# NOTE ON THE RELATION BETWEEN RADIUS AND DIAMETER OF A GRAPH

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(Received April 14, 1994)

*Summary.* The known relation between the standard radius and diameter holds for graphs, but not for digraphs. We show that no upper estimation is possible for digraphs. We also give some remarks on distances, which are either metric or non-metric.

*Keywords*: graph, digraph, strong digraph, radius, diameter

*MSC 1991*: 05C12; 05C20

## 1. Introduction

It is known that for a graph $G$ the inequalities $r(G) \leqslant d(G) \leqslant 2r(G)$ hold, where $r(G)$ and $d(G)$ mean the radius and diameter of $G$, respectively (see [1] and [2]). Paper [3] notes that if we have any distance on graphs which is a metric and then define eccentricity, radius, and diameter as usual in terms of this metric, then the corresponding inequality holds. In [3] and also in [1], p. 217 a strong digraph $D$ with radius $r(D) = 2$ and diameter $d(D) = 5$ is given. So the known upper estimation $d(D) \leqslant 2r(D)$ does not hold for strong digraphs.

In this note we prove that for any natural numbers $r \geqslant 1$, $d \geqslant 2r$ there exists a strong digraph $D$ with radius $r$ and diameter $d$. (Distance and eccentricity are the usual ones.) Moreover, we give some assertion for distances that are metric or non-metric.

Basic notions not defined here are used according to [1]. A digraph $D$ is *strong*, if every two nodes are mutually reachable; it is *unilateral*, if for any two nodes at least one is reachable from the other.

For digraphs, the basic distance concepts are defined analogously to those for graphs except that we must consider the directions on the arcs. Thus the *distance* from $u$ to $v$ is the length of a shortest (directed) $u - v$ path. The *eccentricity* of a node $v$ in a digraph $D$, $e_D(v)$, is its distance to a farthest node in $D$. (For a strong digraph the eccentricities are all finite.) The *radius* is the minimum eccentricity and the *diameter* is the maximum. The *center* $C(D)$ of a digraph $D$ is the set of all nodes with minimum eccentricity. Any distance $d(u, v)$ between nodes $u$ and $v$ in a digraph $D$ is a *metric*, if it satisfies the following three conditions:

(1) $d(u, v) \geqslant 0$ for all nodes $u$ and $v$, and $d(u, v) = 0$ if and only if $u = v$;
(2) $d(u, v) = d(v, u)$ for all nodes $u$ and $v$;
(3) $d(u, w) \leqslant d(u, v) + d(v, w)$ for all nodes $u, v$, and $w$.

We note that to any distance in digraphs (metric or non-metric) we will apply the eccentricity, radius and diameter as usual, in terms of this distance.

## 2. Results

Now we give the basic theorem.

**Theorem 1.** *Let $r, d$ be natural numbers and $r \geqslant 1$, $d \geqslant 2r$. Then there exists a strong digraph $D$ with radius $r$ and diameter $d$.*

Proof. Let $r \geqslant 1$, $i \geqslant 0$, $d = 2r + i$ be natural numbers. The sought strong digraph $D$ is constructed in Fig. 1.

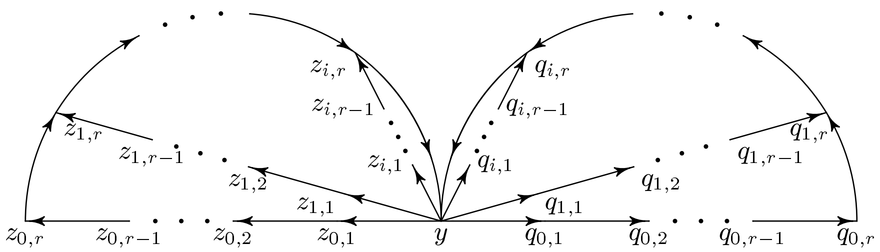


Fig. 1

Directly one can verify that for every $u \in V(D)$ one has

$$\min e(u) = e(y) = r = r(D),$$

$$\max e(u) = e(z_{0,1}) = e(q_{0,1}) = r - 1 + i + 1 + r = 2r + i = d(D)$$

(from $z_{0,1}$ to $q_{j,r}$ or from $q_{0,1}$ to $z_{j,r}$, for $j = 1, 2, \cdots, i$). The theorem holds. $\square$

In the constructed digraph $D$, the node $y$ is a cut-node. The following remark provides a construction without cut-nodes.

**Assertion 1.** *Let $r, d$ be natural numbers and $r \geqslant 1$, $d \geqslant 3r$. Then there exists a strong digraph $D$ with radius $r$, diameter $d$ and without cut-nodes.*

Proof. Let $r \geqslant 1$, $i \geqslant 0$, $d = 3r + i$. Let $C$ be an oriented circuit of length $2i+4$. Digraph $D$ is constructed in Fig. 2.

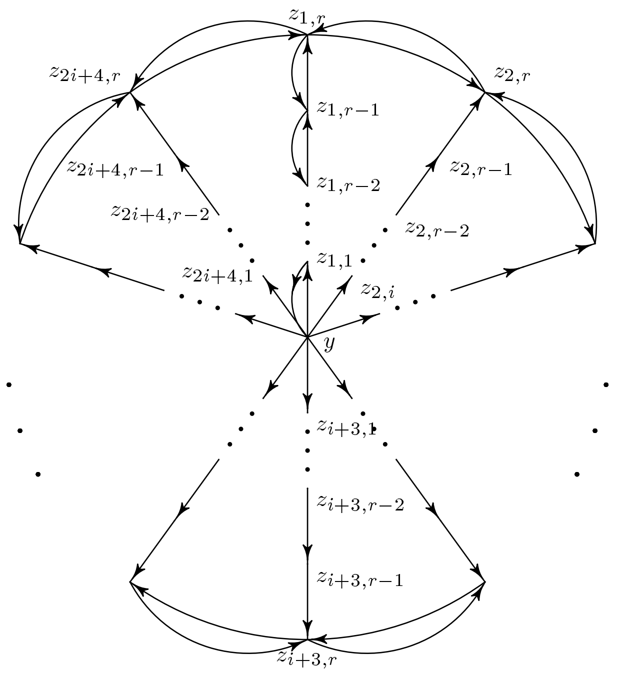


Fig. 2

In this strong digraph one can verify that for every $u \in V(D)$ one has

$$\min e(u) = e(y) = r = r(D),$$

$$\max e(u) = e(z_{i+3,1}) = 3r + i = d = d(D)$$

and the extrema are reached for nodes $z_{k,r-1}$ for $k \neq 1$, $k \neq i+3$. The assertion holds. $\square$

Now we give some remarks about other distances on digraphs which may be either metric or non-metric. First of all, for strong digraphs we show a result analogous to that given for graphs in [3].

**Assertion 2.** *Let $D$ be a strong digraph and $d$ its distance, which is a metric. Then the inequality $r(D) \leqslant d(D) \leqslant 2r(D)$ holds.*

Proof. Let the notions of eccentricity, radius and diameter of $D$ correspond to the distance $d$ and let them be defined analogously to the case of graphs. Let $a, b$ be such nodes of $D$ that their distance satisfies $d(a,b) = d(D)$. Let $w \in C(D)$. Then

$$d(a,b) \leqslant d(a,w) + d(w,b) = d(w,a) + d(w,b) \leqslant 2r(D).$$

The assertion holds. □

**Assertion 3.** *Let $D$ be a strong digraph with at least two nodes. Then there exists a distance $d$ on $D$ such that $d$ is not a metric and the inequality $r(D) \leqslant d(D) \leqslant 2r(D)$ holds.*

Proof. Let $D$ be a strong digraph with at least two nodes and let $w \in V(D)$ be its fixed node. For any two nodes $u, v$ of $D$ let $\bar{d}_w(u,v)$ be the length of a shortest oriented walk going from $u$ to $v$ through the node $w$. The sought distance $d_w$ can be defined as

$$d_w(u,v) = \bar{d}_w(u,v) + \bar{d}_w(v,u).$$

The distance $d_w$ is not a metric, because $d_w(u,u) \neq 0$ for every node $u \neq w$. (Using the distance $d_w$ one can define the eccentricity, radius and diameter analogously to the standard case.) Now we prove that $w$ is a central node of $D$.

Let $a \in V(D)$, $a \neq w$ and let $b \in V(D)$. Further, let $W_1(a,b) = (a = x_0,\ x_1, \ldots, x_i, x_{i+1}, \ldots, x_n = b)$ be a shortest oriented walk from $a$ to $b$ through $w$ and let the node $x_i = w$ and $x_i$ be the last occurence of $w$ in the walk $W_1(a,b)$. Analogously, let $W_2(b,a) = (b = y_0, y_1, \ldots, y_j, y_{j+1}, \ldots, y_m = a)$ be a shortest oriented walk from $b$ to $a$ through $w$. Let $y_j$ be the first occurence $w$ in the walk $W_2(b,a)$. Then the length of the walk $W_1$ going from $x_i = w$ to $b$ is $\leqslant \bar{d}_w(a,b)$ and analogously the length of the walk $W_2$ going from $b$ to $y_j = w$ is $\leqslant \bar{d}_w(b,a)$. Thus we obtain $d_w(a,b) = \bar{d}_w(a,b) + \bar{d}_w(b,a) \geqslant \bar{d}_w(w,b) + \bar{d}_w(b,w) = d_w(w,b)$, for any $b \in V(D)$.

Therefore, $e(a) \geqslant e(w)$ and $e(w) = r(D)$.

Directly from the definitions one obtains that $r(D) \leqslant d(D)$. Let $a, b \in V(D)$ be such that $d_w(a,b) = d(D)$. Then

$$\begin{aligned} d_w(a,b) &= \bar{d}_w(a,b) + \bar{d}_w(b,a) = \\ &= \bar{d}_w(a,w) + \bar{d}_w(w,b) + \bar{d}_w(b,w) + \bar{d}_w(w,a) = \\ &= (\bar{d}_w(a,w) + \bar{d}_w(w,a)) + (\bar{d}_w(w,b) + \bar{d}_w(b,w)) \leqslant \\ &\leqslant r(D) + r(D) = 2r(D). \end{aligned}$$

The assertion holds. □

Analogously to the case of digraphs in Assertion 3 one can prove the following assertion for graphs.

**Assertion 4.** *Let $G$ be a connected graph with at least two vertices. Then there exists a distance $d$ on $D$ such that $d$ is not a metric and the inequality $r(G) \leqslant d(G) \leqslant 2r(G)$ holds.*

*References*

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