)^{-1}V^T$ is a projector of rank r, (11) holds.

The assertion itself then follows from (12) by performing the multiplication of $\alpha A + \beta UXV^T$ and $\alpha^{-1}B + \beta^{-1}P(V^TP)^{-1}X^{-1}(Q^TU)^{-1}Q^T$. The rest is obvious. □

R e m a r k 2.4. It is easily checked that B satisfies

$$ABA = A, \quad BAB = B,$$

i.e., B is a $(1,2)$-inverse of A.

Lemma 2.5. *Let A be a nonsingular $n \times n$ matrix, let r be a positive integer less than n. If U, V are $n \times (n-r)$ matrices such that $V^TA^{-1}U$ is nonsingular, then the decomposition*

$$A = A_0 + U(V^TA^{-1}U)^{-1}V^T,$$

for $A_0 = A - U(V^TA^{-1}U)^{-1}V^T$, is quasidirect.

In addition, $A_0(A^{-1}U) = 0$, $(V^TA^{-1})A_0 = 0$.

P r o o f. Immediate since all U, V and $U(V^TA^{-1}U)^{-1}V^T$ have rank $n-r$, whereas A_0 has rank at most r. □

Theorem 2.6. *Let A be a nonsingular $n \times n$ matrix, let r be a positive integer less than n. Let X be a nonsingular $r \times r$ matrix, U, V $n \times (n-r)$ matrices such that $V^T A^{-1} U$ as well as $X + (V^T A^{-1} U)^{-1}$ are nonsingular. Then $A + UXV^T$ is nonsingular and its inverse is*

$$B + A^{-1}U(V^T A^{-1} U)^{-1}(X + (V^T A^{-1} U)^{-1})^{-1}(V^T A^{-1} U)^{-1} V^T A^{-1}, \tag{15}$$

where B is the matrix for which

$$\begin{pmatrix} A & U \\ V^T & 0 \end{pmatrix} \begin{pmatrix} B & * \\ * & * \end{pmatrix} = I_{2n-r}. \tag{16}$$

Proof. By Lemma 2.5, A can be written as a quasidirect sum $A_0 + U(V^T A^{-1} U)^{-1} V^T$, and $A_0 P = 0$, $Q^T A_0 = 0$, where $P = A^{-1}U$ and $Q^T = V^T A^{-1}$. We have thus

$$(A + UXV^T)^{-1} = (A_0 + U(X + (V^T A^{-1} U)^{-1})V^T)^{-1},$$

so that (15) follows from Theorem 2.3 for $\alpha = \beta = 1$ and appropriately chosen matrices A and X. The fact that in (16) the matrix A can replace A_0 follows from $V^T B = 0$. □

For illustration, let us formulate the case $r = 1$ as a corollary.

Corollary 2.7. *Let A be a nonsingular $n \times n$ matrix, let u, v be column vectors with n coordinates such that $v^T A^{-1} u \neq 0$. If ξ is a number, then $A + u\xi v^T$ is nonsingular if and only if $\xi \neq -(v^T A^{-1} u)^{-1}$. In that case,*

$$(A + u\xi v^T)^{-1} = B + (\xi + (v^T A^{-1} u)^{-1})^{-1}(v^T A^{-1} u)^{-2} A^{-1} u v^T A^{-1},$$

where B is the matrix for which

$$\begin{pmatrix} A & u \\ v^T & 0 \end{pmatrix} \begin{pmatrix} B & * \\ * & * \end{pmatrix} = I_{n+1}.$$

We intend now to combine the results on the generalized inverses with the previous ones.

Theorem 2.8. *Let A be a real or complex $m \times n$ matrix of rank r. Let V be an $n \times (n-r)$ matrix of rank $n-r$ for which $AV = 0$, let U be an $m \times (m-r)$ matrix of rank $m-r$ for which $U^*A = 0$. Then the matrix*

$$\begin{pmatrix} A + UXV^* & U \\ V^* & 0 \end{pmatrix} \tag{17}$$

is nonsingular for every $r \times r$ matrix X, and its inverse is

$$\begin{pmatrix} A^+ & V(V^*V)^{-1} \\ (U^*U)^{-1}U^* & X \end{pmatrix}, \tag{18}$$

where A^+ is the Moore-Penrose inverse of A.

P r o o f. Since

$$\begin{pmatrix} A + UXV^* & U \\ V^* & 0 \end{pmatrix} = \begin{pmatrix} I & UX \\ 0 & I \end{pmatrix} \begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix},$$

the inverse is by Theorem 1.2

$$\begin{pmatrix} A^+ & V(V^*V)^{-1} \\ (U^*U)^{-1}U^* & 0 \end{pmatrix} \begin{pmatrix} I & -UX \\ 0 & I \end{pmatrix},$$

i.e. (18) since $A^+U = 0$ by Theorem 1.2. □

Theorem 2.9. *Let A be a real or complex $n \times n$ matrix of rank $r < n$. Let $AV = 0$ and $U^*A = 0$, where U and V are $n \times (n-r)$ matrices of rank $n-r$. Let X be a nonsingular $(n-r) \times (n-r)$ matrix and let P, Q be $n \times (n-r)$ matrices such that $Q^*V = 0$ as well as $U^*P = 0$.*

Then the matrix $A + PXQ^*$ has rank at most r, and exactly r if and only if the matrix $X^{-1} + Q^*A^+P$ is nonsingular. In this case,

$$(A + PXQ^*)^+ = A^+ - A^+P(X^{-1} + Q^*A^+P)^{-1}Q^*A^+. \tag{19}$$

P r o o f. By Remark 1.3, we can suppose without loss of generality that both U and V are normalized, i.e. that $U^*U = I$ and $V^*V = I$. Since $U^*(A + PXQ^*) = 0$ as well as $(A + PXQ^*)V = 0$, the rank of $A + PXQ^*$ is at most r. The matrix

$$\begin{pmatrix} A + PXQ^* & U \\ V^* & 0 \end{pmatrix}$$

can be written as

$$\begin{pmatrix} A & U \\ V^* & 0 \end{pmatrix} + \begin{pmatrix} P \\ 0 \end{pmatrix} X (Q^* \ 0). \tag{20}$$

By Woodbury's formula (7), its inverse exists if and only if

$$X^{-1} + (Q^* \ 0) \begin{pmatrix} A^+ & V \\ U^* & 0 \end{pmatrix} \begin{pmatrix} P \\ 0 \end{pmatrix}$$

is nonsingular, i.e., if and only if $X^{-1} + Q^* A^+ P$ is nonsingular. But this occurs if and only if the rank of $A + PXQ^*$ is r as follows from Theorem 1.2.

Now, the inverse of (20) can be written in the form

$$\begin{pmatrix} A^+ & V \\ U^* & 0 \end{pmatrix} - \begin{pmatrix} A^+ & V \\ U^* & 0 \end{pmatrix} \begin{pmatrix} P \\ 0 \end{pmatrix} (X^{-1} + Q^* A^+ P)^{-1} (Q^* \ 0) \begin{pmatrix} A^+ & V \\ U^* & 0 \end{pmatrix}.$$

On the other hand, this matrix is, by Theorem 1.2,

$$\begin{pmatrix} (A + PXQ^*)^+ & V \\ U^* & 0 \end{pmatrix}.$$

Thus (19) follows by comparison of the upper-left corner matrices. □

3. Concluding remarks

Theorems 2.3, 2.6 and 2.9 present formulae extending in some sense Woodbury's formula. It would be desirable to use them in the case that the given matrix A is nonsingular but very badly conditioned to improve the situation from the (partial) knowledge of "almost annihilating" vectors.

Observe also that Theorem 2.3 implies the following maybe surprising result:

Theorem 3.1. *Let A be an $n \times n$ matrix of rank $r < n$. Let $AP = 0$ and $Q^T A = 0$, where P and Q are $n \times (n-r)$ matrices of rank $n-r$. Let X be a nonsingular $(n-r) \times (n-r)$ matrix and let U, V be $n \times (n-r)$ matrices such that both the matrices $V^T P$ and $Q^T U$ are nonsingular.*

Then the set of triples (x, y, z), $xyz \neq 0$, which satisfy

$$\det(xA + yUXV^T + zI) = 0$$

coincides with the set of those, again nonzero, triples which satisfy

$$\det(x^{-1}B + y^{-1}P(V^T P)^{-1} X^{-1} (O^T U)^{-1} Q^T + z^{-1} I) = 0,$$

where B is a $(1,2)$-inverse of A for which $BU = 0$ and $V^T B = 0$.

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