FUNCTIONAL MONADIC n-VALUED ŁUKASIEWICZ ALGEBRAS

A. V. Figallo, C. Sanza, A. Ziliani, Bahía Blanca

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Abstract. Some functional representation theorems for monadic n-valued Łukasiewicz algebras (qLk_n-algebras, for short) are given. Bearing in mind some of the results established by G. Georgescu and C. Vraciu (Algebre Boole monadice si algebre Łukasiewicz monadice, Studii Cercet. Mat. 23 (1971), 1027–1048) and P. Halmos (Algebraic Logic, Chelsea, New York, 1962), two functional representation theorems for qLk_n-algebras are obtained. Besides, rich qLk_n-algebras are introduced and characterized. In addition, a third theorem for these algebras is presented and the relationship between the three theorems is shown.

Keywords: monadic n-valued Łukasiewicz algebra, monadic Boolean algebra

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Introduction

Monadic Boolean algebras were introduced by P. Halmos in the fifties ([9], [10]) as an algebraic counterpart of the one variable fragment of classical predicate logic. One of his well known results related to the present paper is that every monadic Boolean algebra can be embedded into one constructed by the collection of all functions from a set X into a complete Boolean algebra B.

Many papers deal with the problem of determining functional representation theorems for several classes of algebras along the lines of the above mentioned Halmos representation theorem. For instance, in 2002, G. Bezhanishvili and J. Harding ([3]) established a representation theorem for monadic Heyting algebras, thus resolving a question posed by A. Monteiro and O. Varsavsky ([18]) 45 years before.

In 1971, in a very interesting but not widely known paper, G. Georgescu and C. Vraciu ([8]) introduced monadic n-valued Łukasiewicz algebras (qLk_n-algebras, for

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short) and studied their relationship to monadic Boolean algebras. From the results obtained by these authors, a functional representation theorem for these algebras is deduced without following Halmos's reasoning.

The aim of this article is to give other functional representation theorems for qLk_n-algebras taking into account Halmos's results and to point out the relationship between the theorems obtained and the one which arose from [8].

In Section 1, we briefly summarize the main definitions and results needed throughout the paper. In Section 2, the core of this paper, we establish a first functional representation theorem for qLk_n-algebras from the results established in [8]. By applying Halmos's functional representation theorem for monadic Boolean algebras to the set of Boolean elements of a qLk_n-algebra, we obtain another representation theorem. As a consequence of this theorem, we prove that every qLk_n-algebra can be embedded into a complete one. Finally, we introduce and characterize the notion of rich qLk_n-algebras and give a third representation theorem for them. At the end of this section, we present some final conclusions showing the relationship between these representation theorems.

1. Preliminaries

We refer the reader to the bibliography listed here as [2], [5], [11] for specific details of the many basic notions and results of universal algebra including distributive lattices and Boolean algebras considered in this paper.

A monadic Boolean algebra is a pair $(A, \exists)$ where A is a Boolean algebra and $\exists$ is a unary operation on A which fulfils the following identities:

(Q0) $\exists 0 = 0$,
(Q1) $x \wedge \exists x = x$,
(Q2) $\exists(x \wedge \exists y) = \exists x \wedge \exists y$.

A constant of a monadic Boolean algebra A (see [10]) is a Boolean endomorphism c on A such that

(c1) $c \circ \exists = \exists$,
(c2) $\exists \circ c = c$.

This mapping possesses the following properties:

(p1) $c \circ c = c$: From (c2) and (c1) we have $c \circ c = c \circ (\exists \circ c) = (c \circ \exists) \circ c = \exists \circ c = c$.
(p2) $c(x) \leqslant \exists x$ for all $x \in A$: Since $x \leqslant \exists x$ and c is a Boolean endomorphism on A we infer that $c(x) \leqslant c(\exists x)$. Then by (c1) we conclude (p2).

In particular, a constant c is a witness to an element z of A if $\exists z = c(z)$, and we will denote it by c_z. Besides, a monadic Boolean algebra A is rich if for any $x \in A$ there exists a witness to x.

Some of Halmos's main results on monadic Boolean algebras listed below are relevant to our work.

(H1) Every monadic Boolean algebra is a subalgebra of a rich one ([10, Theorem 11]).

(H2) If A is a monadic Boolean algebra, then there exists a set X and a Boolean algebra B such that

(i) A is isomorphic to a subalgebra S of the functional Boolean algebra B^X where Y^Z denotes the set of all functions from Z into Y,

(ii) for each $f \in S$ there exists $x \in X$ such that $\exists f(x) = f(x)$ ([10, Theorem 12]).

On the other hand, a De Morgan algebra is a pair $(A, \sim)$ where A is a bounded distributive lattice and $\sim$ is an involutive dual endomorphism of A [4, Definition 2.6] (see also [12], [17]).

In 1941, G. Moisil ([13], [14], [15]) introduced and developed the theory of n-valued Łukasiewicz algebras. Later, R. Cignoli studied them in detail in his Ph.D. Thesis [6]. According to this author, these algebras are defined as follows:

An n-valued Łukasiewicz algebra (Lk_n-algebra, for short), where n is an integer, $n \geqslant 2$, is an algebra $(A, \sim, \varphi_1, \ldots, \varphi_{n-1})$ such that $(A, \sim)$ is a De Morgan algebra and φ_i, with $1 \leqslant i \leqslant n-1$, are unary operations on A fulfilling the conditions

(L1) $\varphi_i(x \vee y) = \varphi_i x \vee \varphi_i y$,

(L2) $\varphi_i x \vee \sim \varphi_i x = 1$,

(L3) $\varphi_i \varphi_j x = \varphi_j x$,

(L4) $\varphi_i \sim x = \sim \varphi_{n-i} x$,

(L5) $i \leqslant j$ implies $\varphi_i x \leqslant \varphi_j x$,

(L6) $\varphi_i x = \varphi_i y$ for all i, $1 \leqslant i \leqslant n-1$, implies $x = y$.

It is well known that the most important example of an Lk_n-algebra is the chain of n rational fractions $C_n = \{\frac{j}{n-1}, 0 \leqslant j \leqslant n-1\}$ endowed with the natural lattice structure and the unary operations $\sim$ and φ_i, defined as follows: $\sim (\frac{j}{n-1}) = 1 - \frac{j}{n-1}$ while $\varphi_i(\frac{j}{n-1}) = 0$ if $i + j < n$ and $\varphi_i(\frac{j}{n-1}) = 1$ in the other cases.

The properties announced here for these algebras will be used throughout the paper.

(M1) $\varphi_i A = B(A)$ for all i, $1 \leqslant i \leqslant n-1$, where $B(A)$ is the set of all Boolean elements of A, see [6].

(M2) Let X be an arbitrary nonempty set. Then A^X is an Lk_n-algebra where the operations are defined componentwise.

(M3) A filter F of A is a Stone filter (or s-filter) if the hypothesis $x \in F$ implies $\varphi_1(x) \in F$. We will denote by $\mathscr{F}_A$ the lattice of all s-filters of A.

(M4) The congruence lattice $\mathrm{Con}\,(A)$ of A is $\{R(F) \colon F \in \mathscr{F}_A\}$ where $R(F) = \{(x, y) \in A \times A \colon$ there exists $f \in F$ such that $x \wedge f = y \wedge f\}$, see [6]. Besides, if

$F \in \mathscr{F}_A$ and $x \in A$, we will denote by A/F the quotient algebra of A by $R(F)$ and by $|x|_F$ the equivalence class of x modulo $R(F)$.

(M5) A is centred if for each i, $1 \leqslant i \leqslant n-1$, there exists an element c_i such that $\varphi_j(c_i) = 0$ if $i > j$ and $\varphi_j(c_i) = 1$ otherwise ([4]).

In 1971, the notion of monadic n-valued Łukasiewicz algebras was introduced by G. Georgescu and C. Vraciu [8]. Such an algebra is an Lk_n-algebra together with a unary operation, denoted by $\exists$, which verifies (Q0), (Q1), (Q2) and the following additional identity:

(Q3) $\exists\varphi_i x = \varphi_i \exists x$ for all i, $1 \leqslant i \leqslant n-1$.

Besides, these authors proved

(GV1) If A is a qLk_n-algebra, then $(B(A), \exists)$ is a monadic Boolean algebra ([8, Lemma 1.4]).

Let B be a partially ordered set and $C(n, B)$ the set of all increasing functions from $\{1, 2, \ldots, n-1\}$ into B with n integer, $n \geqslant 2$.

The next two properties mentioned in the proofs of Proposition 1.5 and Corollary 1.6 in [8] can be formulated in the following way:

(GV2) Let B be a monadic Boolean algebra. Then $(C(n, B), \sim, \Phi_1, \ldots, \Phi_{n-1}, \exists)$ is a qLk_n-algebra where for all $f \in C(n, B)$ and $i, j \in \{1, 2, \ldots, n-1\}$ the operations $\sim$, Φ_j are defined as follows:
$(\sim f)(i) = (f(n-i))'$ where x' stands for the Boolean complement of x,
$(\Phi_j f)(i) = f(j)$, and the remaining operations are defined componentwise.

(GV3) If A is a qLk_n-algebra, then the mapping $\alpha\colon A \longrightarrow C(n, B(A))$ defined by $\alpha(a)(i) = \varphi_i a$ for each $i \in \{1, 2, \ldots, n-1\}$ is a one-to-one qLk_n-homomorphism.

Taking into account (GV2) and (GV3), Georgescu and Vraciu obtained the following characterization of centred qLk_n-algebras.

(GV4) Let A be a qLk_n-algebra. Then the following conditions are equivalent:

(i) A is centred,
(ii) A is isomorphic to $C(n, B(A))$ [8, Proposition 2.2].

On the other hand, let X be a nonempty set and C_n^X the Lk_n-algebra obtained as in (M2). Following [1], we shall denote by $C^*_{n,X}$ the monadic functional Lk_n-algebra $(C_n^X, \exists)$ such that the unary operation $\exists$ is defined by means of the formula $(\exists f)(x) = \bigvee f(X)$, where $\bigvee f(X)$ denotes the supremum of $f(X) = \{f(y)\colon\ y \in X\}$. Furthermore, $\exists C^*_{n,X} = \{e_j\}_{0 \leqslant j \leqslant n-1}$ and $B(C^*_{n,X}) \simeq 2^X$ where $e_j(x) = j/(n-1)$ for all $x \in X$ and 2 is the Boolean algebra with two elements.

For further information on Lk_n-algebras and qLk_n-algebras, the reader is referred to [1], [4], [6], [7], [8], [13], [14], [15], [16], [20], [21].

It is also pertinent to remark that the assertions established in (GV2) and (GV3) determine the following functional representation theorem for qLk$_n$-algebras:

Theorem 2.1. *Every qLk$_n$-algebra A can be embedded into the functional qLk$_n$-algebra $C(n, B(A))$.*

Besides, by virtue of (GV4) such embedding is onto if and only if A is a centred qLk$_n$-algebra.

Remark 2.1. Taking into account [1], it is straightforward to prove that the simple qLk$_n$-algebra $C^*_{n,X}$ is centred. On the other hand, it is well known that every simple qLk$_n$-algebra is a subalgebra of $C^*_{n,X}$. However, there exist subalgebras of the latter, i.e. simple qLk$_n$-algebras, which are not centred. Indeed, let us consider the qLk$_3$-algebra $C^*_{3,X}$ where $|X| = 2$ which is described as follows:

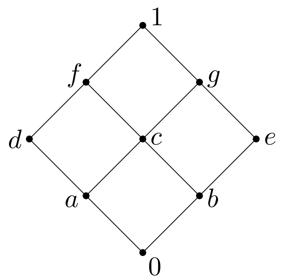

Figure 1

x	$\sim x$	$\varphi_1 x$	$\varphi_2 x$	$\exists x$
$0 = (0, 0)$	1	0	0	0
$a = (0, \frac{1}{2})$	g	0	d	c
$b = (\frac{1}{2}, 0)$	f	0	e	c
$c = (\frac{1}{2}, \frac{1}{2})$	c	0	1	c
$d = (0, 1)$	e	d	d	1
$e = (1, 0)$	d	e	e	1
$f = (\frac{1}{2}, 1)$	b	d	1	1
$g = (1, \frac{1}{2})$	a	e	1	1
$1 = (1, 1)$	0	1	1	1

Table 1

Taking into account [1, Lemma II.2.5] we find that the subalgebra $S = \{0, d, e, 1\}$ of $C^*_{3,X}$ is simple since $B(S) \cap \exists S = \{0, 1\}$. Nonetheless, S is not centred.

Corollary 2.1. *Every simple qLk$_n$-algebra is a subalgebra of $C(n, 2^X)$.*

Proof. By (GV4) and [1] the centred qLk$_n$-algebra $C^*_{n,X}$ is isomorphic to $C(n, 2^X)$, and taking into account that every simple qLk$_n$-algebra is a subalgebra of $C^*_{n,X}$, we conclude the proof. □

Remark 2.2. The following assertions are easily verified:

(i) If B is a Boolean algebra and X is an arbitrary nonempty set, then $C(n, B)^X$ is an Lk$_n$-algebra where the operations are defined componentwise.

(ii) If B is a complete Boolean algebra, then $C(n, B)$ is a complete Lk$_n$-algebra where the operations are defined as in (GV2).

Bearing in mind a well known result on Boolean algebras, (ii) in Remark 2.2 and Theorem 2.1, we have

Corollary 2.2. *Every* qLk_n*-algebra can be embedded into a complete one.*

Proposition 2.1. *Let B be a complete Boolean algebra and X an arbitrary nonempty set. Then $(C(n,B)^X, \sim, \varphi_1, \ldots, \varphi_{n-1}, \exists)$ is a complete* qLk_n*-algebra where $(\exists f)(x) = \bigvee f(X)$ for all $f \in C(n,B)^X$ and the other operations are defined componentwise.*

P r o o f. From Remark 2.2 we have that $C(n,B)^X$ is a complete Lk_n-algebra. On the other hand, from (ii) in Remark 2.2 it follows that $\exists$ is well-defined on $C(n,B)^X$. It is now a straightforward task to show that identities (Q0)–(Q3) hold true. $\square$

The purpose of Theorem 2.2 is to give another representation theorem for qLk_n-algebras by applying the afore mentioned results due to Halmos.

Theorem 2.2. *For every* qLk_n*-algebra A, there exists a nonempty set X and a Boolean algebra B such that A can be embedded into $C(n,\exists B)^X$ and $B(A)$ is a subalgebra of B.*

P r o o f. From (H1) we can assert that $B(A)$ is a subalgebra of a rich monadic Boolean algebra B. Let X be a set of constants of B containing at least one witness to x for each $x \in B$. Let $K = \exists B$ and let $\Theta\colon B \longrightarrow K^X$ be the mapping defined by $\Theta(z)(c) = c(z)$ for all $c \in X$. Then Θ is a monadic Boolean isomorphism between B and $\Theta(B)$ (see [10, Theorem 12]). We now consider the mapping $\Psi\colon A \longrightarrow C(n,K)^X$ defined as $(\Psi(x)(c))(i) = \Theta(\varphi_i x)(c)$ for each $x \in A$, $c \in X$ and $i \in \{1,2,\ldots,n-1\}$. Taking into account the definition of Θ and properties (L5), (L6), (L1) and (L3) we show by a routine verification that Ψ is a one-to-one morphism of bounded lattices which commutes with φ_i for all $i \in \{1,2,\ldots,n-1\}$. On the other hand, since Θ is a Boolean morphism, from (L4), (M1), (M2) and (GV2) we infer that

$$\begin{aligned}
((\Psi(\sim x))(c))(i) &= (\Theta(\varphi_i(\sim x)))(c) = (\Theta(\sim \varphi_{n-i}(x)))(c) \\
&= (\Theta((\varphi_{n-i}(x))'))(c) = (\Theta(\varphi_{n-i}(x)))'(c) \\
&= ((\Theta(\varphi_{n-i}(x)))(c))' = ((\Psi(x)(c))(n-i))' \\
&= (\sim (\Psi(x)(c)))(i) = ((\sim \Psi(x))(c))(i)
\end{aligned}$$

for all $c \in X$ and $i \in \{1,2,\ldots,n-1\}$. Therefore, $\Psi(\sim x) = \sim \Psi(x)$.

If $f \in \operatorname{Im}\Psi$ then (1) $\Psi(a) = f$ for some $a \in A$. Let g_a be the function defined by $g_a(i) = \exists\varphi_i a$ for all $i \in \{1,2,\ldots,n-1\}$. It is simple to check that $g_a \in C(n,K)$. Besides, (2) $f(c) \leqslant g_a$ for all $c \in X$. Indeed, (1) and (p2) imply that $f(c)(i) =$

$\Theta(\varphi_i a)(c) = c(\varphi_i a) \leqslant \exists\varphi_i a = g_a(i)$ for all $i \in \{1, 2, \ldots, n-1\}$. On the other hand, if $h \in C(n, K)$ is such that $f(c) \leqslant h$ for all $c \in X$, then $(f(c))(i) \leqslant h(i)$ for each $i \in \{1, 2, \ldots, n-1\}$. In particular, $f(c_{\varphi_i a})(i) \leqslant h(i)$ from which we have that $g_a \leqslant h$. From this last assertion and (2) we conclude that $g_a = \bigvee\{f(c)\colon c \in X\}$. Thus, for each $f \in \operatorname{Im}\Psi$ we define $(\exists f)(c) = \bigvee\{f(c)\colon c \in X\}$ for all $c \in X$. Finally, Ψ commutes with $\exists$. Indeed, taking into account (Q3) we have that $((\exists(\Psi(x)))(c))(i) = \exists\varphi_i x = c(\varphi_i\exists x) = \Theta(\varphi_i\exists x)(c) = ((\Psi(\exists x))(c))(i)$ for all $x \in A$, $c \in X$ and $i \in \{1, 2, \ldots, n-1\}$. □

Remark 2.3. (i) Let A be a qLk_n-algebra. Then there is no loss of generality in assuming that the Boolean algebra $K = \exists B(A)$ is complete. Hence, by Proposition 2.1 we have that $C(n, K)^X$ is a complete qLk_n-algebra and so by Theorem 2.2, A can be embedded into a complete one which is different from that obtained in Corollary 2.2.

(ii) In the particular case when a qLk_n-algebra A is such that $A = B(A)$, Theorem 2.2 coincides with Halmos's functional representation theorem cited in (H2).

(iii) The proof of the representation theorem for monadic ϑ-valued Łukasiewicz-Moisil algebras established in [4] is based on the fact that every monadic Boolean algebra can be embedded into a functional algebra B^X where B is a complete Boolean algebra and X is a nonempty set. In this proof, B and X are not explicitly described and the fact that B is complete is essential, whereas the notion of the rich algebra is fundamental for the proof of Theorem 2.2.

Next, our attention is focused on generalizing Halmos's representation theorem quoted in (H2) to certain qLk_n-algebras.

For this purpose, we extend the notion of the constant indicated in Section 1 to the case of qLk_n-algebras as follows. A constant of a qLk_n-algebra A is an Lk_n-endomorphism c on A such that $c \circ \exists = \exists$ and $\exists \circ c = c$. Clearly, from this definition we have that $c(A) = \exists A$ and therefore, $c\colon A \longrightarrow \exists A$ is an Lk_n-epimorphism such that c is the identity on the range of $\exists$.

The notions of the witness and the rich qLk_n-algebra are similar to those given for monadic Boolean algebras.

Lemma 2.1. *Let A be a rich qLk_n-algebra and X a set of constants of A containing at least one witness to a for each $a \in A$. Then the following conditions are equivalent:*

(i) $c(a) = 1$ *for all* $c \in X$,

(ii) $a = 1$.

Proof. From (i) we have that $c(\sim a) = 0$ for all $c \in X$. Then $0 = c_{\sim a}(\sim a) = \exists \sim a$ and so we conclude that $a = 1$. The converse implication is obvious. □

With this tool, the announced functional representation theorem may be now established.

Theorem 2.3. *For every rich* qLk_n*-algebra* A *there exists a nonempty set* X *such that* A *can be embedded into* $(\exists A)^X$.

Proof. Let X be a set of constants of A containing at least one witness to a for each $a \in A$. Then $X \neq \emptyset$ and by (M2) we deduce that $(\exists A)^X$ is a functional Lk_n-algebra where the operations $\wedge$, $\vee$, $\sim$ and φ_i for all $i \in \{1, 2, \ldots, n-1\}$ are defined as usual. Let $\tau\colon A \longrightarrow (\exists A)^X$ be defined by $\tau(x)(c) = c(x)$ for all $c \in X$. It is easy to prove that τ is an Lk_n-homomorphism. Besides, τ is one-to-one. Indeed, suppose that $\tau(a) = \tau(b)$, then for all $c \in X$ we have $c(a) = c(b)$ from which we obtain $c(\varphi_i(a)) = c(\varphi_i(b))$ for all $i \in \{1, 2, \ldots, n-1\}$. Therefore, for all $c \in X$, $c(\sim \varphi_i(a) \vee \varphi_i(b)) = c(\sim \varphi_i(b) \vee \varphi_i(a)) = 1$ holds for all $i \in \{1, 2, \ldots, n-1\}$ and by Lemma 2.1 we infer that $\sim \varphi_i(a) \vee \varphi_i(b) = \sim \varphi_i(b) \vee \varphi_i(a) = 1$ for all $i \in \{1, 2, \ldots, n-1\}$. This last assertion and (L6) allow us to conclude that $a = b$. On the other hand, let $f = \tau(a)$ for some $a \in A$. Thus, for all $c \in X$ we have $f(c) = c(a) \leqslant \exists a$. Furthermore, if $k \in \exists A$ is such that $f(c) \leqslant k$ for all $c \in X$, then $f(c_a) \leqslant k$ and so we obtain that $\exists a \leqslant k$. Hence, $\bigvee\{f(c)\colon c \in X\} = \exists a$.

Defining for each $f \in \operatorname{Im}\tau$, $(\exists f)(c) = \exists a$ for all $c \in X$ and setting $f = \tau(a)$, it is straightforward to prove that $\tau(\exists x) = \exists(\tau(x))$ for all $x \in A$. Therefore, $\operatorname{Im}\tau$ is a qLk_n-algebra isomorphic to A. □

Taking into account the well known results on monadic Boolean algebras and the fact that every qLk_n-algebra is a monadic Boolean one whenever $\varphi_i x = x$ for all $i \in \{1, 2, \ldots, n-1\}$, we can assert that there are qLk_n-algebras which are not rich. This last statement gives us a reason to characterize rich qLk_n-algebras.

Theorem 2.4. *Let* A *be a* qLk_n*-algebra. Then the following conditions are equivalent:*

(i) *A is rich,*

(ii) *for all $a \in A$ there is an s-filter F_a such that the natural mapping $q_a\colon A \longrightarrow A/F_a$ restricted to $\exists A$ is an Lk_n-isomorphism and $q_a(\exists a) = q_a(a)$.*

Proof. (i) $\Rightarrow$ (ii): By hypothesis for each $a \in A$ there is a witness c_a to a. Therefore, $F_a = c_a^{-1}(1)$ is an s-filter of A and the natural Lk_n-homomorphism $q_a\colon A \longrightarrow A/F_a$ restricted to $\exists A$ is one-to-one and onto. In order to prove this last assertion, it is enough to verify that each equivalence class contains a unique element of $\exists A$. It follows from the definition of c_a that for each $x \in A$ there is $k \in \exists A$ such that $c_a(x) = k$. This implies that $c_a(x) = c_a(k)$ and then $|x|_{F_a} = |k|_{F_a}$. Besides, if we assume that there is $k_1 \in \exists A$ such that $|k|_{F_a} = |k_1|_{F_a}$ we have that

$c_a(k) = c_a(k_1)$. Hence, $k = k_1$. On the other hand, $c_a(\exists a) = c_a(a)$ which allows us to conclude $q_a(\exists a) = q_a(a)$.

(ii) $\Rightarrow$ (i): Let $a \in A$. By hypothesis, $c = q_a|_{\exists A}^{-1} \circ q_a$ is an Lk_n-epimorphism from A into $\exists A$ where $q_a|_{\exists A}$ is the restriction of q_a to $\exists A$. Hence, $\exists c(x) = c(x)$ for all $x \in A$ and furthermore, $c(\exists x) = q_a|_{\exists A}^{-1}(q_a(\exists x)) = \exists x$ for all $x \in A$. Finally, since $q_a(\exists a) = q_a(a)$ we conclude that c is a witness to a. $\square$

Corollary 2.3. *$C^*_{n,X}$ is rich.*

Proof. If $g \in C^*_{n,X}$, then there exists $x_0 \in X$ such that $g(x_0) = \bigvee g(X)$. Let $k_g \in 2^X$ be defined as follows: $k_g(x) = 1$ if $x = x_0$ and $k_g(x) = 0$ otherwise. Since $k_g \in B(C^*_{n,X})$ we have that $F_g = [k_g)$ is an s-filter of $C^*_{n,X}$. In order to show that $q\colon\ C^*_{n,X} \longrightarrow C^*_{n,X}/F_g$ restricted to $\exists C^*_{n,X}$ is an Lk_n-isomorphism we only prove that $q|_{\exists C^*_{n,X}}$ is onto. Let $|h|_{F_g} \in C^*_{n,X}/F_g$ and suppose that $h(x_0) = \frac{j}{n-1}$. Then

$$h(x) \wedge k_g(x) = \begin{cases} h(x_0) & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{j}{n-1} & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases} = e_j(x) \wedge k_g(x).$$

Therefore, in each equivalence class there is an element of $\exists C^*_{n,X}$ and so, $q|_{\exists C^*_{n,X}}$ is onto. Finally, we have that

$$(\exists g)(x) \wedge k_g(x) = \begin{cases} \bigvee g(X) = g(x_0) & \text{if } x = x_0 \\ 0 & \text{otherwise} \end{cases} = g(x) \wedge k_g(x).$$

Hence, $q(\exists g) = q(g)$. $\square$

Remark 2.4. Let A be an Lk_n-algebra, $k \in A$ and let $[0, k]$ be the set $\{x \in A\colon\ 0 \leqslant x \leqslant k\}$. If $k \in B(A)$, it is easy to prove that $([0, k], -, \varphi_1, \ldots, \varphi_{n-1})$ is an Lk_n-algebra where $-x = \sim x \wedge k$. Besides, the mapping $h_k\colon\ A \longrightarrow [0, k]$ defined by $h_k(x) = x \wedge k$ is an Lk_n-epimorphism where $[k) = \{x \in A\colon\ k \leqslant x\}$ is the kernel of h_k.

Corollary 2.4. *Let A be a finite $q\mathrm{Lk}_n$-algebra. Then the following conditions are equivalent:*

(i) *A is rich,*

(ii) *for all $a \in A$, there is $k_a \in B(A)$ such that h_{k_a} restricted to $\exists A$ is an Lk_n-isomorphism and $h_{k_a}(\exists a) = h_{k_a}(a)$.*

Proof. (i) $\Rightarrow$ (ii): From the hypothesis and Theorem 2.4 we have that for each $a \in A$ there is an s-filter $F_a = [k_a)$ for some $k_a \in B(A)$. Then by Remark 2.4, the mapping $\gamma\colon\ A/[k_a) \longrightarrow [0, k_a]$ is an Lk_n-isomorphism and $h_{k_a} = \gamma \circ q_a$. From this last assertion and Theorem 2.4, we conclude (ii).

(ii) ⇒ (i): By (ii), there exists $k_a \in B(A)$ for each $a \in A$. Then by Remark 2.4, we infer that there is an Lk_n-isomorphism $\gamma\colon A/[k_a) \longrightarrow [0, k_a]$ such that $\gamma \circ q_{k_a} = h_{k_a}$ where $q_{k_a}\colon A \longrightarrow A/[k_a)$ is the natural Lk_n-homomorphism. This last equality implies that (1) $q_{k_a} = \gamma^{-1} \circ h_{k_a}$, from which using (ii) it is easy to verify that the restriction of q_{k_a} to $\exists A$ is one-to-one. Therefore, $q_{k_a}|_{\exists A}$ is an Lk_n-isomorphism. Besides, from (1) and (ii) we deduce that $q_{k_a}(\exists a) = q_{k_a}(a)$, and so in view of Theorem 2.4 we conclude that A is rich. □

Final remarks. (i) Observe that the case $n = 3$ in Theorem 2.3 coincides with the functional representation for monadic 3-valued Łukasiewicz algebras given by L. Monteiro in [19].

(ii) Here we show the relationship between Theorems 2.1, 2.2 and 2.3 obtained above in the case of rich qLk_n-algebras. For this purpose, let X_A be a set of constants of A containing at least one witness to a for each $a \in A$ and $X_{B(A)} = \{c^* = c|_{B(A)} : c \in X_A$ and c is a witness to at least one x for each $x \in B(A)\}$. By Theorem 2.3 we have that $\tau(a) = (c(a))_{c \in X_A}$ for each $a \in A$. If $\alpha^*\colon A^{X_A} \longrightarrow C(n, B(A))^{X_A}$ is the mapping defined by $\alpha^*((a_c)_{c\in X_A}) = (\alpha(a_c))_{c \in X_A}$ where α is the function defined in Theorem 2.1 and $P\colon C(n, \exists B(A))^{X_A} \to C(n, \exists B(A))^{X_{B(A)}}$ is the mapping defined by $P((f(c))_{c\in X_A}) = (f(c^*))_{c^* \in X_{B(A)}}$, then

$$\begin{aligned}
P(\alpha^*(\tau(a))) &= P(\alpha^*((c(a))_{c\in X_A})) = P((\alpha(c(a)))_{c\in X_A}) \\
&= P(((\varphi_j(c(a)))_{j\in\{1,2,\ldots,n-1\}})_{c\in X_A}) = P(((c(\varphi_j(a)))_{j\in\{1,2,\ldots,n-1\}})_{c\in X_A}) \\
&= ((c^*(\varphi_j(a)))_{j\in\{1,2,\ldots,n-1\}})_{c^*\in X_{B(A)}} = \Psi(a),
\end{aligned}$$

where Ψ is the map given in Theorem 2.2. This means that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\tau"] \arrow[ddr, "\Psi"'] & (\exists A)^{X_A} \arrow[d, "\alpha^*|_{\exists A}"] \\
& C(n, \exists B(A))^{X_A} \arrow[d, "P"] \\
& C(n, \exists B(A))^{X_{B(A)}}
\end{tikzcd}$$

The following example makes clear the relationship between the three representation theorems given above. Let us consider the rich qLk_3-algebra shown in Figure 1 where the operations are defined in Table 1.

Taking into account Theorems 2.1, 2.2 and 2.3 the mappings $\alpha\colon A \longrightarrow C(\{1,2\}, \{0,d,e,1\})$, $\Psi\colon A \longrightarrow C(\{1,2\},\{0,1\})^X$ and $\tau\colon A \longrightarrow \exists A^X$ are defined by

x	$\alpha(x)$	$\Psi(x)$	$\tau(x)$
0	$(0,0)$	$((0,0),(0,0))$	$(0,0)$
a	$(0,d)$	$((0,1),(0,0))$	$(c,0)$
b	$(0,e)$	$((0,0),(0,1))$	$(0,c)$
c	$(0,1)$	$((0,1),(0,1))$	(c,c)
d	(d,d)	$((1,1),(0,0))$	$(1,0)$
e	(e,e)	$((0,0),(1,1))$	$(0,1)$
f	$(d,1)$	$((1,1),(0,1))$	$(1,c)$
g	$(e,1)$	$((0,1),(1,1))$	$(c,1)$
1	$(1,1)$	$((1,1),(1,1))$	$(1,1)$

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Authors' addresses: *Aldo V. Figallo*, Departamento de Matemática, Universidad Nacional del Sur, 8000 Bahía Blanca, Argentina, Instituto de Ciencias Básicas, Universidad Nacional de San Juan, 5400 San Juan, Argentina, e-mail: `matfiga@criba.edu.ar`; *Claudia Sanza*, Departamento de Matemática, Universidad Nacional del Sur, 8000 Bahía Blanca, Argentina, e-mail: `csanza@criba.edu.ar`; *Alicia N. Ziliani*, Departamento de Matemática, Universidad Nacional del Sur, 8000 Bahía Blanca, Argentina, Instituto de Ciencias Básicas, Universidad Nacional de San Juan, 5400 San Juan, Argentina, e-mail: `aziliani@criba.edu.ar`.