EXACT 2-STEP DOMINATION IN GRAPHS

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Summary. For a vertex v in a graph G, the set $N_2(v)$ consists of those vertices of G whose distance from v is 2. If a graph G contains a set S of vertices such that the sets $N_2(v)$, $v \in S$, form a partition of $V(G)$, then G is called a 2-step domination graph. We describe 2-step domination graphs possessing some prescribed property. In addition, all 2-step domination paths and cycles are determined.

Keywords: 2-step domination graph

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1. Introduction

Two vertices u and v in a graph G for which the distance $d(u,v) = 2$ are said to 2-*step dominate* each other. The set of vertices of G that are 2-step dominated by v is denoted by $N_2(v)$; that is,

$$N_2(v) = \{u \in V(G) \mid d(v,u) = 2\}.$$

A set S of vertices of G is called a 2-*step domination set* if $\bigcup\limits_{v \in S} N_2(v) = V(G)$. A 2-step domination set S such that the sets $N_2(v)$, $v \in S$, are pairwise disjoint is called an *exact* 2-*step domination set.* If a graph G has an exact 2-step domination set, then G is called an *exact* 2-*step domination graph* or, for brevity, a 2-*step domination graph.* Each of the graphs G_1, G_2, and G_3 of Figure 1 is a 2-step domination graph

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with an exact 2-step domination set $S_1 = \{u_1, u_2, u_3, u_4\}$, $S_2 = \{v_1, v_2, v_3, v_4\}$, and $S_3 = \{w_1, w_2, w_3, w_4\}$, respectively. We adopt the convention of drawing a vertex with a solid circle if the vertex belongs to the exact 2-step domination set under discussion. In general we follow the graph theoretic notation and terminology of the books [1], [2].

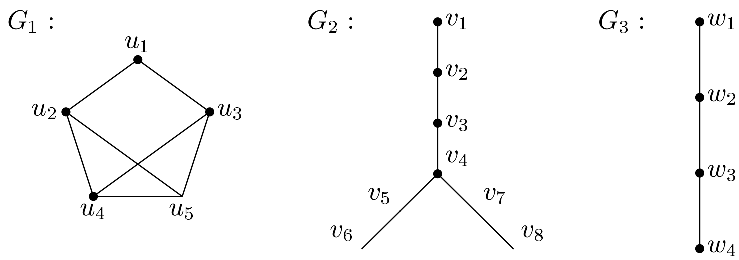

Figure 1. Three 2-step domination graphs.

2. Construction 2-step domination graphs

Our primary problem is to determine which graphs are 2-step domination graphs. If G is a graph of order p containing a vertex v of degree $p-1$, then no vertex of G 2-step dominates v. This observation yields the next result. We denote the radius and diameter of a graph G by $\operatorname{rad} G$ and $\operatorname{diam} G$, and the maximum degree of G by $\Delta(G)$.

Lemma 1. *If G is a 2-step domination graph, then $\operatorname{rad} G \geqslant 2$.*

According to Lemma 1 then, $\Delta(G) \leqslant p-2$ for every 2-step domination graph G of order p. No further reduction in the bound for $\Delta(G)$ is possible. For example, if $p = 2n$, the graph $\overline{nK_2}$ is a $(p-2)$-regular 2-step domination graph in which the only exact 2-step domination set consists of the entire vertex set. The path P_4 (the graph G_3 of Figure 1) also has the property that it is a 2-step domination graph whose unique exact 2-step domination set is the vertex set of the graphs. In fact, these are the only connected graphs with this property.

Theorem 2. *A connected graph G is a 2-step domination graph with exact 2-step domination set $V(G)$ if and only if $G \simeq P_4$ or $G \simeq \overline{nK_2}$ for some $n \geqslant 2$.*

P r o o f. First, the graphs $\overline{nK_2}$, $n \geqslant 2$, and P_4 have the desired property. Conversely, suppose that G is a connected 2-step domination graph with exact 2-step domination set $V(G)$. Necessarily, every vertex v of G has a unique vertex at distance 2 from v. Hence, $\operatorname{diam} G \geqslant 2$. If $\operatorname{diam} G \geqslant 4$, then G contains an induced subgraph isomorphic to P_5, the central vertex of which is at distance 2 from two vertices; so this is impossible. There remain two cases.

Case 1. $\operatorname{diam} G = 2$. Then, for every vertex v of G there is a unique vertex distinct from v and not adjacent to v. Hence p is even, say $p = 2n \geqslant 4$, and $G \simeq \overline{nK_2}$.

Case 2. $\operatorname{diam} G = 3$. In this case, G contains an induced path $P_4\colon v_1, v_2, v_3, v_4$ and hence $d(v_1, v_4) = 3$. Thus each of v_1 and v_3 is the unique vertex at distance 2 from the other, as is the case for v_2 and v_4. We claim that v_1 is an end-vertex of G. If this is not the case, then G contains a vertex x distinct from v_2 adjacent to v_1. If $xv_2 \notin E(G)$, then $d(v_2, x) = 2$, which is impossible; so $xv_2 \in E(G)$. Necessarily, $xv_3 \in E(G)$ as well; for otherwise, $d(v_3, x) = 2$. However, then, $xv_4 \in E(G)$; for otherwise, $d(v_4, x) = 2$. The existence of the path v_1, x, v_4, then contradicts the fact that $d(v_1, v_4) = 3$. Thus, as claimed, v_1 is an end-vertex of G. Similarly, v_4 is an end-vertex of G.

We now claim that each of v_2 and v_3 has degree 2. If this is not the case, then v_2, say, is adjacent to a vertex x different from v_1 and v_3; but then $d(v_1, x) = 2$, which is impossible. Consequently, $G \simeq P_4$. □

The fact that the graphs $\overline{nK_2}$, $n \geqslant 2$, are $(2n-2)$-regular 2-step domination graphs shows that r-regular 2-step domination graphs exist for every even integer $r \geqslant 2$. We next show that such is the case for odd values of r as well.

Let S consist of $2n$ vertices of the graph nC_4, $n \geqslant 2$, two adjacent vertices from each component. Then S is an exact 2-step domination set in the complement $\overline{nC_4}$. Since $\overline{nC_4}$ is $(4n-3)$-regular, r-regular 2 step domination graphs exist for $r \equiv 1 \pmod 4$. It remains to show the existence of r-regular 2-step domination graphs, where $r \equiv 3 \pmod 4$.

For $n \geqslant 0$, define the vertex set of the graph G'_n (as shown in Figure 2) by

$$V(G'_n) = \{u, u'\} \cup \{v, v'\} \cup \{w, w'\} \cup V \cup V',$$

where $V = \{v_1, v_2, \ldots, v_{4n+2}\}$ and $V' = \{v'_1, v'_2, \ldots, v'_{4n+2}\}$ and the edge set of G'_n by

$$E(G'_n) = \{uu', vw, v'w'\} \cup \{ux, wx \mid x \in V\} \cup \{u'x, w'x \mid x \in V'\}.$$

Next let $F \simeq F' \simeq \overline{K_1 \cup (2n+1)K_2}$, where $V(F) = V \cup \{v\}$ and $V(F') = V' \cup \{v'\}$, such that $deg_F v = \deg_{F'} v' = 4n+2$. Now define the graph G_n by $V(G_n) = V(G'_n)$ and

$$E(G_n) = E(G'_n) \cup E(F) \cup E(F').$$

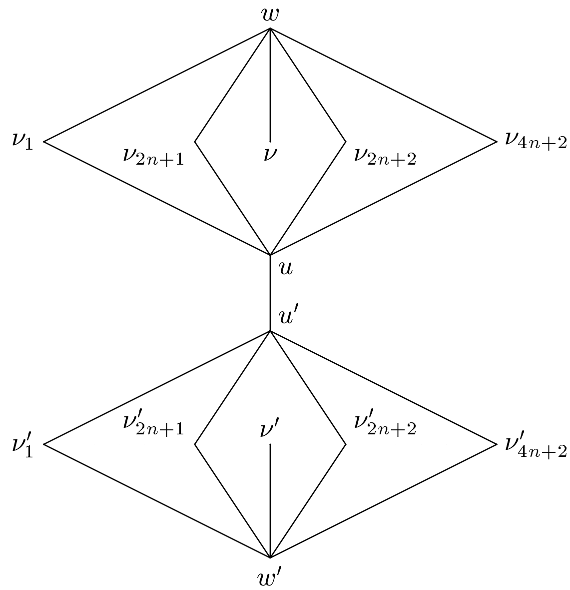

Figure 2. The graph G'_n.

The graph G_n is a $(4n+3)$-regular 2-step domination graph with exact 2-step domination set $\{u, u', w, w'\}$. We now summarize these observations.

Theorem 3. *For every integer $r \geqslant 2$, there exists an r-regular 2-step domination graph.*

The *composition* $G[H]$ of graphs G and H is constructed by replacing each vertex of G by a copy of H and each edge v_iv_j of G by the join H_i+H_j ($H_i \simeq H_j \simeq H$) of these respective copies of H. This operation has been often extended to the *generalized composition* $G[H_1, H_2, \ldots, H_p]$ of the labeled graph G with $V(G) = \{v_1, v_2, \ldots, v_p\}$ determined by any p graphs H_i. Again, each vertex v_i of G is replaced by H_i and each edge v_iv_j by the join $H_i + H_j$. This is illustrated in Figure 3.

With the aid of the generalized composition, we can construct new 2-step domination graphs from given 2-step domination graphs.

Theorem 4. *Let G be a 2-step domination graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$. For positive integers $n_1, n_2, \ldots, n_p$, the generalized composition $G[K_{n_1}, K_{n_2}, \ldots, K_{n_p}]$ is a 2-step domination graph.*

Proof. Since G is a 2-step domination graph, there exists an exact 2-step domiantion set S, say, without loss of generality, $S = \{v_1, v_2, \ldots, v_k\}$. For $i = 1, 2, \ldots, k$, let H_i be a graph such that $H_i \simeq K_{n_i}$ and let v'_i be a vertex of H_i. Then $S' =$

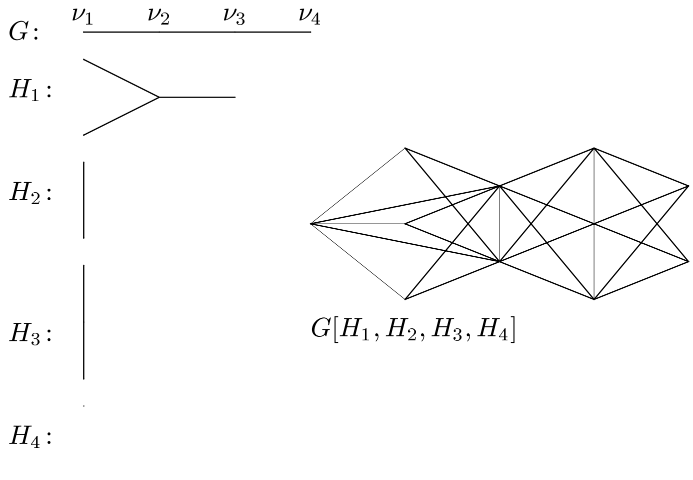

Figure 3. Construction of $G[H_1, H_2, H_3, H_4]$.

$\{v'_1, v'_2, \ldots, v'_k\}$ is an exact 2-step domination set of the graph $G[H_1, H_2, \ldots, H_p]$. □

Since the path P_4 is a 2-step domination graph (in which every vertex belongs to a 2-step domination set), by varying the orders of four complete graphs, we have the following.

Corollary 5. *For every integer $n \geqslant 4$, there exists a 2-step domination graph of order n.*

Furthermore, the proof of Theorem 4 shows that the graph $P_4[K_n, K_n, K_n, K_n]$ illustrates the fact that for every positive integer n, there exists a 2-step domination graph whose vertex set can be partitioned into n subsets, each of which is an exact 2-step domination set.

We now describe some additional examples of 2-step domination graphs. First we present some other terms, whose definitions are expected. A set S of vertices of a graph G is an *exact* 1*-step domination set* if the union $\bigcup N(v)$ of the open neighborhoods of the vertices v of S is $V(G)$ and the sets $N(v)$, $v \in S$, are pairwise disjoint. A graph then is a 1*-step domination graph* if it contains an exact 1-step domination set. The graphs shown in Figure 4 are 1-step domination graphs. So the complete bipartite graphs $K_{m,n}$, for any pair m, n of positive integers, are 1-step domination graphs.

Our special interest is in disconnected 1-step domination graphs.

Theorem 6. *A disconnected graph G is a 1-step domination graph if and only if its complement $\overline{G}$ is a 2-step domination graph.*

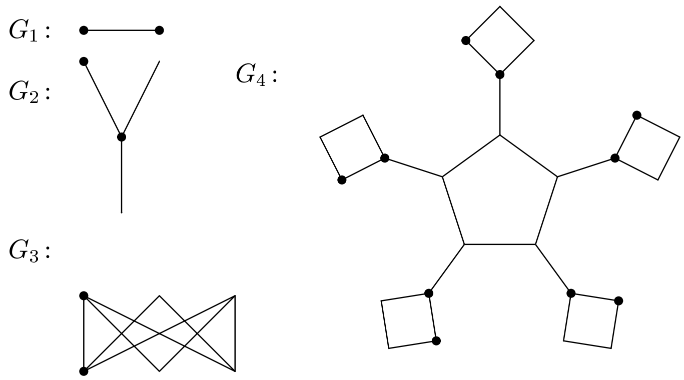

Figure 4. Four 1-step domination graphs.

P r o o f. Let G be a disconnected graph. Suppose first that G is a 1-step domination graph. Then $\operatorname{diam}\overline{G} = 2$ and the vertices adjacent to a vertex v of G are precisely the vertices at distance 2 from v in $\overline{G}$. Thus if S is an exact 1-step domination set of G, then S is an exact 2-step domination set of $\overline{G}$. Conversely, if $\overline{G}$ is a 2-step domination graph, then G is a 1-step domination graph. $\square$

If G is a disconnected graph whose four components G_i, $1 \leqslant i \leqslant 4$, are given in Figure 4, then by Theorem 6, $\overline{G}$ is a 2-step domination graph. We already observed in Theorem 2 that $\overline{nK_2}$, $n \geqslant 2$, is a 2-step domination graph. We have now seen several examples of 2-step domination graphs. If S is an exact 2-step domination set of a 2-step domination graph G, then, of course, $S \subseteq V(G)$, but there need not be any relationship between the numbers $|S|$ and $|V(G)|$.

Theorem 7. *For any rational number a/b, with $0 < a/b \leqslant 1$, there exists a 2-step domination graph G with an exact 2-step domination set S such that $|S|/|V(G)| = a/b$.*

P r o o f. Since we have already characterized those 2-step domination graphs G with $|S|/|V(G)| = 1$, we assume that $0 < a/b < 1$. We have already noted that the graph $H \simeq \overline{2aK_2}$ is a 2-step domination graph. Let G be the generalized composition obtained by replacing some vertex of H by the graph $K_{4b-4a+1}$ (and replacing all other vertices by K_1). By Theorem 4, G is a 2-step domination graph with $|S| = 4a$ and $|V(G)| = 4b$. Consequently, $|S|/|V(G)| = a/b$. $\square$

We now determine all those paths and cycles that are 2-step domination graphs. We begin by showing that if $m \equiv 1$, 2, or 3 (mod 8), then P_m is not a 2-step domination graph.

Theorem 8. *For every nonnegative integer n, none of the paths P_{8n+1}, P_{8n+2}, and P_{8n+3} are 2-step domination graphs.*

Proof. Suppose that the result is false. Since none of P_1, P_2, and P_3 are 2-step domination graphs, there is a smallest positive integer m (of the form $8n+1$, $8n+2$, or $8n+3$) such that P_m is a 2-step domination graph. Suppose that P_m is the path $v_1, v_2, \ldots, v_m$. Let S be an exact 2-step domination set of P_m. We consider three cases.

Case 1. Suppose that $m = 8n+1$. We now consider two subcases.

Subcase 1.1. Assume that four consecutive vertices among v_1, v_2, v_3, v_4, v_5, v_6 belongs to S. If $v_1, v_2, v_3, v_4 \in S$, then the vertices $v_1, v_2, \ldots, v_6$ of P_{8n+1} are 2-step dominated by the vertices v_1, v_2, v_3, v_4. Consequently, $P_{8n-5} = P_{8(n-1)+3}$ is a 2-step domination graph, contrary to assumption.

Suppose next that $v_2, v_3, v_4, v_5 \in S$. Then the vertices $v_1, v_2, \ldots, v_7$ of P_{8n+1} are 2-step dominated by the vertices v_2, v_3, v_4, v_5. This implies that $P_{8n-6} = P_{8(n-1)+2}$ is a 2-step domination graph, which is impossible. Similarly, we cannot have $v_3, v_4, v_5, v_6 \in S$.

Subcase 1.2. Assume that $v_1 \in S$. Since v_1 and v_2 must be 2-step dominated by elements of S, it follows that $v_3, v_4 \in S$. We can assume that $v_2 \notin S$; otherwise, the situation is covered by Subcase 1.1. Since v_4 is 2-step dominated by some vertex, $v_6 \in S$. Because $v_5 \notin S$ and v_7 is 2-step dominated by some vertex, $v_9 \in S$. If $n = 1$, we have a contradiction; if $n \geqslant 2$, we are repeating this Subcase with the path $P_{8(n-1)+1}$. Continuing in this manner, we see that $v_{8n+1} \in S$ but that v_{8n+1} is 2-step dominated by no vertex, producing a contradiction.

If neither $v_1 \in S$ nor four consecutive vertices among v_1, v_2, v_3, v_4, v_5, v_6 belong to S, then we must still have $v_3, v_4 \in S$ in order to have v_1 and v_2 2-step dominated. Now since v_3 must be 2-step dominated, $v_5 \in S$. In order for v_4 to be 2-step dominated, either $v_2 \in S$ or $v_6 \in S$, producing four consecutive vertices among v_1, v_2, v_3, v_4, v_5, v_6 in S. That is, Subcases 1.1 and 1.2 are exhaustive.

The proofs of the cases where $m = 8n+2$ and $m = 8n+3$ are similar and are, therefore, omitted. □

We next complete the problem for paths by showing that all other paths are 2-step domination graphs.

Theorem 9. *For every positive integer n, P_{8n} is a 2-step domination graph, and for every nonnegative integer n, P_{8n+4}, P_{8n+5}, P_{8n+6}, and P_{8n+7} are 2-step domination graphs.*

P r o o f. Consider the path $P_m\colon v_1, v_2, \ldots, v_m$, where m is of the form described in the statement of the theorem. For $m < 8$, Figure 5 shows that each path P_m is a 2-step domination graph. For $j = 4, 5, 6, 7$, denote by S_j the exact 2-step domination set of the path P_j.

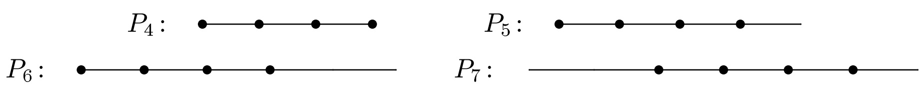

Figure 5.

We now make some observations that will be useful to us later. For the path P_{8n}, $n \geqslant 1$, an exact 2-step domination set $S_1 = \{v_i \mid i \equiv 3, 4, 5, 6 \pmod 8\}$ is described in Figure 6. The set $S_2 = \{v_i \mid i \equiv 1, 2, 3, 4 \pmod 8\}$ is also shown in Figure 6. It is not an exact 2-step domination set, but in this case, every vertex of P_{8n} is 2-step dominated except v_{8n-1} and v_{8n}.

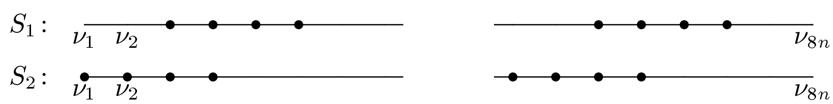

Figure 6.

The set S_1 shows that P_{8n}, $n \geqslant 1$, is a 2-step domination graph. Now label the vertices of the paths P_j $(j = 4, 5, 6, 7)$ in Figure 5 from left to right as v_{8n+1}, v_{8n+2}, $\ldots$, v_{8n+j}. The paths P_{8n+j} can be formed by taking the union of P_{8n} (see Figure 6) and P_j and joining v_{8n} and v_{8n+1}. The set $S_2 \cup S_j$ is an exact 2-step domiantion set for P_{8n+j} for $j = 4, 5, 6$; while $S_1 \cup S_7$ is an exact 2-step domination set for P_{8n+7}. $\square$

Corollary 10. *The path P_m is a 2-step domiantion graph if an only if $m = 0, 4, 5, 6,$ or $7 \pmod 8$,*

In order to characterize the 2-step domination cycles, we begin with a preliminary result.

Lemma 11. *If a cycle $C_n\colon v_1, v_2, \ldots, v_n, v_1$ $(n \geqslant 4)$ is a 2-step domination graph with exact 2-step domination set S, then there is an integer i $(1 \leqslant i \leqslant n)$ such that* *either* (1) v_i, v_{i+1}, v_{i+2}, $v_{i+3} \in S$ *or* (2) v_i, v_{i+2}, $v_{i+3} \in S$ *and* $v_{i+1} \notin S$ *(where all addition is performed modulo* n*).*

Proof. If $n = 4$, then $S = \{v_1, v_2, v_3, v_4\}$ is the only exact 2-step domination set, and the result follows. Thus we may assume that $n \geqslant 5$. Suppose that there are no vertices v_1, v_{i+1}, v_{i+2}, v_{i+3} for which (1) or (2) holds.

Every vertex $v_j \in S$ $(1 \leqslant j \leqslant n)$ is 2-step dominated by either v_{j-2} or v_{j+2}. Hence, without loss of generality, we may assume that v_1, $v_3 \in S$. By our assumption, there are now two possibilities for v_2 and v_4.

Case 1. v_2, $v_4 \notin S$. Hence $v_n \in S$ and so $v_{n-2} \in S$. (See Figure 7a.) If $v_{n-1} \in S$, then (1) is satisfied; while if $v_{n-1} \notin S$, (2) is satisfied, producing a contradiction.

Case 2. $v_2 \in S$ and $v_4 \notin S$. (See Figure 7b.) Since v_2 is not 2-step dominated by v_4, it follows that $v_n \in S$. Thus, v_n, v_1, v_2, $v_3 \in S$, producing a contradiction. □

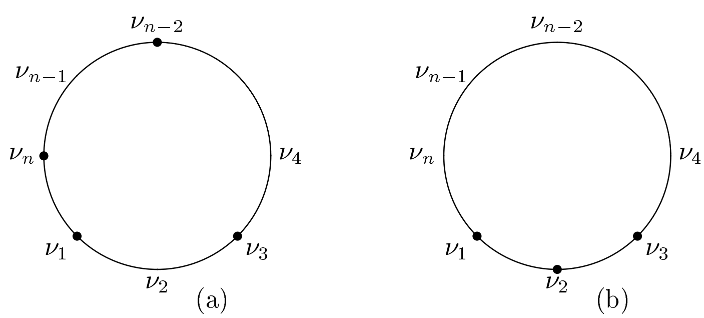

Figure 7.

We can now describe all 2-step domination cycles.

Theorem 12. *A cycle* C_n *is a 2-step domination graph if and only if* $n = 4$ *or* $n \equiv 0 \pmod 8$.

Proof. We have already seen that C_4 is a 2-step domination graph. It is straightforward to see that for other values of $m < 8$, the cycle C_m is not a 2-step domination graph. Now let $C_{8n}\colon v_1, v_2, \ldots, v_{8n}, v_1$ $(n \geqslant 1)$ be a cycle. The set $S = \{v_i \mid i \equiv 1, 2, 3, 4 \pmod 8\}$ is an exact 2-step domination set.

For the converse, assume that $C_m\colon v_1, v_2, \ldots, v_m, v_1$ is a 2-step domination graph with $m \geqslant 8$ and with exact 2-step domiantion set S. By Lemma 11, we can assume, without loss of generality, that either (1) $v_1, v_2, v_3, v_4 \in S$ or (2) $v_1, v_3, v_4 \in S$ and $v_2 \notin S$. If (1) occurs, then $v_5, v_6, v_7, v_8 \notin S$. If $m > 8$, then the vertices of P_m must repeat in this manner in groups of 8, that is, $v_i \in S$ if $i \equiv 1$, 2, 3, 4 (mod 8) and

$v_i \notin S$ otherwise. Thus $m \equiv 0 \pmod 8$. If (2) occurs, then $v_5, v_7, v_8 \notin S$ and $v_6 \in S$. If $m > 8$, then the vertices of P_m must repeat in this manner as well. In any case, $m \equiv 0 \pmod 8$. □

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