LAPLACE EQUATION IN THE HALF-SPACE WITH A NONHOMOGENEOUS DIRICHLET BOUNDARY CONDITION

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Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. We deal with the Laplace equation in the half space. The use of a special family of weigted Sobolev spaces as a framework is at the heart of our approach. A complete class of existence, uniqueness and regularity results is obtained for inhomogeneous Dirichlet problem.

Keywords: the Laplace equation, weighted Sobolev spaces, the half space, existence, uniqueness, regularity

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1. Introduction

The purpose of this paper is to solve the problem

$$\text{(P)} \qquad \begin{cases} -\Delta u = f & \text{in } \ \mathbb{R}^N_+, \\ \quad u = g & \text{on } \ \Gamma = \mathbb{R}^{N-1}, \end{cases}$$

with the Dirichlet boundary condition on Γ. The approach is based on the use of a special class of weighted Sobolev spaces for describing the behavior at infinity. Many authors have studied the Laplace equation in the whole space $\mathbb{R}^N$ or in an exterior domain. The main difference is due to the nature of the boundary and one of difficulties is to obtain the appropriate spaces of traces. However, the half-space has a useful symmetric property.

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Problem (P) has been investigated in weighted Sobolev spaces by several authors, but only in the Hilbert cases ($p = 2$) and without the critical cases corresponding to the logarithmic factor (cf. [2], [4]). We can also mention the book by Simader, Sohr [5] where the Dirichlet problem for the Laplacian is investigated.

Let Ω be an open subset of $\mathbb{R}^N$, $N \geqslant 2$. Let $x = (x_1, \dots, x_N)$ be a typical point of $\mathbb{R}^N$ and $r = |x| = (x_1^2 + \dots + x_N^2)^{1/2}$. We use two basic weights:

$$\varrho = (1 + r^2)^{1/2} \quad \text{and} \ \lg \varrho = \ln(2 + r^2).$$

As usual, $\mathcal{D}(\mathbb{R}^N)$ denotes the spaces of indefinitely differentiable functions with a compact support and $\mathcal{D}'(\mathbb{R}^N)$ denotes its dual space, called the space of distributions. For any nonnegative integers n and m, real numbers $p > 1$, α and β, setting

$$k = k(m, N, p, \alpha) = \begin{cases} -1 & \text{if} \quad \frac{N}{p} + \alpha \notin \{1, \dots, m\}, \\ m - \frac{N}{p} - \alpha & \text{if} \quad \frac{N}{p} + \alpha \in \{1, \dots, m\}, \end{cases}$$

we define the following space:

$$\begin{aligned} W^{m,p}_{\alpha,\beta}(\Omega) = \{u \in \mathcal{D}'(\Omega);\ 0 \leqslant |\lambda| \leqslant k,\ \varrho^{\alpha - m + |\lambda|}(\lg \varrho)^{\beta - 1} D^\lambda u \in L^p(\Omega); \\ k + 1 \leqslant |\lambda| \leqslant m,\ \varrho^{\alpha - m + |\lambda|}(\lg \varrho)^{\beta} D^\lambda u \in L^p(\Omega)\}. \end{aligned} \tag{1.1}$$

In case $\beta = 0$, we simply denote the space by $W^{m,p}_{\alpha}(\Omega)$. Note that $W^{m,p}_{\alpha,\beta}(\Omega)$ is a reflexive Banach space equipped with its natural norm

$$\begin{aligned} \|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big[\sum_{0 \leqslant |\lambda| \leqslant k} \|\varrho^{\alpha - m + |\lambda|}(\lg \varrho)^{\beta - 1} D^\lambda u\|^p_{L^p(\Omega)} \\ + \sum_{k+1 \leqslant |\lambda| \leqslant m} \|\varrho^{\alpha - m + |\lambda|}(\lg \varrho)^{\beta} D^\lambda u\|^p_{L^p(\Omega)} \Big]^{1/p}. \end{aligned}$$

We also define the semi-norm

$$|u|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big(\sum_{|\lambda| = m} \|\varrho^{\alpha}(\lg \varrho)^{\beta} D^\lambda u\|^p_{L^p(\Omega)} \Big)^{1/p},$$

and for any integer q, we denote by P_q the space of polynomials in N variables of a degree smaller than or equal to q, with the convention that P_q is reduced to $\{0\}$ when q is negative. The weights defined in (1.1) are chosen so that the corresponding space satisfies two properties:

$$\mathcal{D}(\overline{\mathbb{R}^N_+}) \quad \text{is dense in} \quad W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+), \tag{1.2}$$

and the following Poincaré-type inequality holds in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$.

Theorem 1.1. *Let α and β be two real numbers and $m \geqslant 1$ an integer not satisfying simultaneously*

$$\frac{N}{p} + \alpha \in \{1, \ldots, m\} \quad \text{and} \quad (\beta - 1)p = -1. \tag{1.3}$$

Then the semi-norm $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ defines on $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/P_{q'}$ a norm which is equivalent to the quotient norm,

$$\forall u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+),\ \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/P_{q'}} \leqslant c|u|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)} \tag{1.4}$$

with $q' = \inf(q, m-1)$, where q is the highest degree of the polynomials contained in $W^{m,p}_{\alpha}(\mathbb{R}^N_+)$,

Proof. First, we construct a linear continuous extension operator such that

$$P\colon W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \to W^{m,p}_{\alpha,\beta}(\mathbb{R}^N) \tag{1.5}$$

satisfying

$$\|Pu\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N)} \leqslant \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}. \tag{1.6}$$

Since

$$\forall u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N),\ \|u\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N)/P_{q'}} \leqslant c|u|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N)} \tag{1.6}$$

holds [cf. 1], it automatically implies the statement of our theorem. $\square$

Now, we define the space

$$\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) = \overline{\mathcal{D}(\mathbb{R}^N_+)}^{\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}};$$

the dual space of $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is denoted by $W^{-m,p'}_{-\alpha,-\beta}(\mathbb{R}^N_+)$, where p' is the conjugate of p: $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 1.2. *Under the assumptions of Theorem 1.1, the semi-norm $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ is a norm on $\overset{\circ}{W}{}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ such that it is equivalent to the full norm $\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$.*

We recall now some properties of weighted Sobolev spaces $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$. We have the algebraic and topological imbeddings

$$W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \subset W^{m-1,p}_{\alpha-1,\beta}(\mathbb{R}^N_+) \subset \ldots \subset W^{0,p}_{\alpha-m,\beta}(\mathbb{R}^N_+)$$

if $\frac{N}{p}+\alpha \notin \{1,\dots,m\}$. When $\frac{N}{p}+\alpha = j \in \{1,\dots,m\}$, then we have:

$$W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \subset \ldots \subset W^{m-j+1,p}_{\alpha-j+1,\beta}(\mathbb{R}^N_+) \subset W^{m-j,p}_{\alpha-j,\beta-1}(\mathbb{R}^N_+) \subset \ldots \subset W^{0,p}_{\alpha-m,\beta-1}(\mathbb{R}^N_+).$$

Note that in the first case, the mapping $u \to \varrho^\gamma u$ is an isomorphism from $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ onto $W^{m,p}_{\alpha-\gamma,\beta}(\mathbb{R}^N_+)$ for any integer m. Moreover, in both cases and for any multi-index $\lambda \in \mathbb{N}^N$, the mapping

$$u \in W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) \to D^\lambda u \in W^{m-|\lambda|,p}_{\alpha,\beta}(\mathbb{R}^N_+)$$

is continuous.

Finally, it can be readily checked that the highest degree q of the polynomials contained in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is given by

$$q = \begin{cases} m-(\frac{N}{p}+\alpha)-1 & \text{if} \begin{cases} \frac{N}{p}+\alpha \in \{1,\dots,m\} \text{ and } (\beta-1)p \geqslant -1 \\ \frac{N}{p}+\alpha \in \{j \in Z; j \leqslant 0\} \text{ and } \beta p \geqslant -1 \end{cases} \\ [m-(\frac{N}{p}+\alpha)] & \text{otherwise,} \end{cases}$$

where $[s]$ denotes the integer part of s.

In the sequel, for any integer $q \geqslant 0$, we will use the following polynomial spaces:

— P_q (P_q^Δ) is the space of polynomials (respectively, harmonic polynomials) of degree $\leqslant q$,

— P'_q is the subspace of polynomials in P_q depending only on the $N-1$ first variables, $x' = (x_1,\dots,x_{N-1})$,

— A_q^Δ (N_q^Δ) is the subspace of polynomials P_q^Δ satisfying the condition $p(x',0)=0$ (respectively, $\frac{\partial p}{\partial x_N}(x',0)=0$) or equivalently odd with respect to x_N (even with respect to x_N), with the convention that $P_q, P_q^\Delta, P'_q, \ldots$ are reduced to $\{0\}$ when q is negative.

2. The spaces of traces

In order to define the traces of functions of $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$, we introduce for any $\sigma \in]0,1[$ the space

$$(2.1) \qquad W_0^{\sigma,p}(\mathbb{R}^N) = \Big\{u \in \mathcal{D}'(\mathbb{R}^N);\ w^{-\sigma}u \in L^p(\mathbb{R}^N), \\ \int_0^{+\infty} t^{-1-\sigma p}\,\mathrm{d}t \int_{\mathbb{R}^N} |u(x+te_i)-u(x)|^p\,\mathrm{d}x < \infty\Big\},$$

where

$$w = \begin{cases} \varrho & \text{if } \frac{N}{p} \neq \sigma, \\ \varrho(\lg \varrho)^{1/\sigma} & \text{if } \frac{N}{p} = \sigma, \end{cases}$$

and $e_1, \ldots, e_N$ is a canonical basis of $\mathbb{R}^N$. It is a reflexive Banach space equipped with its natural norm

$$\|u\|_{W_0^{\sigma,p}(\mathbb{R}^N)} = \Big(\Big\|\frac{u}{w^\sigma}\Big\|^p_{L^p(\mathbb{R}^N)} + \sum_{i=1}^N \int_0^\infty t^{-1-\sigma p}\,\mathrm{d}t \int_{\mathbb{R}^N} |u(x+te_i) - u(x)|^p\,\mathrm{d}x\Big)^{1/p}$$

which is equivalent to the norm

$$\Big(\Big\|\frac{u}{w^\sigma}\Big\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+\sigma p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p}.$$

For any $s \in \mathbb{R}^+$, we set

$$W_0^{s,p}(\mathbb{R}^N) = \Big\{u \in W^{[s],p}_{[s]-s}(\mathbb{R}^N);\ \forall|\lambda| = [s],\ D^\lambda u \in W_0^{s-[s],p}(\mathbb{R}^N)\Big\}. \tag{2.2}$$

It is a reflexive Banach space equipped with the norm

$$\|u\|_{W_0^{s,p}(\mathbb{R}^N)} = \|u\|_{W^{[s],p}_{[s]-s}(\mathbb{R}^N)} + \sum_{|\lambda|=s} \|D^\lambda u\|_{W_0^{s-[s],p}(\mathbb{R}^N)}.$$

We notice that this definition and the next one coincide with the definition in the first section when $s = m$ is a nonnegative integer. For any $s \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}$, we then set

$$W_\alpha^{s,p}(\mathbb{R}^N) = \Big\{u \in W^{[s],p}_{[s]+\alpha-s}(\mathbb{R}^N),\ \forall|\lambda| = [s],\ \varrho^\alpha D^\lambda u \in W_0^{s-[s],p}(\mathbb{R}^N)\Big\}. \tag{2.3}$$

Finally, for any integer $m \geqslant 1$, we define the space

$$\begin{aligned} X_0^{m,p}(\mathbb{R}^N_+) = \Big\{&u \in \mathcal{D}'(\mathbb{R}^N_+);\ 0 \leqslant |\lambda| \leqslant k,\ \varrho'^{|\lambda|-m}(\lg\varrho')^{-1}D^\lambda u \in L^p(\mathbb{R}^N_+),\\ &k+1 \leqslant |\lambda| \leqslant m,\ \varrho'^{|\lambda|-m}D^\lambda u \in L^p(\mathbb{R}^N_+)\Big\} \end{aligned} \tag{2.4}$$

with $\varrho' = (1+|x'|^2)^{1/2}$ and $\lg\varrho' = \ln(2+|x'|^2)$. It is a reflexive Banach space. We can prove that

$$\mathcal{D}(\overline{\mathbb{R}^N_+}) \quad \text{is dense in} \quad X_0^{m,p}(\mathbb{R}^N_+).$$

We observe that the functions from $X_0^{m,p}(\mathbb{R}^N_+)$ and $W_0^{m,p}(\mathbb{R}^N_+)$ have the same traces on $\Gamma = \mathbb{R}^{N-1}$ (see below). If u is a function, we denote its traces on $\Gamma = \mathbb{R}^{N-1}$ by $x' \in \mathbb{R}^{N-1}$, $\gamma_0 u(x') = u(x',0), \ldots, \gamma_j u(x') = \frac{\partial^j u}{\partial x_N^j}(x',0)$.

As in [3], we can prove the following trace lemma:

Lemma 2.1. *For any integer $m \geqslant 1$ and real number α, the mapping*

$$\gamma\colon\ \mathcal{D}(\overline{\mathbb{R}^N_+}) \to \prod_{j=0}^{m-1} \mathcal{D}(\mathbb{R}^{N-1})$$
$$u \mapsto (\gamma_0 u, \ldots, \gamma_{m-1} u)$$

can be extended by continuity to a linear and continuous mapping still denoted by γ from $W^{m,p}_{\alpha}(\mathbb{R}^N_+)$ to $\prod\limits_{j=0}^{m-1} W_{\alpha}^{m-j-\frac{1}{p},p}(\mathbb{R}^{N-1})$. Moreover, γ is onto and

$$\operatorname{Ker} \gamma = \mathring{W}^{m,p}_{\alpha}(\mathbb{R}^N_+).$$

3. The Laplace equation

The aim of this section is to study the problem (P):

$$\text{(P)} \qquad \begin{cases} -\Delta u = f & \text{in } \ \mathbb{R}^N_+, \\ u = g & \text{in } \ \Gamma = \mathbb{R}^{N-1}. \end{cases}$$

Theorem 3.1. *Let $\ell \geqslant 0$ be an integer and assume that*

$$\frac{N}{p'} \notin \{1, \ldots, \ell\} \tag{3.1}$$

with the convention that this set is empty if $\ell = 0$. For any f in $W^{-1,p}_{\ell}(\mathbb{R}^N_+)$ and g in $W_{\ell}^{\frac{1}{p'},p}(\Gamma)$ satisfying the compatibility condition

$$\forall \varphi \in A^{\Delta}{}_{[\ell+1-\frac{N}{p'}]},\ \langle f, \varphi \rangle_{W^{-1,p}_{\ell} \times W^{1,p'}_{-\ell}} = \Big\langle g, \frac{\partial \varphi}{\partial \gamma_N} \Big\rangle_{\Gamma} \tag{3.2}$$

where $\langle \cdot, \cdot \rangle_{\Gamma}$ denotes the duality between $W_{\ell}^{\frac{1}{p'},p}(\Gamma)$ and $W_{-\ell}^{-\frac{1}{p'},p'}(\Gamma)$, problem (P) has a unique solution $u \in W^{1,p}_{\ell}(\mathbb{R}^N_+)$ and there exists a constant C independent of u, f and g such that

$$\|u\|_{W^{1,p}_{\ell}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{-1,p}_{\ell}(\mathbb{R}^N_+)} + \|g\|_{W_{\ell}^{\frac{1}{p'},p}(\Gamma)}). \tag{3.3}$$

P r o o f. First, the kernel of the operator

$$(-\Delta, \gamma_0)\colon\ W^{1,p}_{\ell}(\mathbb{R}^N_+) \to W^{-1,p}_{\ell}(\mathbb{R}^N_+) \times W_{\ell}^{\frac{1}{p'},p}(\Gamma)$$

is precisely the space $A^{\Delta}_{[\ell+1-N/p']}$ for any integer ℓ and $A^{\Delta}_{[\ell+1-\frac{N}{p'}]}$ is reduced to $\{0\}$ when $\ell \geqslant 0$. Thanks to Lemma 2.1, let $u_g \in W^{1,p}_{\ell}(\mathbb{R}^N_+)$ be the lifting function of g such that

$$u_g = g \text{ on } \Gamma \text{ and } \|u_g\|_{W^{1,p}_{\ell}(\mathbb{R}^N_+)} \leqslant C_1 \|g\|_{W^{\frac{1}{p'},p}_{\ell}(\Gamma)}.$$

Then problem (P) is equivalent to

$$\begin{cases} -\Delta v = f + \Delta u_g & \text{in} \quad \mathbb{R}^N_+, \\ v = 0 & \text{on} \quad \Gamma. \end{cases} \tag{3.4}$$

Set $h = f + \Delta u_g$. For any $\varphi \in W^{1,p'}_{-\ell}(\mathbb{R}^N)$ set

$$\sqcap\varphi(x', x_N) = \varphi(x', x_N) - \varphi(x', -x_N) \quad \text{if} \quad x_N > 0.$$

It is clear that $\sqcap\varphi \in \overset{\circ}{W}{}^{1,p'}_{-\ell}(\mathbb{R}^N_+)$. Then h can be extended to $h_\pi \in W^{-1,p}_{\ell}(\mathbb{R}^N)$ defined by

$$\varphi \in W^{1,p'}_{-\ell}(\mathbb{R}^N), \ h_\pi(\varphi) = \langle h, \sqcap\varphi \rangle_{W^{-1,p}_{\ell}(\mathbb{R}^N_+) \times W^{1,p'}_{-\ell}(\mathbb{R}^N_+)}.$$

Moreover,

$$\|h_\pi\|_{W^{-1,p}_{\ell}(\mathbb{R}^N)} = \|h\|_{W^{-1,p}_{\ell}(\mathbb{R}^N_+)}.$$

Let q be a polynomial in $P^{\Delta}_{[\ell+1-N/p']}$. We can write it in the form

$$q = r + s, \ r \in A^{\Delta}_{[\ell+1-N/p']} \text{ and } s \in N^{\Delta}_{[\ell+1-N/p]}.$$

Then,

$$\langle h_\pi, q \rangle = \langle f + \Delta u_g, r \rangle_{W^{-1,p}_{\ell}(\mathbb{R}^N_+) \times W^{1,p'}_{-\ell}(\mathbb{R}^N_+)}$$

and applying the Green formula we get

$$\begin{aligned} \langle \Delta u_g, r \rangle &= -\int_{\mathbb{R}^N_+} \nabla u_g \cdot \nabla r \, \mathrm{d}x \\ &= -\Big\langle g, \frac{\partial r}{\partial x_N} \Big\rangle_{W^{\frac{1}{p'},p}_{\ell}(\Gamma) \times W^{-\frac{1}{p'},p'}_{-\ell}(\Gamma)} \end{aligned}$$

(note that $\Delta r = 0$ in $\mathbb{R}^N_+$ and $r = 0$ on Γ). Thus, $h_\pi \in W^{-1,p}_{\ell}(\mathbb{R}^N)$ and it satisfies

$$\forall q \in P^{\Delta}_{[\ell+1-N/p']}, \ \langle h_\pi, q \rangle = 0.$$

Recall that (cf. [1]) since (3.1) holds, the operators

$$\Delta\colon \ W^{1,p}_{\ell}(\mathbb{R}^N) \to W^{-1,p}_{\ell} \perp P^{\Delta}_{[\ell+1-\frac{N}{p'}]} \text{ if } \ell \geqslant 1,$$

$$\Delta\colon \ W^{1,p}_{0}(\mathbb{R}^N)/P_{[1-\frac{N}{p}]} \to W^{-1,p}_{0}(\mathbb{R}^N) \perp P_{[1-\frac{N}{p'}]} \text{ if } \ell = 0$$

are isomorphisms. Hence, there exists $\tilde{v}$ in $W^{1,p}_{\ell}(\mathbb{R}^N)$ such that $-\Delta\tilde{v} = h_\pi$. Now we remark that the function $w = \frac{1}{2}\sqcap\tilde{v}$ belongs to $W^{1,p}_{\ell}(\mathbb{R}^N_+)$ and

$$-\Delta w = h \quad \text{in} \quad \mathbb{R}^N_+ \quad \text{and} \quad w = 0 \quad \text{on} \quad \Gamma,$$

i.e. w is a solution of (3.4). $\square$

R e m a r k. The kernel $A^{\Delta}_{[-\ell+1-N/p]}$ is reduced to $\{0\}$ if $\ell \geqslant 0$ and to $P_{[1-N/p]}$ if $\ell = 0$.

With similar arguments, we can prove the following theorem:

Theorem 3.2. *Let $\ell \geqslant 1$ be an integer and assume that*

$$\frac{N}{p} \notin \{1, \ldots, -\ell\}. \tag{3.5}$$

Then for any f in $W^{-1,p}_{-\ell}(\mathbb{R}^N_+)$ and g in $W^{\frac{1}{p'},p}_{-\ell}(\Gamma)$, problem (P) has a unique solution $u \in W^{1,p}_{-\ell}(\mathbb{R}^N_+)/A^{\Delta}_{[\ell+1-N/p]}$ and there exists a constant C independent of u, f and g such that

$$\inf_{q\in A^{\Delta}_{[\ell+1-\frac{N}{p}]}} \|u+q\|_{W^{1,p}_{-\ell}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{-1,p}_{-\ell}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'},p}_{-\ell}(\Gamma)}).$$

Theorem 3.3. *Let m be a nonnegative integer, let g belong to $W^{\frac{1}{p'}+m,p}_{m}(\Gamma)$ and assume that*

$$f \in W^{-1+m,p}_{m}(\mathbb{R}^N_+) \text{ if } \frac{N}{p'} \neq 1 \text{ or } m = 0, \tag{3.6}$$

or

$$f \in W^{-1+m,p}_{m}(\mathbb{R}^N_+) \cap W^{-1,p}_{0}(\mathbb{R}^N_+) \text{ if } \frac{N}{p'} = 1 \text{ and } m \neq 0. \tag{3.7}$$

Then problem (P) has a unique solution $u \in W^{1+m,p}_{m}(\mathbb{R}^N_+)$ and u satisfies

$$\|u\|_{W^{m+1,p}_{m}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{-1+m,p}_{m}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m}(\Gamma)}) \text{ if } \frac{N}{p'} \neq 1 \text{ or } m = 0$$

and

$$\|u\|_{W^{m+1,p}_{m}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{1,p}_{0}(\mathbb{R}^N_+)} + \|f\|_{W^{-1+m,p}_{m}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m}(\Gamma)}) \quad \text{if } \frac{N}{p'} = 1 \text{ and } m \neq 0.$$

P r o o f. First, we observe that for any integer $m \geqslant 0$ we have the inclusion

$$W_m^{-1+m,p}(\mathbb{R}^N_+) \subset W_0^{-1,p}(\mathbb{R}^N_+)$$

if $\frac{N}{p'} \neq 1$ or $m = 0$. Thus, under the assumptions (3.6) or (3.7) and thanks to Theorem 3.1, there exists a unique solution $u \in W_0^{1,p}(\mathbb{R}^N_+)$ of problem (P). Let us prove by induction that

$$g \in W_m^{\frac{1}{p'}+m,p}(\Gamma) \text{ and } f \text{ satisfies (3.6) or (3.7)} \Longrightarrow u \in W_m^{m+1,p}(\mathbb{R}^N_+). \tag{3.8}$$

For $m = 0$, (3.8) is valid. Assume that (3.8) is valid for $0, 1, \ldots, m$ and suppose that $g \in W_{m+1}^{\frac{1}{p'}+m+1,p}(\Gamma)$ and $f \in W_{m+1}^{m,p}(\mathbb{R}^N_+)$ with $\frac{N}{p'} \neq 1$ (a similar argument can be used for f satisfying (3.7)). Let us prove that $u \in W_{m+1}^{m+2,p}(\mathbb{R}^N_+)$. We observe first that

$$W_{m+1}^{m,p}(\mathbb{R}^N_+) \subset W_m^{m-1,p}(\mathbb{R}^N_+) \text{ and } W_{m+1}^{\frac{1}{p'}+m+1,p}(\Gamma) \subset W_m^{\frac{1}{p'}+m,p}(\Gamma),$$

hence u belongs to $W_m^{m+1,p}(\mathbb{R}^N_+)$ thanks to the induction hypothesis. Now, for $i = 1, \ldots, N-1$,

$$\Delta(\varrho\partial_i u) = \varrho\partial_i f + \frac{2}{\varrho} r \cdot \nabla(\partial_i u) + \Bigl(\frac{2}{\varrho} + \frac{1}{\varrho^3}\Bigr)\partial_i u.$$

Thus, $\Delta(\varrho\partial_i u) \in W_m^{m-1,p}(\mathbb{R}^N_+)$ and $\gamma_0(\varrho\partial_i u) \in W_m^{m+1,p}(\mathbb{R}^{N-1})$. Applying the induction hypothesis, we can deduce that

$$\partial_i u \in W_{m+1}^{m+1,p}(\mathbb{R}^N_+) \text{ for } i = 1, \ldots, N-1.$$

It remains to prove that $v = \partial_N u \in W_{m+1}^{m+1,p}(\mathbb{R}^N_+)$. This is a consequence of the fact that v belongs to $W_m^{m,p}(\mathbb{R}^N_+)$ and

$$\partial_i\partial_N u = \partial_N\partial_i u \in W_{m+1}^{m,p}(\mathbb{R}^N_+), \; i = 1, \ldots, N-1,$$
$$\partial_N(\partial_N u) = \Delta u - \sum_{i=1}^{N-1} \partial_i^2 u \in W_{m+1}^{m,p}(\mathbb{R}^N_+).$$

We can conlude that $u \in W_{m+1}^{m+2,p}(\mathbb{R}^N_+)$. $\square$

Corollary 3.4. *Let $\ell \geqslant 1$ and $m \geqslant 1$ be two integers.*
(i) *Under the assumption*

$$\frac{N}{p'} \notin \{1, \ldots, \ell + 1\},$$

for any $f \in W^{m-1,p}_{m+\ell}(\mathbb{R}^N_+)$ *and* $g \in W^{\frac{1}{p'}+m,p}_{m+\ell}(\Gamma)$ *satisfying the compatibility condition* (3.2) *there exists a unique solution* $u \in W^{m+1,p}_{m+\ell}(\mathbb{R}^N_+)$ *of* (P) *and* u *satisfies*

$$\|u\|_{W^{m+1,p}_{m+\ell}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{m-1,p}_{m+\ell}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m+\ell}(\Gamma)})$$

where $C = C(m,p,\ell,N)$ *is a constant independent of* u, f *and* g.

(ii) *Under the assumption*

$$m \geqslant \ell \quad \text{or} \quad \frac{N}{p} \notin \{1,\dots,\ell-m\},$$

for any $f \in W^{m-1,p}_{m-\ell}(\mathbb{R}^N_+)$ *and* $g \in W^{\frac{1}{p'}+m,p}_{m-\ell}(\Gamma)$ *there exists a unique solution* $u \in W^{m+1,p}_{m-\ell}(\mathbb{R}^N_+)/A^{\Delta}_{[1+\ell-N/p]}$ *of* (P) *and* u *satisfies*

$$\inf_{q \in A^{\Delta}_{[1+\ell-N/p]}} \|u+q\|_{W^{m+1,p}_{m-\ell}(\mathbb{R}^N_+)} \leqslant C(\|f\|_{W^{m-1,p}_{m-\ell}(\mathbb{R}^N_+)} + \|g\|_{W^{\frac{1}{p'}+m,p}_{m-\ell}(\Gamma)}).$$

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