EXISTENCE OF NONOSCILLATORY SOLUTIONS OF A CLASS OF NONLINEAR DIFFERENCE EQUATIONS WITH A FORCED TERM

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Abstract. In this paper, necessary and sufficient conditions for the existence of nonoscillatory solutions of the forced nonlinear difference equation

$$\Delta(x_n - p_n x_{\tau(n)}) + f(n, x_{\sigma(n)}) = q_n$$

are obtained. Examples are included to illustrate the results.

Keywords: difference equations, nonlinear, forced term, nonoscillation

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1. Introduction

In this paper, we consider the nonlinear difference equation with a forced term

$$\Delta(x_n - p_n x_{\tau(n)}) + f(n, x_{\sigma(n)}) = q_n,\ n = 1, 2, 3, \ldots, \tag{1}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $f\colon \mathbb{N}\times\mathbb{R} \to \mathbb{R}$ is a continuous function, and $\tau, \sigma\colon \mathbb{N} \to \mathbb{N}$ with $\lim\limits_{n\to\infty} \sigma(n) = +\infty$, $\lim\limits_{n\to\infty} \tau(n) = +\infty$; $\{p_n\}$, $\{q_n\}$ are real sequences. A solution of (1) is a real sequence x_n defined for all $n \geqslant \min\{N_0, \min\limits_{n\geqslant N_0} \sigma(n), \min\limits_{n\geqslant N_0} \tau(n)\}$ and satisfying (1) for all $n \geqslant N_0$. A nontrivial solution $\{x_n\}$ of (1) is said to be *oscillatory* if for any $N \geqslant N_0$ there exists $n > N$ such that $x_{n+1}x_n \leqslant 0$. Otherwise, the solution is said to be *nonoscillatory*.

Difference equations of neutral type have been studied by a number of authors in recent years, for example, see [2–11,13] and the references contained therein. Various

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authors have obtained results guaranteeing the oscillation of equation (1), and we cite the papers [2, 6–8]. In this paper we are interested in obtaining necessary and sufficient conditions for the existence of nonoscillatory solutions of (1).

2. Main results

Let X denote the Banach space l_∞^N of all bounded real sequences $x = \{x_n\}$, $n \geqslant N$, with the norm $\|x\| = \sup\limits_{n \geqslant N} |x_n|$. We will use the following assumptions:

(i) $|f(n,x)| \leqslant |f(n,y)|$, provided $|x| \leqslant |y|$;

(ii) for each closed interval $L = [d_1, d_2]$ $(0 < d_1 < d_2)$, there exists $L(n)$ such that

$$|f(n,x) - f(n,y)| \leqslant L(n)|x-y|, \; x, y \in L,$$

and $\sum\limits_{i=N}^{\infty} L(i) < \infty$;

(iii) $xf(n,x) \geqslant 0 \quad (x \neq 0)$;

(iv) $\sum\limits_{i=N}^{\infty} |q_i| < \infty$;

(v) there exists $r \in (0,1)$ such that

$$0 \leqslant p_n \leqslant 1 - r, \; n \geqslant N;$$

(vi) there exists $r \in (0,1)$ such that

$$r - 1 \leqslant p_n \leqslant 0, \; n \geqslant N;$$

(vii) $|p_n| \leqslant 1 - r$, $n \geqslant N$, $r \in (\frac{1}{2}, 1)$;

(viii) $p_n \equiv 1$.

Theorem 1. *Suppose that* (i), (ii) *and* (iv) *hold. Further suppose that either* (v) *or* (vi) *holds. If*

$$\sum_{n=N}^{\infty} |f(n,d)| < \infty \text{ for some } d \neq 0, \tag{2}$$

then Eq. (1) *has a bounded nonoscillatory solution* $\{x_n\}$ *such that* $\liminf\limits_{n\to\infty} |x_n| > 0$.

P r o o f. Define a subset Ω of X as follows:

$$\Omega = \{\{x_n\} \subset X\colon \; d_1 \leqslant x_n \leqslant |d|, \; n \geqslant N\}$$

and an operator T on Ω:

$$Tx_n = \begin{cases} c_1 + p_n x_{\tau(n)} + \sum\limits_{i=n}^{\infty} f(i, x_{\sigma(i)}) - \sum\limits_{i=n}^{\infty} q_i, & n \geqslant N_1, \\ Tx_{N_1}, & N \leqslant n < N_1, \end{cases}$$

where $0 < d_1 < r|d|$, c_1 and N_1 satisfy the following conditions: If (v) holds, $d_1 < c_1 < r|d|$ and N_1 is sufficiently large such that $\tau(n) \geqslant N$, $\sigma(n) \geqslant N$ as $n \geqslant N_1 \geqslant N$ and

$$\sum_{n=N_1}^{\infty} f(n, d) + \sum_{n=N_1}^{\infty} |q_n| \leqslant \min\{c_1 - d_1, r|d| - c_1\}$$

and

$$\sum_{i=N_1}^{\infty} L(i) \leqslant \frac{r}{2}.$$

If (vi) holds, $d_1 + (1 - r)|d| < c_1 < \frac{1}{2}(d_1 + (2 - r)|d|)$, N_1 is sufficiently large such that $\tau(n) \geqslant N$, $\sigma(n) \geqslant N$ as $n \geqslant N_1$ and

$$\sum_{n=N_1}^{\infty} |f(n, d)| + \sum_{n=N_1}^{\infty} |q_n| \leqslant c_1 - d_1 - (1 - r)|d|.$$

First, we claim that $T\Omega \subset \Omega$.

If (v) holds, then for any $x \in \Omega$, $n \geqslant N_1$ we have

$$\begin{aligned} Tx_n &= c_1 + p_n x_{\tau(n)} + \sum_{i=n}^{\infty} f(i, x_{\sigma(i)}) - \sum_{i=n}^{\infty} q_i \\ &\geqslant c_1 - \sum_{i=N_1}^{\infty} |f(i, x_{\sigma(i)})| - \sum_{i=N_1}^{\infty} |q_i| \\ &\geqslant c_1 - (c_1 - d_1) = d_1 \end{aligned}$$

and

$$Tx_n \leqslant c_1 + p_n|d| + \sum_{i=N_1}^{\infty} |f(i, x_{\sigma(i)})| + \sum_{i=N_1}^{\infty} |q_i| \leqslant c_1 + (1 - r)|d| + (r|d| - c_1) = |d|.$$

If (vi) holds, then for $n \geqslant N_1$ we have

$$\begin{aligned} Tx_n &\geqslant c_1 + |d|p_n - \sum_{i=N_1}^{\infty} |f(i, d)| - \sum_{i=N_1}^{\infty} |q_i| \\ &\geqslant c_1 - (1 - r)|d| - (c_1 - d_1 - (1 - r)|d|) = d_1 \end{aligned}$$

and

$$
\begin{aligned}
Tx_n &\leqslant c_1 + \sum_{i=N_1}^{\infty} |f(i,d)| + \sum_{i=N_1}^{\infty} |q_i| \\
&\leqslant c_1 + c_1 - d_1 - (1-r)|d| \\
&< d_1 + (2-r)|d| - d_1 - (1-r)|d| \\
&= |d|.
\end{aligned}
$$

Therefore $T\Omega \subset \Omega$.

Next, we claim that T is a compression mapping on Ω. In fact, for $x, y \in \Omega$, $n \geqslant N_1$, we have

$$
\begin{aligned}
|Tx_n - Ty_n| &= |p_n(x_{\tau(n)} - y_{\tau(n)}) + \sum_{i=N_1}^{\infty} (f(i, x_{\sigma(i)}) - f(i, y_{\sigma(i)}))| \\
&\leqslant |p_n| \sup_{n \geqslant N} |x_n - y_n| + \sum_{i=N_1}^{\infty} L(i)|x_{\sigma(i)} - y_{\sigma(i)}| \\
&\leqslant \left(|p_n| + \sum_{i=N_1}^{\infty} L(i)\right) \sup_{n \geqslant N} |x_n - y_n| \\
&\leqslant \left(1 - r + \frac{r}{2}\right) \|x - y\| \\
&= \left(1 - \frac{r}{2}\right) \|x - y\|,
\end{aligned}
$$

which implies that

$$
\|Tx - Ty\| \leqslant \left(1 - \frac{r}{2}\right) \|x - y\|.
$$

By the Banach fixed point theorem, T has a fixed point $\bar{x} = \{\bar{x}_n\} \in \Omega$. Obviously, $\bar{x}$ is a bounded nonoscillatory solution of (1) with $\liminf\limits_{n\to\infty} |\bar{x}_n| \geqslant d_1 > 0$. The proof is complete. $\square$

The following lemmas show the necessity of condition (2) for the existence of a nonoscillatory solution $\{x_n\}$ with $\liminf\limits_{n\to\infty} |x_n| > 0$.

Lemma 1. *Assume that* (i), (iii), (iv) *and* (vi) *hold. If* (1) *has a nonoscillatory solution* $\{x_n\}$ *with* $\liminf\limits_{n\to\infty} |x_n| > 0$, *then* (2) *holds.*

Proof. Without loss of generality, assume that $x_n > d > 0$, $n \geqslant N$. Let $y_n = x_n - p_n x_{\tau(n)} > 0$. Then

$$
\Delta y_n = q_n - f(n, x_{\sigma(n)}).
$$

If (2) does not hold, summing the last equation we obtain

$$y_n - y_{N_1} \leqslant \sum_{i=N_1}^{n-1} q_i - \sum_{i=N_1}^{n-1} f(i, x_{\sigma(i)}) \leqslant \sum_{i=N_1}^{n-1} |q_i| - \sum_{i=N_1}^{n-1} f(i, d) \to -\infty,\ n \to \infty.$$

Then $\lim_{n\to\infty} y_n = -\infty$, a contradiction. The proof is complete. □

Lemma 2. *Assume that* (i), (iii), (iv), (v) *and* $\tau(n) \leqslant n$, $n \geqslant N$ *hold. Then the conclusion of Lemma* 1 *is true.*

Proof. Assume that $x_n \geqslant d > 0$, $n \geqslant N$ is a positive solution of (1). If (2) does not hold, as in the proof of Lemma 1, we have $\lim_{n\to\infty} y_n = -\infty$. Then x_n is unbounded. Therefore there exists a sequence $\{n_k\}$ with $\lim_{k\to\infty} n_k = \infty$ such that $x_{n_k} = \max_{n\leqslant n_k} x_n$. Then

$$y_{n_k} = x_{n_k} - p_{n_k} x_{\tau(n_k)} \geqslant (1 - p_{n_k}) x_{n_k} > 0,$$

a contradiction. The proof is complete. □

Combining the above results we obtain

Theorem 2. *Assume that* (i), (ii), (iii), (iv) *and* (vi) *hold. Then* (2) *is a necessary and sufficient condition for* (1) *to have a nonoscillatory solution* $\{x_n\}$ *with* $\liminf_{n\to\infty} |x_n| > 0$.

Theorem 3. *Assume that* (i), (ii), (iii), (iv), (v) *and* $\tau(n) \leqslant n$, $n \geqslant N$ *hold. Then* (2) *is a necessary and sufficient condition for* (1) *to have a nonoscillatory solution* $\{x_n\}$ *with* $\liminf_{n\to\infty} |x_n| > 0$.

Now we consider the case that p_n is oscillatory in (1).

Theorem 4. *Assume that* (i), (ii), (iv), (vii) *and* (2) *hold. Then* (1) *has a bounded nonoscillatory solution* $\{x_n\}$ *with* $\liminf_{n\to\infty} |x_n| > 0$.

Proof. Let $\Omega = \{\{x_n\} \in X\colon\ d_1 \leqslant x_n \leqslant |d|, n \geqslant N\}$, where $0 < d_1 < (2r-1)|d|$. Define an operator T by (3), where c_1 satisfies $d_1 + (1-r)|d| < c_1 < r|d|$ and N_1 is sufficiently large such that when $n \geqslant N_1 \geqslant N, \tau(n) \geqslant N, \sigma(n) \geqslant N$ and

$$\sum_{i=N_1}^{\infty} |f(i, d)| + \sum_{i=N_1}^{\infty} |q_i| \leqslant \min\{c_1 - d_1 - (1-r)|d|, r|d| - c_1\}$$

and

$$\sum_{i=N_1}^{\infty} L(i) \leqslant \frac{r}{2}.$$

The rest of the proof is similar to that of Theorem 1. □

Similarly to Lemma 2 we can prove the following assertion.

Lemma 3. *Assume that* (i), (iii), (iv), (vii) *and* $\tau(n) \leqslant n$, $n \geqslant N$ *hold. Then the conclusion of Lemma* 1 *is true.*

Combining Theorem 4 and Lemma 3 we obtain

Theorem 5. *Assume that* (i), (ii), (iii), (iv), (vii) *and* $\tau(n) \leqslant n$, $n \geqslant N$ *hold. Then* (2) *is a necessary and sufficient condition for* (1) *to have a nonoscillatory solution* $\{x_n\}$ *with* $\liminf\limits_{n\to\infty} |x_n| > 0$.

Remark 1. Theorems 1–5 are discrete analogues of the corresponding results for the neutral differential equation [12].

Finally, we consider the case (viii).

Theorem 6. *Assume that* (i) *and* (viii) *hold. Further assume that* $\tau(n)$ *is increasing,* $\tau(n) < n$ *for all large* n, *and*

$$\sum_{n=N}^{\infty} n|f(n,d)| < \infty \text{ for some } d \neq 0$$

and

$$\sum_{n=N}^{\infty} n|q_n| < \infty. \tag{3}$$

Then (1) *has a bounded nonoscillatory solution.*

Let

$$\tau^0(n_0) = n_0,\ \tau^{n+1}(n_0) = \tau(\tau^n(n_0)),\ n = 0, 1, 2, \ldots,$$
$$\tau^{n-1}(n_0) = \tau^{-1}(\tau^n(n_0)),\ n = 0, -1, -2, \ldots.$$

By a known result [13, *Lemma* 2.3], (3) *is equivalent to*

$$\sum_{j=0}^{\infty} \sum_{n=\tau^{-j}(n_0)}^{\infty} |f(n,d)| < \infty$$

and

$$\sum_{j=0}^{\infty} \sum_{n=\tau^{-j}(n_0)}^{\infty} |q_n| < \infty. \tag{4}$$

Proof of Theorem 6. In view of (4), we can choose a sufficiently large n_0 such that

$$\sum_{j=0}^{\infty} \sum_{n=\tau^{-j}(n_0)}^{\infty} |f(n,d)| \leqslant \frac{1}{2}$$

and

$$\sum_{j=0}^{\infty} \sum_{n=\tau^{-j}(n_0)}^{\infty} |q_n| \leqslant \frac{1}{2}. \tag{5}$$

Define

$$H_n = \begin{cases} \sum_{i=n}^{\infty} |f(i,d)| + \sum_{i=n}^{\infty} |q_i|, & t \geqslant n_0, \\ \frac{n-\tau(n_0)}{n_0-\tau(n_0)} H(n_0), & \tau(n_0) \leqslant n \leqslant n_0, \\ 0, & n \leqslant \tau(n_0). \end{cases}$$

Clearly, $H_n\colon N \to R$. Define

$$y_n = \sum_{m=0}^{\infty} H_{\tau^m(n)}, \ n \geqslant n_0. \tag{6}$$

It is easy to see that $y_n - y_{\tau(n)} = H_n, \ n \geqslant \tau^{-1}(n_0)$ and

$$0 < y_n \leqslant 1, \ n \geqslant n_0. \tag{7}$$

Define a set $\Omega \subset X$ by

$$\Omega = \{\{x_n\} \subset X\colon 0 \leqslant x_n \leqslant y_n, \ n \geqslant n_0\}$$

and an operator S on Ω by

$$Sx_n = \begin{cases} x_{\tau(n)} + \sum_{i=n}^{\infty} f(i, x_{\sigma(i)}) - \sum_{i=n}^{\infty} q_i, & n \geqslant n_0, \\ \frac{Sx_{n_0} n y_n}{n_0 y_{n_0}} + y_n\Big(1 - \frac{n}{n_0}\Big), & \tau(n_0) \leqslant n \leqslant n_0. \end{cases}$$

By (5)–(7), $S\Omega \subset \Omega$.

Define a sequence of sequences $\{x_n^k\}_{k=0}^{\infty}$ as follows:

$$x_n^0 = y_n, \ \ x_n^k = Sx_n^{k-1}, \ n \geqslant n_0, \ k = 1, 2, \ldots.$$

By induction, we can prove that

$$y_n = x_n^0 \geqslant x_n^1 \geqslant \ldots, \ n \geqslant n_0.$$

Then there exists a sequence $\{u_n\} \in \Omega$ such that $\lim\limits_{k\to\infty} x_n^k = u_n$ and $u_n > 0$ for $n \geqslant n_0, u_n = Su_n$, i.e.,

$$u_n = u_{\tau(n)} + \sum_{i=n}^{\infty} f(i, u_{\sigma(n)}) - \sum_{i=n}^{\infty} q_i.$$

Hence

$$\Delta(u_n - u_{\tau(n)}) + f(n, u_{\sigma(n)}) = q_n.$$

The proof is complete.

Remark 2. We can establish a result similar to Theorem 6 for the neutral differential equation

$$(x(t) - x(\tau(t)))' + f(t, x(\sigma(t))) = q(t).$$

Example 1. Consider the equation

$$\Delta(x_n - \tfrac{3}{5}x_{\tau(n)}) + n^{-2}x^3_{\sigma(n)} = \mathrm{e}^{-n}, \ n \geqslant N \tag{8}$$

where $\lim\limits_{n\to\infty} \tau(n) = \infty$, $\lim\limits_{n\to\infty} \sigma(n) = \infty, q_n = \mathrm{e}^{-n}$ and $f(n,x) = n^{-2}x^3$.

For $x, y \in L = [d_1, d_2]$ $(0 < d_1 < d_2)$ and $d > d_2$ we have

$$|f(n,x) - f(n,y)| = n^{-2}|x^2 + xy + y^2||x - y| \leqslant 3d^2n^{-2}|x - y|.$$

Let $L(n) = 3d^2n^{-2}$. Then $\sum\limits_{i=N}^{\infty} L(i) < \infty$ and $\sum\limits_{i=N}^{\infty} |f(i,d)| = \sum\limits_{i=N}^{\infty} |d|^3 i^{-2} < \infty$. By Theorem 1, (8) has a nonoscillatory solution $\{x_n\}$ with $\liminf\limits_{n\to\infty} |x_n| > 0$.

Example 2. Consider the equation

$$\Delta(x_n - (-\tfrac{1}{3})^n x_{n-1}) + (-\tfrac{1}{3})^{n-1}\tfrac{4n^2-2n+1}{3(n-1)^2(n^2+1)}x_n = \tfrac{4}{3}(-\tfrac{1}{3})^n - \tfrac{2n+1}{n^2(n+1)^2}, \tag{9}$$

where $p_n = (-\frac{1}{3})^n$ is oscillatory and satisfies (vii), $f(n,x) = (-\frac{1}{3})^{n-1}\frac{4n^2-2n+1}{3(n-1)^2(n^2+1)}x$ and satisfies (i) and (ii), $q_n = \frac{4}{3}(-\frac{1}{3})^n - \frac{2n+1}{n^2(n+1)^2}$ satisfies (iv), $\tau(n) = n - 1 < n$ and

$$\sum_{n=N}^{\infty} |f(n,d)| = \sum_{n=N}^{\infty} (\tfrac{1}{3})^{n-1} \frac{4n^2 - 2n + 1}{3(n-1)^2(n^2+1)}|d| < \infty.$$

By Theorem 4, (9) has a bounded nonoscillatory solution $\{x_n\}$ with $\liminf\limits_{n\to\infty} |x_n| > 0$. In fact, $\{x_n\} = \{1 + n^{-2}\}$ is such a solution of (9).

E x a m p l e 3. Consider the difference equation

$$\Delta(x_n - x_{n-3}) + \frac{1}{n(n+1)(n-3)} x_{n-2} = \frac{6n-5}{(n+1)n(n-2)(n-3)}. \tag{10}$$

It is easy to see that Eq. (10) satisfies all assumptions of Theorem 6. Therefore (10) has a bounded nonoscillatory solution. In fact, $\{x_n\} = \{\frac{1}{n}\}$ is such a solution of (10).

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