UNIFORMLY CONVEX FUNCTIONS ON BANACH SPACES

J. BORWEIN[1], A. J. GUIRAO[2], P. HÁJEK[3], AND J. VANDERWERFF

ABSTRACT. We study the connection between uniformly convex functions $f : X \to \mathbb{R}$ bounded above by $\|\cdot\|^p$, and the existence of norms on X with moduli of convexity of power type. In particular, we show that there exists a uniformly convex function $f : X \to \mathbb{R}$ bounded above by $\|\cdot\|^2$ if and only if X admits an equivalent norm with modulus of convexity of power type 2.

1. INTRODUCTION

Uniformly convex functions on Banach spaces were introduced by Levitin and Poljak in [11]. Their properties were studied in depth by Zălinescu [14], and then later Azé and Penot [1] studied their duality with uniformly smooth convex functions. The monograph [15] provides a systematic development of these topics. Additionally, related properties of convex functions and their applications have been studied in papers such as [2, 3, 4, 5]. In particular, [3] examines various properties of $\|\cdot\|^r$ when $\|\cdot\|$ is a uniformly convex norm. In this note, we will present a related result that determines when functions of the form $f = \|\cdot\|^r$ are uniformly convex. We also examine a more general converse problem: if $f : X \to \mathbb{R}$ is uniformly convex and bounded above by $\|\cdot\|^r$, does X admit a norm with a modulus of convexity of power type related to r?

We work with a real Banach space X with dual X^*, and let B_X and S_X denote the closed unit ball and sphere respectively. The *modulus of convexity* of a norm $\|\cdot\|$ on X is defined for $\epsilon \in [0, 2]$ by

$$\delta_{\|\cdot\|}(\epsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : x, y \in X, \|x\| = \|y\| = 1, \|x - y\| = \epsilon\right\}.$$

The norm $\|\cdot\|$ is *uniformly convex* if $\delta_{\|\cdot\|}(\epsilon) > 0$ for all $\epsilon \in (0, 2]$; additionally, $\|\cdot\|$ has *modulus of convexity of power type p* if there exists $C > 0$ so that $\delta_{\|\cdot\|}(\epsilon) \geq C\epsilon^p$ for $\epsilon \in [0, 2]$. The *modulus of smoothness* of the norm $\|\cdot\|$ is defined for $\tau > 0$ by

$$\rho_{\|\cdot\|}(\tau) = \sup\left\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1\right\}.$$

A norm is *uniformly smooth* if $\lim_{\tau\to 0^+} \rho_{\|\cdot\|}(\tau)/\tau = 0$; additionally, $\|\cdot\|$ has *modulus of smoothness of power type p* if there exists $C > 0$ such that $\rho_{\|\cdot\|}(\tau) \leq C\tau^p$ for $\tau > 0$. See [6, Chapter IV] for more information on these notions.

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We now introduce the like-named concepts for convex functions whose definitions are different from—but motivated by—the norm cases. Given a convex function $f : X \to (-\infty, +\infty]$ we define its *modulus of convexity* as the function $\delta_f : (0, +\infty) \to [0, +\infty]$ given by

$$\delta_f(t) := \inf\left\{\frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{x+y}{2}\right) : \|x - y\| = t,\ x, y \in \operatorname{dom} f\right\},$$

where the infimum over the empty set is $+\infty$. We say that f is *uniformly convex* when $\delta_f(t) > 0$ for all $t > 0$; additionally f has a *modulus of convexity of power type p* if there exists $C > 0$ so that $\delta_f(t) \geq Ct^p$ for all $t > 0$.

Similarly we consider the *modulus of smoothness* of the convex function $f : X \to \mathbb{R}$ as the function $\rho_f : (0, +\infty) \to [0, +\infty]$ defined by

$$\rho_f(t) := \sup\left\{\frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{x+y}{2}\right) : \|x - y\| = t\right\}.$$

We will say f is *uniformly smooth* if $\lim_{t\to 0^+} \rho_f(t)/t = 0$; additionally f has a *modulus of smoothness of power type q* if there is a constant $C > 0$, so that $\rho_f(t) \leq Ct^q$ for all $t > 0$.

This terminology may cause some confusion, because, for example, $f = \|\cdot\|$ is never uniformly convex as a function, even when $\|\cdot\|$ is a uniformly convex norm. Therefore, it is important to note the context in which the terms are used. Moreover, the concepts of uniform smoothness and uniform convexity for functions are sometimes defined using the *gage of uniform convexity* and *gage of uniform smoothness* respectively as found in [15]; it is important to note that these alternate definitions using the respective gages are equivalent to those just given; cf. [14, Remark 2.1] and [15, p. 205].

Finally, the *Fenchel conjugate* of $f : X \to (-\infty, +\infty]$ is the function $f^* : X^* \to [-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{x^*(x) - f(x) : x \in X\}.$$

It is through this concept that duality between uniform convexity and uniform smoothness is studied in the context of convex functions; see [1, 15].

2. Uniform Convexity of Functions and Norms

This section will demonstrate for $2 \leq p < \infty$ that $f(\cdot) = \|\cdot\|^p$ is uniformly convex if and only if the norm $\|\cdot\|$ has modulus of convexity of power type p.

Lemma 2.1. *Let $0 < r \leq 1$, then $|t^r - s^r| \leq |t - s|^r$ for all s, $t \in [0, \infty)$.*

Proof. First, for $x \geq 0$, $(1 + x)^r \leq 1 + x^r$ (see [13, Example 4.20]). Setting $x = (t - s)/s$ with $t \geq s > 0$, and then multiplying by s^r, we get $t^r \leq s^r + (t - s)^r$. The conclusion follows from this. □

Theorem 2.2. *For $1 < q \leq 2$, the following are equivalent in a Banach space $(X, \|\cdot\|)$.*
(a) The norm $\|\cdot\|$ has modulus of smoothness of power type q.
(b) The derivative of $f(\cdot) = \|\cdot\|^q$ satisfies a $(q-1)$-Hölder condition.
(c) The function $f(\cdot) = \|\cdot\|^q$ has modulus of smoothness of power type q.
(d) The function $f(\cdot) = \|\cdot\|^q$ is uniformly smooth.

Proof. (a) $\Rightarrow$ (b): Assume that $\|\cdot\|$ has modulus of smoothness of power type q. Given $x \in X \setminus \{0\}$, let ϕ_x denote a support functional of x, that is, $\phi_x \in S_{X^*}$ and $\phi_x(x) = \|x\|$. According to [6, Lemma IV.5.1], $\|\cdot\|$ has a (Fréchet) derivative satisfying a $(q-1)$-Hölder-condition on its sphere; this implies that each $x \neq 0$ has a unique support functional, and there exists $C > 0$ such that

$$\|\phi_x - \phi_y\| \leq C\|x - y\|^{q-1} \quad \text{for all } x, y \in S_X. \tag{2.1}$$

Let $f(\cdot) = \|\cdot\|^q$. Then $f'(0) = 0$, and $f'(x) = q\,\|x\|^{q-1}\,\phi_x$ for $x \neq 0$. Thus if $x = 0$ or $y = 0$, then $\|f'(x) - f'(y)\| \leq q\,\|x - y\|^{q-1}$. Let $x, y \in X \setminus \{0\}$. Then

$$\begin{aligned} f'(x) - f'(y) &= q\,\|x\|^{q-1}\,\phi_x - q\,\|y\|^{q-1}\,\phi_y \\ &= q\,\|x\|^{q-1}\,(\phi_x - \phi_y) + \big(q\,\|x\|^{q-1} - q\,\|y\|^{q-1}\big)\phi_y. \end{aligned} \tag{2.2}$$

Using Lemma 2.1 we also compute

$$\left| q\,\|x\|^{q-1} - q\,\|y\|^{q-1} \right| \leq q \big|\, \|x\| - \|y\| \,\big|^{q-1} \leq q\,\|x - y\|^{q-1}\,. \tag{2.3}$$

We now work on an estimate for $q\,\|x\|^{q-1}\,(\phi_x - \phi_y)$. We may and do assume that $0 < \|y\| \leq \|x\|$. If $\|y\| \leq \|x\|/2$, then

$$q\,\|x\|^{q-1}\,\|\phi_x - \phi_y\| \leq 2q\,\|x\|^{q-1} \leq q2^q\,\|x - y\|^{q-1}\,. \tag{2.4}$$

If $\|y\| \geq \|x\|/2$, consider $x' = \lambda x$ where $\lambda = \|y\|/\|x\|$, so that $\|x'\| = \|y\|$. Then

$$\|x' - y\| \leq \|x' - x\| + \|x - y\| = \|x\| - \|y\| + \|x - y\| \leq 2\,\|x - y\|\,. \tag{2.5}$$

Now let $\alpha = \|y\|$. Observe that ϕ_x and ϕ_y are also support functionals for $\alpha^{-1}x'$ and $\alpha^{-1}y$ respectively. Applying (2.1), the fact that $\|x\| \leq 2\alpha$, and (2.5) we obtain

$$\begin{aligned} \|\phi_x - \phi_y\| &\leq C\|\alpha^{-1}x' - \alpha^{-1}y\|^{q-1} \leq \frac{C}{\alpha^{q-1}}\|x' - y\|^{q-1} \\ &\leq \frac{C2^{q-1}}{\|x\|^{q-1}}(2\|x - y\|)^{q-1} = \frac{C4^{q-1}}{\|x\|^{q-1}}\|x - y\|^{q-1}. \end{aligned}$$

Consequently, $q\,\|x\|^{q-1}\,\|\phi_x - \phi_y\| \leq C4^{q-1}q\,\|x - y\|^{q-1}$. This inequality and (2.4) show there exists $K > 0$ such that

$$q\,\|x\|^{q-1}\,\|\phi_x - \phi_y\| \leq K\,\|x - y\|^{q-1} \quad \text{for all } x, y \in X \setminus \{0\}. \tag{2.6}$$

Combining (2.2), (2.3) and (2.6) shows that f' satisfies a $(q-1)$-Hölder-condition.

(b) $\Rightarrow$ (c) follows from [15, Corollary 3.5.7] (see also [6, Lemma V.3.5]) and (c) $\Rightarrow$ (d) is trivial, so we prove (d) $\Rightarrow$ (a). Suppose $\|\cdot\|$ does not have modulus of smoothness of power type q. Then using [6, Lemma IV.5.1] there are x_n, $y_n \in S_X$ such that $\|x_n - y_n\| \to 0$ while

$$\|\phi_{x_n} - \phi_{y_n}\| \geq n\,\|x_n - y_n\|^{q-1}\,.$$

Let $\delta_n = \|x_n - y_n\|$ and define $u_n = \frac{1}{\delta_n\sqrt{n}}x_n$ and $v_n = \frac{1}{\delta_n\sqrt{n}}y_n$. Then $\|u_n - v_n\| = \frac{1}{\sqrt{n}} \to 0$. However

$$\begin{aligned}
\|f'(u_n) - f'(v_n)\| &= \left\| q\,\|u_n\|^{q-1}\,\phi_{u_n} - q\,\|v_n\|^{q-1}\,\phi_{v_n} \right\| \\
&= \left\| q\,\|u_n\|^{q-1}\,\phi_{x_n} - q\,\|v_n\|^{q-1}\,\phi_{y_n} \right\| \\
&= \frac{q}{\delta_n^{q-1} n^{\frac{q-1}{2}}}\,\|\phi_{x_n} - \phi_{y_n}\| \\
&\geq \frac{q}{\delta_n^{q-1} n^{\frac{q-1}{2}}}\left(n\delta_n^{q-1}\right) = qn^{\frac{3-q}{2}} \to \infty.
\end{aligned}$$

Consequently, f' is not uniformly continuous, and so [15, Theorem 3.5.6] (see also [6, Lemma V.3.5]) shows that that $f(\cdot) = \|\cdot\|^q$ is not a uniformly smooth function. □

The results in [1] enable us to derive the dual version of Theorem 2.2 for uniformly convex functions.

Theorem 2.3. *Let $(X, \|\cdot\|)$ be a Banach space, and let $2 \leq p < \infty$. Then the following are equivalent.*
(a) The norm $\|\cdot\|$ on X has modulus of convexity of power type p.
(b) The function $f(\cdot) = \|\cdot\|^p$ has modulus of convexity of power type p.
(c) The function $f(\cdot) = \|\cdot\|^p$ is uniformly convex.

Proof. (a) ⇒ (b): Let us assume that $\|\cdot\|$ has modulus of convexity of power type p, then the modulus of smoothness of the dual norm on X^*, which we denote in this proof as $\|\cdot\|_*$, is of power type q where $\frac{1}{p} + \frac{1}{q} = 1$; see [6, Proposition IV.1.12]. By Theorem 2.2 the function $g(\cdot) = \frac{1}{q}\|\cdot\|_*^q$ has modulus of smoothness of power type q. The Fenchel conjugate of g is $g^*(\cdot) = \frac{1}{p}\|\cdot\|^p$, see [1, 15]. Now g^*—and hence $\|\cdot\|^p$—has a modulus of convexity of power type p according to [1] (see also [15, Corollary 3.5.11]).

(b) ⇒ (c) is trivial, so we prove (c) ⇒ (a). Indeed, assuming that $f(\cdot) = \|\cdot\|^p$ is a uniformly convex function, then [1] shows that f^*, defined by

$$f^*(x^*) = \sup_{x \in X}\{x^*(x) - f(x)\}, \quad \text{for} \quad x^* \in X^*$$

(and hence $\|\cdot\|_*^q$) is a uniformly smooth function. According to Theorem 2.2, $\|\cdot\|_*$ has modulus of smoothness of power type q; therefore $\|\cdot\|$ has modulus of convexity of power type p, see [6, Proposition IV.1.12]. □

We conclude this section by confirming that the spaces with nontrivial uniformly convex functions are those that admit equivalent uniformly convex norms.

Theorem 2.4. *Let $(X, \|\cdot\|)$ be a Banach space. Then the following are equivalent.*
(a) There exists a l.s.c. uniformly convex function $f : X \to (-\infty, +\infty]$ that is continuous at the origin.
(b) X admits an equivalent uniformly convex norm.
(c) There exist $p \geq 2$ and an equivalent norm $\|\cdot\|$ on X so that the function $f = \|\cdot\|^p$ is uniformly convex.

Proof. (a) ⇒ (b): By replacing f with the function $x \mapsto \frac{f(x)+f(-x)}{2}$ we may and do assume that f is centrally symmetric, and by shifting f we assume $f(0) = 0$. It then follows that $f(x) \geq 0$ for all $x \in X$. Then for $r > 1$ and $h \in X$, with $\|h\| = r$, we have

$$\frac{1}{2}f(h) + \frac{1}{2}f(0) - f\left(\frac{h}{2}\right) \geq \delta_f(r) \geq \delta_f(1) > 0;$$

and thus $f(h) \geq 2\delta_f(1)$. Let us consider the norm $|||\cdot|||$ whose unit ball is $B = \{x : f(x) \leq \delta_f(1)\}$. The continuity of f at 0 implies $0 \in \operatorname{int}B$, and from the above we obtain that $B \subset B_{(X,\|\cdot\|)}$. Thus, $|||\cdot|||$ is an equivalent norm on X.

Consider $x_n, y_n \in X$ such that $|||x_n||| = |||y_n||| = 1$ and $|||x_n + y_n||| \to 2$. Because f is Lipschitz on B, we have that $f\left(\frac{x_n+y_n}{2}\right) \to \delta_f(1)$. Consequently $\frac{1}{2}f(x_n) + \frac{1}{2}f(y_n) - f\left(\frac{x_n+y_n}{2}\right) \to 0$. Thus, the uniform convexity of f ensures that $\|x_n - y_n\| \to 0$ and hence $|||x_n - y_n||| \to 0$

(b) $\Rightarrow$ (c): According to the Enflo-Pisier theorem ([8, 12]), there exist $p \geq 2$ and an equivalent norm $\|\cdot\|$ whose modulus of convexity is of power type p. Consequently, Theorem 2.3 ensures the function $f(\cdot) = \|\cdot\|^p$ is uniformly convex.

(c) $\Rightarrow$ (a): This is trivial. □

3. Growth Rates of Uniformly Convex Functions and Renorming

In this section we will construct a uniformly convex norm whose modulus of convexity is related to the growth rate of a given uniformly convex function on the Banach space. We begin with some preliminary results.

Lemma 3.1. *Let $\|\cdot\|$ be a norm on a Banach space X. Suppose $\|x\| = \|y\|$, and $\|x - y\| \geq \delta$ where $0 < \delta \leq 2\|x\|$. Then $\inf_{t \geq 0} \|x - ty\| \geq \delta/2$.*

Proof. Assume that $\|x - t_0 y\| < \delta/2$ for some $t_0 \geq 0$. Then $|1 - t_0| \|y\| < \delta/2$ and so

$$\|x - y\| \leq \|x - t_0 y\| + |1 - t_0| \|y\| < \delta.$$

which is a contradiction. □

The next lemma will be used later to estimate the modulus of convexity of a norm constructed by using level sets of a symmetric uniformly convex function.

Lemma 3.2. *Let $\{\|\cdot\|_n\}_{n \geq N}$ be a family of norms on $(X, \|\cdot\|)$ satisfying*

$$\frac{1}{2^{n+1}} \|\cdot\| \leq \|\cdot\|_n \leq \frac{1}{2^n} \|\cdot\| \quad \textit{for} \quad n \geq N. \tag{3.1}$$

For each $n \geq N$, suppose there exists $d_n > 0$ so that

$$\left\|\frac{x+y}{2}\right\|_n \leq 1 - d_n, \quad \textit{whenever} \quad \|x\|_n = \|y\|_n = 1 \textit{ and } \|x - y\| \geq 1.$$

Then there exist an equivalent norm $|\cdot|$ on X and $M \in \mathbb{N}$ so that the modulus of convexity of the norm $|\cdot|$ satisfies

$$\delta_{|\cdot|}(t) \geq \frac{d_n}{n^2} \quad \textit{whenever} \quad \frac{1}{2^{n-M-1}} \leq t \leq 2 \textit{ and } n \geq M.$$

Proof. Choose $M \geq \max\{4, N\}$, and define $|\cdot|$ by

$$|\cdot| = \sum_{m=M}^{\infty} \frac{2^{m+1}}{m^2} \|\cdot\|_m .$$

Observe that, $|\cdot| \le \sum_{m=M}^{\infty} \frac{2^{m+1}}{m^2 2^m} \|\cdot\| \le \sum_{m=4}^{\infty} \frac{2}{m^2} \|\cdot\| \le \|\cdot\|$; and then

$$\frac{1}{2^M} \|\cdot\| \le \frac{1}{M^2} \|\cdot\| \le \frac{2^{M+1}}{M^2} \|\cdot\|_M \le |\cdot| \le \|\cdot\| . \tag{3.2}$$

Now suppose that $|x| = |y| = 1$ and $|x - y| \ge \frac{1}{2^{n-M-1}}$ where $n \ge M$ is fixed. Because $|x| = |y| = 1$, it follows from (3.2) that

$$1 \le \|x\| \le 2^M \qquad \text{and} \qquad 1 \le \|y\| \le 2^M. \tag{3.3}$$

We assume, without loss of generality, $\|x\|_n \le \|y\|_n$. Now let us denote $a = \|x\|_n^{-1}$ and $b = \|y\|_n^{-1}$. It follows from (3.1) and (3.3) that $2^{n-M} \le b \le a \le 2^{n+1}$, which in turn implies $|ax - ay| \ge 2$.

According to Lemma 3.1, $|ax - by| \ge 1$, and hence $\|ax - by\| \ge 1$. Thus we compute

$$\begin{aligned} \left\|\frac{ax+ay}{2}\right\|_n &\le \left\|\frac{ax+by}{2}\right\|_n + \frac{1}{2}(a-b)\|y\|_n \\ &\le \frac{1}{2}\|ax\|_n + \frac{1}{2}\|by\|_n + \frac{1}{2}(a-b)\|y\|_n - d_n \\ &= \frac{a}{2}(\|x\|_n + \|y\|_n) - d_n. \end{aligned}$$

This inequality implies

$$\left\|\frac{x+y}{2}\right\|_n \le \frac{1}{2}\|x\|_n + \frac{1}{2}\|y\|_n - \frac{d_n}{a}. \tag{3.4}$$

Thus, using (3.4), and the triangle inequality for $\|\cdot\|_j$ when $j \ne n$, and then that $a \le 2^{n+1}$ we obtain

$$\left|\frac{x+y}{2}\right| \le \sum_{j=M}^{\infty} \frac{2^{j+1}}{2j^2}\|x\|_j + \sum_{j=M}^{\infty} \frac{2^{j+1}}{2j^2}\|y\|_j - \frac{2^{n+1}d_n}{n^2 a} \le 1 - \frac{d_n}{n^2},$$

which finishes the proof. □

We will also use the following important fact from [15] concerning growth rates of uniformly convex functions.

Lemma 3.3. [15, Proposition 3.5.8] *Suppose $f : X \to (-\infty, +\infty]$ is a l.s.c. uniformly convex function. Then* $\liminf_{\|x\|\to\infty} \frac{f(x)}{\|x\|^2} > 0$.

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a Banach space and let $F : [0, +\infty) \to [0, +\infty)$ be a continuous convex function satisfying $F(0) = 0$. Suppose $f : X \to \mathbb{R}$ is a continuous uniformly convex function satisfying $f(x) \le F(\|x\|)$ for all $x \in X$. Then there is an equivalent norm $|\cdot|$ on X such that given any $\gamma > 0$, there are constants $\alpha > 0$ and $\beta > 0$ so that*

$$\delta_{|\cdot|}(t) \ge \frac{\alpha}{F(\beta t^{-1})} t^{\gamma} \qquad \textit{for} \quad 0 < t \le 2.$$

Proof. As before, we may and do assume f is centrally symmetric; notice that this new f will still be bounded above by $F(\|\cdot\|)$. Because $F(0) = 0$, the convexity of F ensures that $F(\lambda t) \le \lambda F(t)$ for $0 \le \lambda \le 1$; in particular F is nondecreasing on $[0, +\infty)$ because it is nonnegative there. Consequently, $F(\|\cdot\|)$ is a convex function.

According to Lemma 3.3, there exists $N > 0$ so that $f(x) \geq 0$ if $\|x\| \geq N$. Now replace f with $[f(\cdot) + F(\|\cdot\|)]/2$; then we have

$$F\left(\frac{\|x\|}{2}\right) \leq \frac{1}{2}F(\|x\|) \leq f(x) \leq F(\|x\|) \text{ whenever } \|x\| \geq N. \tag{3.5}$$

For $n \geq N$, let $\|\cdot\|_n$ be norm whose unit ball is $B_n = \{x : f(x) \leq F(2^n)\}$. It follows from (3.6) that if $\|x\|_n = 1$, then $2^n \leq \|x\| \leq 2^{n+1}$. Consequently,

$$\frac{1}{2^{n+1}} \|\cdot\| \leq \|\cdot\|_n \leq \frac{1}{2^n} \|\cdot\| .$$

Let $M_n = \sup\{f'_+(u,v) : \|u\|_n = 1, \|v\| = 1\}$. If $\|u\|_n = 1$, and $\|v\| = 1$, then $\|u\| \leq 2^{n+1}$ and we compute

$$f'_+(u,v) \leq \frac{f(u + 2^{n+1}v) - f(u)}{2^{n+1}} \leq \frac{F(2 \cdot 2^{n+1}) - 0}{2^{n+1}} = \frac{F(2^{n+2})}{2^{n+1}}. \tag{3.6}$$

It follows that $M_n \leq 2^{-(n+1)}F(2^{n+2})$.

Now suppose $\|x\|_n = \|y\|_n = 1$, and $\|x - y\| \geq 1$. Letting δ_f denote the modulus of convexity of f with respect to $\|\cdot\|$, the uniform convexity of f ensures $\delta_f(1) > 0$. Then denoting $z = \frac{x+y}{2}$ and $z' = z/\|z\|_n$ we obtain $f(x) = f(y) = f(z') = F(2^n)$, and so

$$\begin{aligned} \delta_f(1) &\leq \frac{1}{2}f(x) + \frac{1}{2}f(y) - f\left(\frac{x+y}{2}\right) = f(z') - f(z) \leq f'_+(z', z' - z) \\ &= \|z' - z\| f'_+\left(z', \frac{z'-z}{\|z'-z\|}\right) \leq M_n \|z' - z\| . \end{aligned} \tag{3.7}$$

Consequently, using $\|\cdot\|_n \geq \frac{1}{2^{n+1}} \|\cdot\|$, (3.8) and then the bound on M_n, we obtain

$$\begin{aligned} \left\|\frac{x+y}{2}\right\|_n = 1 - \|z' - z\|_n &\leq 1 - \|z' - z\| \frac{1}{2^{n+1}} \leq 1 - \frac{\delta_f(1)}{M_n} \cdot \frac{1}{2^{n+1}} \\ &\leq 1 - \frac{\delta_f(1)}{F(2^{n+2})}. \end{aligned} \tag{3.8}$$

Applying Lemma 3.2, we find an equivalent norm $|\cdot|$ and $M \geq N$ such that

$$\delta_{|\cdot|}(t) \geq \frac{\delta_f(1)}{F(2^{n+2})} \cdot \frac{1}{n^2}, \quad \text{whenever } n \geq M \text{ and } \frac{1}{2^{n-M-1}} \leq t \leq 2.$$

Given $\gamma > 0$, we fix $n_0 \geq M$ so large that $n^{-2} \geq (2^{-n})^\gamma$ for all $n \geq n_0$. Then

$$\delta_{|\cdot|}(t) \geq \frac{\delta_f(1)}{F(2^{n+2})} \cdot \left(\frac{1}{2^n}\right)^\gamma \quad \text{whenever } n \geq n_0 \text{ and } \frac{1}{2^{n-M-1}} \leq t \leq 2.$$

Let $\alpha = \delta_f(1) \min\left\{\left(\frac{1}{2^{n_0+1}}\right)^\gamma, \left(\frac{1}{2^{M+2}}\right)^\gamma\right\}$ and $\beta = \max\left\{2^{n_0+3}, 2^{M+4}\right\}$. The previous inequality and along with the fact F is nondecreasing, ensure that for $\frac{1}{2^{n_0-M-1}} \leq t \leq 2$ we have

$$\delta_{|\cdot|}(t) \geq \frac{\delta_f(1)}{F(2^{n_0+2})}\left(\frac{1}{2^{n_0}}\right)^\gamma \geq \frac{\delta_f(1)}{F(2^{n_0+3}t^{-1})}\left(\frac{t}{2^{n_0+1}}\right)^\gamma \geq \frac{\alpha t^\gamma}{F(\beta t^{-1})};$$

and for $\frac{1}{2^{n-M-1}} \leq t \leq \frac{1}{2^{n-M-2}}$ where $n \geq n_0 + 1$, we have

$$\delta_{|\cdot|}(t) \geq \frac{\delta_f(1)}{F(2^{n+2})}\left(\frac{1}{2^n}\right)^\gamma \geq \frac{\delta_f(1)}{F(2^{M+4}t^{-1})}\left(\frac{t}{2^{M+2}}\right)^\gamma \geq \frac{\alpha t^\gamma}{F(\beta t^{-1})}.$$

Altogether, $\delta_{|\cdot|}(t) \geq \frac{\alpha}{F(\beta t^{-1})}t^\gamma$ for $0 < t \leq 2$, as desired. □

Corollary 3.5. *Let $(X, \|\cdot\|)$ be a Banach space, and let $p \geq 2$. Suppose $f : X \to \mathbb{R}$ is a continuous uniformly convex function such that $f(x) \leq \|x\|^p$ for all $x \in X$. Then for any $r > p$, X admits an equivalent norm with modulus of convexity of power type r.*

Proof. Apply the previous theorem with $F(t) = t^p$. □

In the case $p = 2$ we will prove the following sharp result.

Theorem 3.6. *Let $(X, \|\cdot\|)$ be a Banach space. Then there is a continuous uniformly convex funtion $f : X \to \mathbb{R}$ satisfying $f(\cdot) \leq \|\cdot\|^2$ if and only if X admits an equivalent norm with modulus of convexity of power type 2.*

Before proving this theorem, we will present a preliminary lemma, and we also refer the reader to [7] for some related information about this case.

Lemma 3.7. *Let X be a Banach space. Suppose $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ are norms on $(X, \|\cdot\|)$ so that*

$$K\|\cdot\| \leq \|\cdot\|_n \leq \|\cdot\|, \tag{3.9}$$

for some $K > 0$ and all $n \in \mathbb{N}$. Then, there exists an equivalent norm $|\cdot|$ such that

$$\delta_{|\cdot|}(t) \geq \liminf \delta_{\|\cdot\|_n}(t), \quad \textit{for} \quad 0 < t < 2.$$

Proof. Let us consider a free (non-principal) ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then $\lim_{\mathcal{U}} \|x\|_n$ exists for each $x \in X$, where $\lim_{\mathcal{U}} \|x\|_n = L$ means for each $\epsilon > 0$, there exists $A \in \mathcal{U}$ such that $|\,\|x\|_n - L| < \epsilon$ for all $n \in A$. Now define $|\cdot| : X \to [0, +\infty)$ by

$$|x| = \lim_{\mathcal{U}} \|x\|_n, \quad \text{for all} \quad x \in X.$$

The definition of $|\cdot|$ together with (3.10) ensure $|\cdot|$ is an equivalent norm on X.

If proceed by reductio ad absurdum, we find $t \in (0, 2)$ such that $\delta_{|\cdot|}(t) < \liminf \delta_{\|\cdot\|_n}(t)$. Since $\delta_{|\cdot|}$ is continuous — see [10]— there exists $t' \in (t, 2)$ such that $\delta_{|\cdot|}(t') < \liminf \delta_{\|\cdot\|_n}(t)$. Then, there exist $x, y \in X$ and a constant $a > 0$ such that $|x| = |y| = 1$, $|x - y| \geq t'$ and $1 - |(x + y)/2| < a < \liminf \delta_{\|\cdot\|_n}(t)$. For this x and y, let $x_n = x/\|x\|_n$ and $y_n = y/\|y\|_n$. By the definition of $|\cdot|$, there exists $A \in \mathcal{U}$ such that $\|x_m - y_m\|_m \geq t$ and $1 - \|(x_m + y_m)/2\|_m < a$ for all $m \in A$. Therefore $\delta_{\|\cdot\|_m}(t) < a < \liminf \delta_{\|\cdot\|_n}(t)$ for all $m \in A$, which yields a contradiction, since $\mathcal{U}$ is free and then A is infinite. □

Proof. (Theorem 3.6) First, if X admits an equivalent norm that has modulus of convexity of power type 2, then it has such a norm $|\cdot|$ satisfying $|\cdot| \leq \|\cdot\|$. According to Theorem 2.3, $f(\cdot) = |\cdot|^2$ is uniformly convex as desired.

Conversely, suppose $f : X, \to \mathbb{R}$ is a uniformly convex function such that $f(\cdot) \leq \|\cdot\|^2$. Proceeding as in Theorem 3.4 when $F(t) = t^2$ we obtain norms $\{\|\cdot\|_n\}_{n\geq N}$ satisfying $\frac{1}{2^{n+1}}\|\cdot\| \leq \|\cdot\|_n \leq \frac{1}{2^n}\|\cdot\|$ and then (3.9) becomes

$$\left\|\frac{x+y}{2}\right\|_n \leq 1 - \frac{\delta_f(1)}{16}\left(\frac{1}{2^n}\right)^2, \text{ whenever } \|x\|_n = \|y\|_n = 1 \text{ and } \|x - y\|_n \geq \frac{1}{2^n}.$$

The previous inequality implies

$$\delta_{\|\cdot\|_n}(2^{-n}) \geq \frac{\delta_f(1)}{16}(2^{-n})^2.$$

According to [9, Corollary 11] there is a universal constant $L > 0$ such that

$$\frac{\delta_{|\cdot|_n}(2^{-n})}{(2^{-n})^2} \leq 4L \frac{\delta_{|\cdot|_n}(\eta)}{\eta^2} \quad \text{for } 2^{-n} \leq \eta \leq 2.$$

Let $R = \frac{\delta_f(1)}{64L}$; then the previous two inequalities imply

$$\delta_{|\cdot|_n}(t) \geq Rt^2 \quad \text{for } 2^{-n} \leq t \leq 2. \tag{3.10}$$

For each $n \geq N$, let us consider the new norm $|\cdot|_n = 2^n \|\cdot\|_n$. These new norms satisfy $\frac{1}{2}\|\cdot\| \leq \|\cdot\|_n \leq \|\cdot\|$ and $\delta_{|\cdot|_n}(\cdot) = \delta_{\|\cdot\|_n}(\cdot)$. Applying Lemma 3.7 and then (3.11) we obtain

$$\delta_{|\cdot|}(t) \geq \liminf_{n\to\infty} \delta_{|\cdot|_n}(t) = \liminf_{n\to\infty} \delta_{\|\cdot\|_n}(t) \geq Rt^2 \text{ for } 0 < t \leq 2,$$

which finishes the proof. □

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Computer Science Faculty, 325, Dalhousie University, Halifax, NS, Canada, B3H 1W5
E-mail address: jborwein@cs.dal.ca

Departamento de Matemáticas, Universidad de Murcia, 30100 Espinardo (Murcia), Spain
E-mail address: ajguirao@um.es

Mathematical Institute, AV ČR, Žitná 25, 115 67 Praha 1, Czech Republic
E-mail address: hajek@math.cas.cz

Department of Mathematics, La Sierra University, Riverside, CA
E-mail address: jvanderw@lasierra.edu