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REFLEXIVITY AND HYPERREFLEXIVITY OF THE SPACE OF LOCALLY INTERTWINING OPERATORS

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Abstract. We characterize the spaces of all local intertwiners $\mathrm{I}(A, B; e)$ that are reflexive (hyperreflexive). We show that if e is not an eigenvector of A, then the reflexivity (hyperreflexivity) of $\mathrm{I}(A, B; e)$ depends only on B and is independent of A and e. This has consequences concerning the reflexivity of the space of intertwiners $\mathrm{I}(A, B)$ and of the commutant of an operator.

1. Introduction

For complex Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, let $B(\mathcal{X}, \mathcal{Y})$ be the Banach space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$; similarly, let $B(\mathcal{X})$ be the Banach algebra of all bounded linear operators on $\mathcal{X}$. The topological dual of $\mathcal{X}$ is denoted by $\mathcal{X}^*$.

Let $A \in B(\mathcal{X})$, $B \in B(\mathcal{Y})$, and $e \in \mathcal{X}$ be given. An operator $S \in B(\mathcal{X}, \mathcal{Y})$ intertwines A and B at e, if $SAe = BSe$. The set of all operators that intertwine A and B at e is denoted by $\mathrm{I}(A, B; e)$. In particular, if $\mathcal{X} = \mathcal{Y}$ and $A = B$, then $\mathrm{C}(A, e) := \mathrm{I}(A, A; e)$ is the local commutant of A at e. Local commutants were introduced and studied by Larson [8], see also [3].

It is obvious that $\mathrm{I}(A, B; e)$ is a linear space of operators and it is not hard to see that $\mathrm{I}(A, B; e)$ is closed in the strong operator topology, which means, by convexity, that it is closed in the weak operator topology as well.

For a linear subspace $\mathcal{S} \subseteq B(\mathcal{X}, \mathcal{Y})$, the reflexive closure of $\mathcal{S}$ is given by

$$\mathrm{Ref}\, \mathcal{S} = \{T \in B(\mathcal{X}, \mathcal{Y});\quad Tx \in [\mathcal{S}x] \quad \text{for all} \;\; x \in \mathcal{X}\},$$

where $\mathcal{S}x = \{Sx;\ S \in \mathcal{S}\}$ is the orbit of $\mathcal{S}$ at x and $[\mathcal{S}x]$ is its closure. It is obvious that $\mathrm{Ref}\, \mathcal{S} \supseteq \mathcal{S}$. If $\mathrm{Ref}\, \mathcal{S} = \mathcal{S}$, then the space $\mathcal{S}$ is said to be reflexive.

In Section 2 we give a complete description of subspaces $\mathrm{I}(A, B; e)$ that are reflexive. It is easy to see that this space is reflexive if e is an eigenvector of A. If e and Ae are linearly independent then the space $\mathrm{I}(A, B; e)$ is reflexive if and only if $\bigcap_{\lambda \in \mathbb{C}} [\mathrm{im}\,(B - \lambda)] = \{0\}$. It is interesting that this condition depends only on B and is independent of A and e. This has applications for the reflexivity of the space of intertwiners between A and B.

Section 3 is devoted to the hyperreflexivity (for the definition see that section). It is well-known that any hyperreflexive subspace of operators is reflexive and that the converse does not hold, see [7], Theorem 6. We shall show that spaces of locally intertwining operators provide natural examples of spaces of operators that are reflexive but not hyperreflexive.

In the last section we discuss the k-reflexivity and k-hyperreflexivity of spaces of local intertwiners.

Key words and phrases. Commutant, local commutant, reflexivity, hyperreflexivity.

2. Reflexivity of the space of locally intertwining operators

In this section we shall characterize those spaces $\mathrm{I}(A, B; e)$ that are reflexive. The following proposition describes the orbits of spaces of local intertwiners.

Proposition 2.1. *Let $A \in B(\mathcal{X})$, $B \in B(\mathcal{Y})$, and $e,\ x \in \mathcal{X} \setminus \{0\}$ be arbitrary.*

(i) If x is not in the linear span of the vectors e and Ae, i.e. $x \notin \vee\{e, Ae\}$, then $\mathrm{I}(A, B; e)x = \mathcal{Y}$.

(ii) If $Ae = \lambda e$ for some $\lambda \in \mathbb{C}$ and x is a scalar multiple of e, then $\mathrm{I}(A, B; e)x = \ker(B - \lambda)$.

(iii) If e and Ae are linearly independent, $\alpha, \beta \in \mathbb{C}$ and $x = \alpha Ae + \beta e$, then $\mathrm{I}(A, B; e)x = \mathrm{im}\,(\alpha B + \beta)$.

Proof. (i) Since $x \notin \vee\{e, Ae\}$ there exists $\xi \in \mathcal{X}^*$ that annihilates $\vee\{e, Ae\}$, that is $\xi \in \big(\vee\{e, Ae\}\big)^{\perp}$ such that $\langle x, \xi\rangle = 1$. Let $y \in \mathcal{Y}$ be artitrary. The operator $y \otimes \xi$, which is given by $(y \otimes \xi)z = \langle z, \xi\rangle y \ \ (z \in \mathcal{X})$, maps x to y and it is in $\mathrm{I}(A, B; e)$ because $(y \otimes \xi)Ae = 0 = B(y \otimes \xi)e$.

(ii) Let $\mu \in \mathbb{C} \setminus \{0\}$ be such that $x = \mu e$. If $S \in \mathrm{I}(A, B; e)$, then $(B - \lambda)Sx = \mu S(Ae - \lambda e) = 0$. Thus, $\mathrm{I}(A, B; e)x \subseteq \ker(B - \lambda)$. For the opposite inclusion, let $y \in \ker(B - \lambda)$ be arbitrary. Then there exists $S \in B(\mathcal{X}, \mathcal{Y})$ such that $Sx = y$. Since $(B - \lambda)Se = \mu^{-1}(B - \lambda)y = 0$ we have $BSe = \lambda Se = SAe$ and $S \in \mathrm{I}(A, B; e)$.

(iii) If $S \in \mathrm{I}(A, B; e)$, then $Sx = S(\alpha Ae + \beta e) = (\alpha B + \beta)Se$, which shows that $\mathrm{I}(A, B; e)x \subseteq \mathrm{im}\,(\alpha B + \beta)$. On the other hand, let $y = (\alpha B + \beta)w$, where $w \in \mathcal{Y}$, be an arbitrary vector in the range $\mathrm{im}\,(\alpha B + \beta)$. Since e and Ae are linearly independent there exist $\xi,\ \eta \in \mathcal{X}^*$ such that $\langle e, \xi\rangle = 1 = \langle Ae, \eta\rangle$ and $\langle Ae, \xi\rangle = 0 = \langle e, \eta\rangle$. Set $S := w \otimes \xi + Bw \otimes \eta$. Then it is easily seen that $S \in \mathrm{I}(A, B; e)$ and $Sx = y$. ☐

Let $\sigma_p(T)$ be the point spectrum (the set of eigenvalues) of a given linear operator $T \in B(\mathcal{X})$. It is well-known that a number λ is in $\sigma_p(T^*)$ if and only if the range $\mathrm{im}\,(T - \lambda)$ is not dense in $\mathcal{X}$. Recall that a nonempty set $\mathcal{S} \subseteq B(\mathcal{X})$ is transitive if, for any $x \neq 0$, the orbit $\mathcal{S}x$ is dense in $\mathcal{X}$.

Corollary 2.2. *Let $A, B \in B(\mathcal{X})$ and $e \in \mathcal{X}$. Assume that e and Ae are linearly independent. Then it is an immediate consequence of Proposition 2.1 that $\mathrm{I}(A, B; e)$ is transitive if and only if the point spectrum of B^* is empty. In particular, the local commutant $\mathrm{C}(A, e)$ is transitive if and only if $\sigma_p(A^*) = \emptyset$.*

Now we describe the reflexive closure of the space of local intertwiners.

Proposition 2.3. *Let $A \in B(\mathcal{X})$, $B \in B(\mathcal{Y})$, and $e \in \mathcal{X}$ be arbitrary. If e and Ae are linearly independent, then*

$$\mathrm{Ref}\,\mathrm{I}(A, B; e) = \{T \in B(\mathcal{X}, \mathcal{Y}); \quad T(A - \lambda)e \in [\mathrm{im}\,(B - \lambda)] \quad \textit{for all} \quad \lambda \in \mathbb{C}\}.$$

Proof. Let $T \in \mathrm{Ref}\,\mathrm{I}(A, B; e)$ be arbitrary. Choose $\lambda \in \mathbb{C}$ and set $x_\lambda = Ae - \lambda e$. By Proposition 2.1 (iii), $\mathrm{I}(A, B; e)x_\lambda \in \mathrm{im}\,(B - \lambda)$. Since $Tx \in [\mathrm{I}(A, B; e)x]$ for any $x \in \mathcal{X}$ we conclude $T(A - \lambda)e = Tx_\lambda \in [\mathrm{I}(A, B; e)x_\lambda] = [\mathrm{im}\,(B - \lambda)]$.

Now, assume that $T \in B(\mathcal{X}, \mathcal{Y})$ satisfies $T(A-\lambda)e \in [\mathrm{im}\,(B-\lambda)]$ for all $\lambda \in \mathbb{C}$. Let $x \in \mathcal{X}$ be arbitrary. It is obvious that $Tx \in [\mathrm{I}(A,B;e)x]$ for $x = 0$. Suppose therefore that $x \neq 0$. If $x \notin [\{e, Ae\}]$, then, by Proposition 2.1 (i), $\mathrm{I}(A,B;e)x = \mathcal{Y}$ which gives $Tx \in [\mathrm{I}(A,B;e)x]$. If x is a scalar multiple of e, say $x = \beta e$ for some $\beta \neq 0$, then $\mathrm{I}(A,B;e)x = \mathrm{im}\,(\beta I) = \mathcal{Y}$, by Proposition 2.1 (iii), and again $Tx \in [\mathrm{I}(A,B;e)x]$. Finally, assume that $x = \alpha Ae + \beta e$ with $\alpha \neq 0$. Then $Tx = \alpha T(A+\beta/\alpha)e \in [\mathrm{im}\,(B+\beta/\alpha)]$. Since, by Proposition 2.1 (iii), $\mathrm{im}\,(B+\beta/\alpha) = \mathrm{I}(A,B;e)(A+\beta/\alpha)e$ we conclude that $Tx \in [\mathrm{I}(A,B;e)x]$. □

Corollary 2.4. *If e and Ae are linearly independent, then* $\mathrm{Ref}\,\mathrm{I}(A,B;e) = B(\mathcal{X},\mathcal{Y})$ *if and only if* $\sigma_p(B^*) = \emptyset$.

Proof. If $\sigma_p(B^*) = \emptyset$, then $[\mathrm{im}\,(B-\lambda)] = \mathcal{Y}$ for all $\lambda \in \mathbb{C}$. Thus, every $T \in B(\mathcal{X},\mathcal{Y})$ satisfies the condition $T(A-\lambda)e \in [\mathrm{im}\,(B-\lambda)]$ $(\lambda \in \mathbb{C})$, which means, by Proposition 2.3, that $T \in \mathrm{I}(A,B;e)$.

On the other hand, if there exists $\lambda \in \sigma_p(B^*)$, then $[\mathrm{im}\,(B-\lambda)] \neq \mathcal{Y}$. Since $(A-\lambda)e$ is a nonzero vector there exists $T \in B(\mathcal{X},\mathcal{Y})$ such that $T(A-\lambda)e \notin [\mathrm{im}\,(B-\lambda)]$. □

It follows from Proposition 2.1 that $\mathrm{I}(A,B;e)$ is reflexive whenever e is an eigenvector of A.

Proposition 2.5. *Let $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$. If $e \in \mathcal{X}$ is an eigenvector of A, then* $\mathrm{I}(A,B;e)$ *is reflexive.*

Proof. Let $Ae = \lambda e$ and assume that $T \in \mathrm{Ref}\,\mathrm{I}(A,B;e)$. Then, by Proposition 2.1, we have $Te \in \ker\,(B-\lambda)$. It follows that $BTe = \lambda Te = TAe$, i.e., $T \in \mathrm{I}(A,B;e)$. □

For an operator $T \in B(\mathcal{X})$ such that $\sigma_p(T^*) \neq \emptyset$, let $\mathrm{Eig}(T^*)$ be the weak-$*$ closure of the subspace of $\mathcal{X}^*$ that is spanned by the eigenvectors of T^*. If $\sigma_p(T^*)$ is empty, then we set $\mathrm{Eig}(T^*) = \{0\}$.

Theorem 2.6. *Let $A \in B(\mathcal{X})$, $B \in B(\mathcal{Y})$, and $e \in \mathcal{X}$ be arbitrary. If e and Ae are linearly independent, then the following is equivalent:*

(i) $\mathrm{I}(A,B;e)$ *is reflexive;*

(ii) $\mathrm{Eig}(B^*) = \mathcal{Y}^*$;

(iii) $\bigcap_{\lambda\in\mathbb{C}}[\mathrm{im}\,(B-\lambda)] = \{0\}$.

Proof. First we shall prove the equivalence of (i) and (ii). If $\mathrm{Eig}(B^*)$ is a proper subspace of $\mathcal{Y}^*$, then there exists a non-zero vector $y \in \mathrm{Eig}(B^*)_\perp$. Let $\xi \in \mathcal{X}^*$ be such that $\langle e,\xi\rangle = 0$ and $\langle Ae,\xi\rangle = 1$. Then $T := y \otimes \xi$ is not in $\mathrm{I}(A,B;e)$, since $TAe = y \neq 0 = BTe$. However, for an arbitrary number λ_0, we have

$$T(A-\lambda_0)e = y \in \mathrm{Eig}(B^*)_\perp = \bigcap_{\lambda\in\mathbb{C}}[\mathrm{im}\,(B-\lambda)] \subseteq [\mathrm{im}\,(B-\lambda_0)],$$

which gives $T \in \mathrm{Ref}\,\mathrm{I}(A,B;e)$, by Proposition 2.3.

For the opposite implication, assume that $\mathrm{Eig}(B^*) = \mathcal{Y}^*$. Let $T \in \mathrm{Ref}\,\mathrm{I}(A,B;e)$ be arbitary. By Proposition 2.3, we have $T(A-\lambda)e \in [\mathrm{im}\,(B-\lambda)]$ for all $\lambda \in \mathbb{C}$. Choose and fix $\lambda_0 \in \sigma_p(B^*)$. Then $\langle T(A-\lambda_0)e,\eta\rangle = 0$ for each $\eta \in \ker\,(B^*-\lambda_0)$. It follows that

$$\langle TAe,\eta\rangle = \lambda_0\langle Te,\eta\rangle = \langle Te, B^*\eta\rangle = \langle BTe,\eta\rangle.$$

Thus, $\langle (BT - TA)e, \eta\rangle = 0$ for all $\eta \in \ker(B^* - \lambda_0)$. Since $\lambda_0 \in \sigma_p(B^*)$ is arbitrary and since $\mathrm{Eig}(B^*) = \mathcal{Y}^*$ we conclude that $(BT - TA)e = 0$, i.e. operator T is in $\mathrm{I}(A, B; e)$.

Now about the equivalence of (ii) and (iii). It is well known that $[\mathrm{im}\,(B-\lambda)] = \ker(B^*-\lambda)_\perp$. Thus, if $x \in [\mathrm{im}\,(B-\lambda)]$, for all $\lambda \in \mathbb{C}$, then $\langle \xi, x\rangle = 0$, for any eigenvector ξ of B^*. It follows that $x \in \mathrm{Eig}(B^*)_\perp$. On the other hand, if $x \in \mathcal{X}$ is not in the intersection $\cap_{\lambda\in\mathbb{C}}[\mathrm{im}\,(B-\lambda)]$, then there exists a number λ_0 such that $x \notin [\mathrm{im}\,(B-\lambda_0)] = \ker(B^* - \lambda_0)_\perp$. Thus, there exists an eigenvector ξ of B^* such that $\langle \xi, x\rangle \neq 0$, which means $x \notin \mathrm{Eig}(B^*)_\perp$. □

Note that conditions (ii) and (iii) do not depend on vector e. Thus, the following assertion holds.

Corollary 2.7. *If* $\mathrm{I}(A, B; e)$ *is reflexive for* $e \in \mathcal{X} \setminus \{0\}$ *that is not an eigenvector for* A, *then* $\mathrm{I}(A, B; f)$ *is reflexive for any* $f \in \mathcal{X}$. □

Clearly

$$\bigcap_{e\in\mathcal{X}} \mathrm{I}(A, B; e) = \mathrm{I}(A, B) := \{S \in B(\mathcal{X}, \mathcal{Y});\ SA = BS\}.$$

Since an arbitary intersection of reflexive spaces is a reflexive space we have the following corollary, which extends Lemma 1 [9].

Corollary 2.8. *Let* $A \in B(\mathcal{X})$ *and* $B \in B(\mathcal{Y})$. *If* $\mathrm{Eig}(B^*) = \mathcal{Y}^*$, *then* $\mathrm{I}(A, B)$ *is reflexive.* □

Note however that the condition $\mathrm{Eig}(B^*) = \mathcal{Y}^*$ is not necessary for reflexivity of $\mathrm{I}(A, B)$. For instance, let N be a normal operator without eigenvalues on a complex Hilbert space $\mathcal{H}$. Then, of course, $\mathrm{Eig}(N^*) = \{0\}$. On the other hand, the commutant $\{N\}'$ is reflexive since it is a von Neumann algebra ([2], Proposition 56.6).

Corollary 2.9. *Let* $A \in B(\mathcal{X})$ *be an arbitrary operator and let* $B \in B(\mathcal{Y})$ *be a non-zero nilpotent operator. If* $\mathrm{I}(A, B; e)$ *is reflexive for some non-zero* $e \in \mathcal{X}$, *then* e *is an eigenvector of* A.

Proof. Since B is non-zero nilpotent the adjoint operator B^* is a non-zero nilpotent as well. It follows that $\mathrm{Eig}(B^*) \neq \mathcal{Y}^*$. By Theorem 2.6, $\mathrm{I}(A, B; e)$ cannot be reflexive if e is not an eigenvector of A. □

Proposition 2.10. *Let* $T \in B(\mathcal{X})$ *and* $S \in B(\mathcal{Y})$ *be operators such that there exists an injective operator* $V \in \mathrm{I}(T, S)$. *If* S *satisfies condition (iii) of Theorem 2.6, then* T *satisfies this condition as well.*

Proof. Assume that T does not satisfy the conditions. Then there exists a non-zero vector $x \in \bigcap_{\lambda\in\mathbb{C}}[\mathrm{im}\,(T-\lambda)]$. The intertwiner V is injective, therefore $Vx \in \mathcal{Y}$ is also a non-zero vector. Let $\lambda \in \mathbb{C}$ be an arbitary number. Since $x \in [\mathrm{im}\,(T-\lambda)]$, there exists a sequence (x_n) in $\mathcal{X}$ such that $\lim_{n\to\infty} \|(T-\lambda)x_n - x\| = 0$. It follows $\lim_{n\to\infty} \|(S-\lambda)Vx_n - Vx\| \leq \|V\| \lim_{n\to\infty} \|(T-\lambda)x_n - x\| = 0$, which gives $Vx \in [\mathrm{im}\,(S-\lambda)]$. We conclude that S does not satisfy condition (iii) of Theorem 2.6. □

Note that the condition in Proposition 2.10 is satisfied if T is a quasi-affine transform of S. In particular, it is weaker than the quasi-similarity of operators T and S.

Now we shall give a description of operators that satisfy the equivalent conditions (ii) and (iii) of Theorem 2.6. Our description is based on the idea presented in [5], Solution 69.

Let Ω be a non-empty set and let $X(\Omega)$ be a Banach space of complex-valued functions on Ω satisfying the following two conditions:

$$\text{(1)} \qquad \begin{aligned} &\text{for each } \omega \in \Omega, \text{ there exists } f \in X(\Omega) \text{ such that } f(\omega) \neq 0; \\ &|f(\omega)| \leq \|f\|, \text{ for } f \in X(\Omega) \text{ and } \omega \in \Omega. \end{aligned}$$

An operator $M \in B(X(\Omega))$ is a multiplication operator if there exists a complex-valued function φ on Ω such that $(Mf)(\omega) = \varphi(\omega)f(\omega)$ for all $\omega \in \Omega$. If M is a multiplication operator, then the corresponding function φ is uniquely determined. In the sequel we shall write M_φ to indicate the connection between a multiplication operator and the corresponding function.

For each $\omega \in \Omega$, define the point evaluation ξ_ω on $X(\Omega)$ by $\langle f, \xi_\omega \rangle = f(\omega)$ $(f \in X(\Omega))$. Since

$$|\langle f, \xi_\omega \rangle| = |f(\omega)| \leq \|f\| \qquad (f \in X(\Omega))$$

each ξ_ω is a linear functional with norm at most 1. By the first condition in (1), each ξ_ω is non-zero and it is not hard to see that the linear span of $\{\xi_\omega;\ \omega \in \Omega\}$ is weak-$*$ dense in $X(\Omega)^*$. Let $M_\varphi \in B(X(\Omega))$ be an arbitrary multiplication operator. Then

$$\langle f, \big(M_\varphi\big)^* \xi_\omega \rangle = \langle M_\varphi f, \xi_\omega \rangle = \varphi(\omega) f(\omega) = \langle f, \varphi(\omega)\xi_\omega \rangle \qquad (f \in X(\Omega))$$

holds for any $\omega \in \Omega$. Thus, each ξ_ω is an eigenvector for $\big(M_\varphi\big)^*$ (with $\varphi(\omega)$ as the corresponding eigenvalue) and consequently $\mathrm{Eig}\big(\big(M_\varphi\big)^*\big) = X(\Omega)^*$.

Now, let $\mathcal{X}$ be a Banach space that is isometrically isomorphic to $X(\Omega)$, i.e. there exists a (bijective) linear isometry $U : \mathcal{X} \to X(\Omega)$. Assume that $T \in B(\mathcal{X})$ is equivalent to a multiplication operator $M_\varphi \in B(X(\Omega))$, which means $T = U^{-1} M_\varphi U$. It is easily seen that the linear span of $\{U^*\xi_\omega;\ \omega \in \Omega\}$ is weak-$*$ dense in $\mathcal{X}^*$ and that $T^* U^* \xi_\omega = \varphi(\omega) U^* \xi_\omega$ $(\omega \in \Omega)$. Thus, $\mathrm{Eig}(T^*) = \mathcal{X}^*$. We have proved one implication in the following theorem.

Theorem 2.11. *Let $\mathcal{X}$ be a Banach space. An operator $T \in B(\mathcal{X})$ satisfies $\mathrm{Eig}(T^*) = \mathcal{X}^*$ if and only if T is equivalent to a multiplication operator M_φ on a Banach space $X(\Omega)$ satisfying* (1).

Proof. Let Ω be the set of all eigenvectors of T^* of norm 1. For each $x \in \mathcal{X}$, let Ux be the complex function on Ω defined by $\big(Ux\big)(\omega) = \langle x, \omega \rangle$. Of course $X(\Omega) := \{Ux;\ x \in \mathcal{X}\}$ is a linear space of complex-valued functions on Ω and $U :\ x \mapsto Ux$ is a linear surjection from $\mathcal{X}$ to $X(\Omega)$. The map U is also injective since the weak-$*$ closed linear span of Ω is $\mathrm{Eig}(T^*) = \mathcal{X}^*$. If we equip $X(\Omega)$ with the norm $\|Ux\| := \|x\|$ $(x \in \mathcal{X})$, then $X(\Omega)$ becomes a Banach space satisfying (1) and U becomes an isometry, which means that $\mathcal{X}$ and $X(\Omega)$ are isometrically isomorphic Banach spaces. Define $\varphi :\ \Omega \to \mathbb{C}$ through $T^*\omega = \varphi(\omega)\omega$ and let $M_\varphi :\ X(\Omega) \to X(\Omega)$ be given by $\big(M_\varphi Ux\big)(\omega) = \varphi(\omega)\big(Ux\big)(\omega)$. Then

$$\big(M_\varphi Ux\big)(\omega) = \varphi(\omega)\langle x, \omega \rangle = \langle x, T^*\omega \rangle = \big(UTx\big)(\omega),$$

which gives $M_\varphi = UTU^{-1}$. Thus, M_φ is bounded and it is a multiplication operator equivalent to T. □

Corollary 2.12. *Let $A \in B(\mathcal{X})$, $B \in B(\mathcal{Y})$, and $e \in \mathcal{X}$ be arbitrary. If e and Ae are linearly independent, then* $\mathrm{I}(A, B; e)$ *is reflexive if and only if B is equivalent to a multiplication operator M_φ on a Banach space $X(\Omega)$ satisfying* (1).

Assume that a multiplication operator M_φ on $X(\Omega)$ (satisfying (1)) is also an algebraic operator. Let $m(z) = (z-\lambda_1)^{r_1}\cdots(z-\lambda_k)^{r_k}$ be the minimal polynomial. It is easily seen that the condition $m\big(M_\varphi\big) = 0$ is equivalent to the condition

$$\big(\varphi(\omega)-\lambda_1\big)^{r_1}\cdots\big(\varphi(\omega)-\lambda_k\big)^{r_k} = 0 \qquad \text{for all} \quad \omega \in \Omega.$$

However, $\big(\varphi(\omega)-\lambda_1\big)^{r_1}\cdots\big(\varphi(\omega)-\lambda_k\big)^{r_k} = 0$ if and only if $\big(\varphi(\omega)-\lambda_1\big)\cdots\big(\varphi(\omega)-\lambda_k\big) = 0$. Thus, if M_φ is an algebraic operator, then each zero of its minimal polynomial is simple. On the other hand, if $\varphi(\Omega) = \{\lambda_1, \ldots, \lambda_k\}$, then M_φ is an algebraic multiplication operator with the minimal polynomial $m(z) = (z-\lambda_1)\cdots(z-\lambda_k)$.

Corollary 2.13 (Cf. [1], Section 3)**.** *If $B \in B(\mathcal{Y})$ is an algebraic operator such that its minimal polynomial has only simple zeroes, then* $\mathrm{I}(A, B; e)$ *is reflexive for any $A \in B(\mathcal{X})$ and any $e \in \mathcal{X}$. On the other hand, if B is algebraic and* $\mathrm{I}(A, B; e)$ *is reflexive for an operator $A \in B(\mathcal{X})$ and a vector $e \in \mathcal{X}$ that is not an eigenvector for A, then the minimal polynomial of B has only simple zeroes.*

Proof. Let $m(z) = (z-\lambda_1)\cdots(z-\lambda_k)$ be the minimal polynomial of B (thus, $\lambda_i \neq \lambda_j$ if $i \neq j$). For each $1 \le i \le k$, let $q_i(z) := m(z)/(z-\lambda_i)$. Since $m(B) = 0$ we have $[\mathrm{im}\,(B-\lambda_i)] \subseteq \ker q_i(B)$ and consequently

$$\bigcap_{\lambda \in \mathbb{C}} [\mathrm{im}\,(B-\lambda)] \subseteq \bigcap_{i=1}^{k} [\mathrm{im}\,(B-\lambda_i)] \subseteq \bigcap_{i=1}^{k} \ker q_i(B).$$

However, the intersection $\bigcap_{i=1}^{k} \ker q_i(B)$ is trivial since the greatest common divisor of the polynomials q_i is equal to 1.

Conversely, suppose that $I(A, B; e)$ is reflexive for some A and e, such that e is not an eigenvector of A. Then B is equivalent to a multiplication operator M_φ, by Theorems 2.6 and 2.11. Of course, M_φ is an algebraic operator with the same minimal polynomial as B. By the observation above, we conclude that the minimal polynomial has only simple zeroes. □

Example 2.14. (1) An operator $B \in B(\mathcal{Y})$ will be called semi-shift if it is bounded below and $\bigcap_{n=1}^{\infty} \mathrm{im}\, B^n = \{0\}$. Any semi-shift satisfies the equivalent conditions of Theorem 2.6. Indeed, there is an open neighbourhood U of 0 such that $B - z$ is bounded below for $z \in U$. Then $\bigcap_{z \in U} \mathrm{im}\,(B-z) = \bigcap_{n=1}^{\infty} \mathrm{im}\, B^n = \{0\}$. Hence the spaces of intertwiners $\mathrm{I}(A, B; e)$ are reflexive for all $A \in B(\mathcal{X})$ and $e \in \mathcal{X}$, which gives the reflexivity of $\mathrm{I}(A, B)$ for any $A \in B(\mathcal{X})$.

(2) In particular, let $B \in B(\mathcal{H})$ be a unilateral weighted shift on a Hilbert space $\mathcal{H}$. Thus, $Be_i = w_i e_{i+1}$ $(i = 0, 1, \ldots)$, where $e_0, e_1, \ldots$ is an orthonormal basis for $\mathcal{H}$ and $w_i \in \mathbb{C}$ form a

bounded sequence. Suppose that $\operatorname{im} B$ is closed, i.e. $\inf_i |w_i| > 0$. Then B is a semi-shift and therefore it satisfies the conditions of Theorem 2.6.

Assumption that $\operatorname{im} B$ is closed is necessary. For example, let B be the weighted shift with weights $w_i = \frac{1}{i+1}$. Then $\|B^n\| = \frac{1}{n!}$ and so B is quasinilpotent. Hence $\bigcap_{z\in\mathbb{C}}[\operatorname{im}(B-z)] = [\operatorname{im} B] = \vee\{e_i;\ i \geq 1\}$ and B does not satisfy the conditions of Theorem 2.6.

(3) Let V be an isometry acting in a Hilbert space $\mathcal{H}$. Let $V = U \oplus S$ be the Wold decomposition of V, where U is unitary and S is a unilateral shift (of some multiplicity). Clearly, the commutant $\{U\}'$ is reflexive since it is a von Neumann algebra and $\{S\}'$ is reflexive by (1). However, in general $\{V\}'$ is not reflexive. For example, let U be the bilateral shift and S the unilateral shift. Then $V = U \oplus S$ may be represented as the operator of multiplication by z in $L^2 \oplus H^2$, where L^2 is considered with respect to Lebesgue measure on the unit circle and H^2 is the Hardy space. For $f_1, f_2 \in L^\infty$, $f_3 \in H^\infty$ the operator of multiplication by the matrix $\begin{bmatrix} f_1 & f_2 \\ 0 & f_3 \end{bmatrix}$ belongs to $\{V\}'$. For $g \in H^2$, $g \neq 0$, we have $\{V\}'(0 \oplus g) \supset \overline{gL^\infty} \oplus 0 = L^2 \oplus 0$. Hence for any $X \in B(H^2, L^2)$ the operator $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \in \operatorname{Ref}\{V\}'$ and $\{V\}'$ is not reflexive.

3. Hyperreflexivity of the space of locally intertwining operators

Let $\mathcal{S} \subseteq B(\mathcal{X}, \mathcal{Y})$ be a closed subspace. For an operator $T \in B(\mathcal{X}, \mathcal{Y})$, define

$$\alpha(T, \mathcal{S}) = \sup\{\operatorname{dist}(Tx, \mathcal{S}x);\ x \in \mathcal{X},\ \|x\| = 1\}.$$

The space $\mathcal{S}$ is said to be hyperreflexive if there is a constant $c > 0$ such that the inequality $\operatorname{dist}(T, \mathcal{S}) \leq c\,\alpha(T, \mathcal{S})$ holds for all $T \in B(\mathcal{X}, \mathcal{Y})$. It is well known that the hyperreflexivity is stronger condition than reflexivity, that is, each hyperreflexive space is reflexive. In this section we shall show that some spaces of local intertwiners can serve as natural examples of spaces that are reflexive but not hyperreflexive.

First we give a characterisation of hyperreflexive spaces of local intertwiners.

Proposition 3.1. *Let $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ be arbitary operators and assume that $Ae = \lambda e$ for some $\lambda \in \mathbb{C}$. Then $\mathrm{I}(A, B; e)$ is hyperreflexive.*

Proof. Without loss of generality we may assume that $\|e\| = 1$. Let $S \in B(\mathcal{X}, \mathcal{Y})$. By Proposition 2.1, we have $\alpha\big(S, \mathrm{I}(A, B; e)\big) = \operatorname{dist}\big(Se, \ker(B - \lambda)\big)$.

We shall prove that $\operatorname{dist}\big(S, \mathrm{I}(A, B; e)\big) = \operatorname{dist}\big(Se, \ker(B-\lambda)\big)$. Let $\varepsilon > 0$ and let $y \in \ker(B-\lambda)$ satisfy $\|Se - y\| < \operatorname{dist}\big(Se, \ker(B - \lambda)\big) + \varepsilon$. Let $y^* \in \mathcal{Y}^*$ satisfy $\langle e, y^*\rangle = 1 = \|y^*\|$. Define $S_0 \in B(\mathcal{X}, \mathcal{Y})$ by $S_0 e = Se - y$ and $S_0|_{\ker y^*} = 0$. Then $S - S_0 \in \mathrm{I}(A, B; e)$ and $\operatorname{dist}\big(S, \mathrm{I}(A, B; e)\big) \leq \|S_0\|$. Let $x \in \mathcal{X}$ have norm 1. Write $x = \alpha e + x_0$ with $\alpha \in \mathbb{C}$ and $x_0 \in \ker y^*$. Then

$$\begin{aligned} \|(S_0)x\| &= \|\alpha(S_0)e\| = |\langle x, y^*\rangle| \cdot \|Se - y\| \\ &\leq \|Se - y\| \leq \operatorname{dist}\big(Se, \ker(B - \lambda)\big) + \varepsilon. \end{aligned}$$

Hence $\operatorname{dist}\big(S, \mathrm{I}(A, B; e)\big) \leq \operatorname{dist}\big(Se, \ker(B - \lambda)\big)$. □

Lemma 3.2. *Let $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ be arbitary operators. Let $e \in \mathcal{X}$ and Ae be linearly independent. Then there exists a constant $k > 0$ such that for any $S \in B(\mathcal{X}, \mathcal{Y})$ it is possible to find $S_0 \in B(\mathcal{X}, \mathcal{Y})$ with the properties*

$$S_0 e = 0, \qquad S - S_0 \in \mathrm{I}(A, B; e) \qquad \text{and} \qquad \|S_0\| \le k\|SAe - BSe\|.$$

Consequently, $\operatorname{dist}\big(S, \mathrm{I}(A, B; e)\big) \le k\|SAe - BSe\|$.

Proof. Since e and Ae are linearly independent there exists $k > 0$ such that $|\beta| \le \frac{k}{2}\|\alpha e + \beta Ae\|$ for arbitrary $\alpha,\ \beta \in \mathbb{C}$. Choose and fix a projection $P \in B(\mathcal{X})$ whose image is $\vee\{e, Ae\}$ and $\|P\| \le 2$. Let $S \in B(\mathcal{X}, \mathcal{Y})$ be arbitrary. Now let $S_0 \in B(\mathcal{X}, \mathcal{Y})$ be defined by conditions

$$S_0 e = 0, \qquad S_0 Ae = SAe - BSe \qquad \text{and} \qquad S_0|_{\ker P} = 0.$$

Since $(S - S_0)Ae = SAe - SAe + BSe = B(S - S_0)e$, the operator $S - S_0$ is in $\mathrm{I}(A, B; e)$. Let $x \in \mathcal{X}$ be an arbitrary vector of norm 1 and let $x = \alpha e + \beta Ae + x_0$ with $x_0 \in \ker P$. Then

$$\begin{aligned}\|S_0 x\| = \|\beta S_0 Ae\| = |\beta| \cdot \|SAe - BSe\| &\le \frac{k}{2}\|\alpha e + \beta Ae\| \cdot \|SAe - BSe\| \\ &= \frac{k}{2}\|Px\| \cdot \|SAe - BSe\| \le k\|SAe - BSe\|.\end{aligned}$$

It follows now that $\operatorname{dist}\big(S, \mathrm{I}(A, B; e)\big) \le \|S_0\| \le k\|SAe - BSe\|$. □

Theorem 3.3. *Let $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$ be arbitary operators and assume that $e \in \mathcal{X}$ and Ae are linearly independent. Then $\mathrm{I}(A, B; e)$ is hyperreflexive if and only if there exists a number $\epsilon > 0$ such that* $\sup\{\operatorname{dist}\big(y, \operatorname{im}(B - \lambda)\big);\ \lambda \in \mathbb{C}\} > \epsilon$, *for all $y \in \mathcal{Y}$, $\|y\| = 1$.*

Proof. Withot loss of generality we assume that $\|e\| = 1$, $\|A\| \le 1$, and $\|B\| \le 1$.

Suppose that for any $\epsilon > 0$ there exists a vector $y_\epsilon \in \mathcal{Y}$ of norm one such that

$$\sup\{\operatorname{dist}\big(y_\epsilon, \operatorname{im}(B - \lambda)\big);\ \lambda \in \mathbb{C}\} < \epsilon. \tag{2}$$

Since e and Ae are linearly independent, there exists $\xi \in \mathcal{X}^*$ such that $\langle \xi, e\rangle = 0$ and $\langle \xi, Ae\rangle = 1$. Let $F_\epsilon := y_\epsilon \otimes \xi$. Thus F_ϵ is a rank-one operator that maps e to 0 and Ae to y_ϵ. We show that $\operatorname{dist}\big(F_\epsilon, \mathrm{I}(A, B; e)\big) \ge 1/2$. Towards contradiction suppose that there exists an operator $S \in \mathrm{I}(A, B; e)$ such that $\|F_\epsilon - S\| < 1/2$. Then $\|Se\| = \|F_\epsilon e - Se\| \le \|F_\epsilon - S\| < 1/2$ and therefore $\|SAe\| = \|BSe\| \le \|B\| \cdot \|Se\| < 1/2$. It follows that

$$\|(F_\epsilon - S)Ae\| = \|y_\epsilon - SAe\| \ge \|y_\epsilon\| - \|SAe\| > 1 - 1/2 = 1/2.$$

Since $\|Ae\| \le 1$ we conclude that $\|F_\epsilon - S\| > 1/2$, which contradicts to the assumption.

We have seen that for any $\epsilon > 0$ there exists a rank-one operator F_ϵ such that $\operatorname{dist}\big(F_\epsilon, \mathrm{I}(A, B; e)\big) \ge 1/2$. Now we shall estimate $\alpha\big(F_\epsilon, \mathrm{I}(A, B; e)\big)$.

If a vector $x \in \mathcal{X}$ is not in $[\{e, Ae\}]$, then $\mathrm{I}(A, B; e)x = \mathcal{Y}$, by Proposition 2.1. Thus, $\operatorname{dist}\big(F_\epsilon x, \mathrm{I}(A, B; e)x\big) = 0$ in this case.

Assume that $x = \alpha Ae + \beta e$, for some $\alpha,\ \beta \in \mathbb{C}$, and $\|x\| = 1$. Of course, there is a number $M > 0$ such that $M \ge |\alpha|$ for all $\alpha \in \mathbb{C}$ that satisfy condition $\|\alpha Ae + \beta e\| = 1$ for some $\beta \in \mathbb{C}$. Note that M does not depend on ϵ. By Proposition 2.1, if $x = \alpha Ae + \beta e$, then $\mathrm{I}(A, B; e)x =$

$\operatorname{im}(\alpha B+\beta)$. Thus, $\operatorname{dist}\big(F_\epsilon x, \mathrm{I}(A,B;e)x\big) = \operatorname{dist}\big(\alpha y_\epsilon, \operatorname{im}(\alpha B+\beta)\big) \le M \operatorname{dist}\big(y_\epsilon, \operatorname{im}(\alpha B+\beta)\big)$ and therefore, by (2), $\operatorname{dist}\big(F_\epsilon x, \mathrm{I}(A,B;e)x\big) < M\epsilon$. We conclude that $\alpha\big(F_\epsilon, \mathrm{I}(A,B;e)\big) < M\epsilon$. Now, since $\lim_{\epsilon\to 0}\alpha\big(F_\epsilon, \mathrm{I}(A,B;e)\big) = 0$ and $\operatorname{dist}\big(F_\epsilon, \mathrm{I}(A,B;e)\big) \ge 1/2$ for any $\epsilon > 0$, the space $\mathrm{I}(A,B;e)$ is not hyperreflexive.

For the opposite implication, let $S \in B(\mathcal{X},\mathcal{Y})$ be arbitrary and let $S_0 \in B(\mathcal{X},\mathcal{Y})$ be an operator that satisfies the conditions from Lemma 3.2, so $\operatorname{dist}\big(S, \mathrm{I}(A,B;e)\big) \le \|S_0\| \le k\|SAe - BSe\|$. Since $S - S_0 \in \mathrm{I}(A,B;e)$ we have $\alpha\big(S, \mathrm{I}(A,B;e)\big) = \alpha\big(S_0, \mathrm{I}(A,B;e)\big)$. By the assumption, there exists $\lambda \in \mathbb{C}$ such that $\operatorname{dist}\big(S_0Ae, \operatorname{im}(B-\lambda)\big) \ge \varepsilon\|S_0Ae\|$. Clearly $\lambda \in \sigma(B)$, and so $|\lambda| \le \|B\|$. Note also that $\mathrm{I}\big(A,B;e\big)(Ae-\lambda e) = \operatorname{im}(B-\lambda)$, by Proposition 2.1. So we have

$$\begin{aligned}\alpha\big(S, \mathrm{I}(A,B;e)\big) &= \alpha\big(S_0, \mathrm{I}(A,B;e)\big) \ge \|Ae-\lambda e\|^{-1}\operatorname{dist}\big(S_0(Ae-\lambda e), \mathrm{I}(A,B;e)(Ae-\lambda e)\big)\\ &\ge \frac{\operatorname{dist}\big(S_0Ae, \operatorname{im}(B-\lambda)\big)}{(\|A\|+\|B\|)\|e\|} \ge \frac{\varepsilon\|S_0Ae\|}{(\|A\|+\|B\|)\|e\|}.\end{aligned}$$

Recall that $S_0Ae = SAe - BSe$ (see proof of the claim) and so $\alpha\big(S, \mathrm{I}(A,B;e)\big) \ge c\|SAe - BSe\|$, where $c = \frac{\varepsilon}{(\|A\|+\|B\|)\|e\|}$. ☐

Example 3.4. Let $\mathcal{Y} = \ell^2$ and let $B \in B(\ell^2)$ be given by

$$B:\ (x_1,\ x_2,\ x_3,\dots) \mapsto \big(x_1,\ \frac{1}{2}x_2,\ \frac{1}{3}x_3,\dots\big).$$

It is easily seen that $\operatorname{im}(B-\frac{1}{n}) = \{(x_i)\in\ell^2;\ x_n = 0\}$, for any $n\in\mathbb{N}$, and that $\operatorname{im}(B-\lambda) = \ell^2$ if $\lambda \ne \frac{1}{n}$ $(\forall n\in\mathbb{N})$. Thus B satisfies condition (iii) of Theorem 2.6 and we conclude that $\mathrm{I}(A,B;e)$ is reflexive for any Banach space $\mathcal{X}$ and arbitary $A\in B(\mathcal{X})$ and $e\in\mathcal{X}$. On the other hand, these spaces are hyperreflexive if and only if e is eigenvector of A or $e=0$. Namely, we shall see that B does not satisfy condition (ii) of Theorem 3.3.

For a positive integer k, let $f^{(k)} = (f_j^{(k)}) \in \ell^2$ be given by

$$f_j^{(k)} = \begin{cases} \frac{1}{k}; & 1\le j\le k^2\\ 0; & k^2 < j.\end{cases}$$

Then $\|f^{(k)}\| = 1$ and $f^{(k)} \in \operatorname{im}(B-\lambda)$ if $\lambda \notin \{1,\ \frac{1}{2},\ \dots,\ \frac{1}{k^2}\}$. Thus $\operatorname{dist}\big(f^{(k)}, \operatorname{im}(B-\lambda)\big) = 0$ if $\lambda \notin \{1,\ \frac{1}{2},\ \dots,\ \frac{1}{k^2}\}$. For $1\le n\le k^2$ we have

$$\operatorname{dist}\big(f^{(k)}, \operatorname{im}(B-\frac{1}{n}\big) = \min\{\|f^{(k)} - (x_j)\|;\quad x_n = 0\} = \frac{1}{k}.$$

We conclude that

$$\sup\{\operatorname{dist}\big(f^{(k)}, \operatorname{im}(B-\lambda)\big);\ \lambda\in\mathbb{C}\} = \frac{1}{k},$$

which means that condition (ii) of Theorem 3.3 is not fulfilled.

4. k-reflexivity and k-hyperreflexivity of the space of locally intertwining operators

Let $\mathcal{X}$ and $\mathcal{Y}$ be complex Banach spaces and let $F(\mathcal{Y},\mathcal{X})$ be the space of all operators of finite rank from $\mathcal{Y}$ to $\mathcal{X}$, that is the linear span of all operators of finite rank. Thus, an operator $F\in B(\mathcal{Y},\mathcal{X})$ is of finite rank if and only if there exist a positive integer n and $x_1,\dots,x_n\in\mathcal{X}$,

$\eta_1, \dots, \eta_n \in \mathcal{Y}^*$ such that $F = x_1 \otimes \eta_1 + \cdots + x_n \otimes \eta_n$. The pair $\big(B(\mathcal{X},\mathcal{Y}), F(\mathcal{Y},\mathcal{X})\big)$ is a dual pair via the pairing

$$\langle T, F\rangle = \langle Tx_1, \eta_1\rangle + \cdots + \langle Tx_n, \eta_n\rangle,$$

where $T \in B(\mathcal{X},\mathcal{Y})$ and $F = x_1 \otimes \eta_1 + \cdots + x_n \otimes \eta_n \in F(\mathcal{Y},\mathcal{X})$ are arbitrary. If $\mathcal{U} \subseteq B(\mathcal{X},\mathcal{Y})$, then let $\mathcal{U}^\perp := \{F \in F(\mathcal{Y},\mathcal{X});\quad \langle S, F\rangle = 0 \quad \text{for all} \quad S \in \mathcal{U}\}$ and, similarly, for $\mathcal{W} \subseteq F(\mathcal{Y},\mathcal{X})$, let $\mathcal{W}_\perp := \{S \in B(\mathcal{X},\mathcal{Y});\quad \langle S, F\rangle = 0 \quad \text{for all} \quad F \in \mathcal{W}\}$.

For a positive integer k, let $F_k(\mathcal{Y},\mathcal{X}) \subseteq F(\mathcal{Y},\mathcal{X})$ be the subset of all operators from $\mathcal{Y}$ to $\mathcal{X}$ whose rank is at most k. Since $F_k(\mathcal{Y},\mathcal{X})_\perp = \{0\}$ and $F_k(\mathcal{Y},\mathcal{X})$ is closed under multiplication by the scalars, $\big(B(\mathcal{X},\mathcal{Y}),\ F(\mathcal{Y},\mathcal{X}),\ F_k(\mathcal{Y},\mathcal{X})\big)$ satisfies the conditions of a reflexive triple (over $\mathbb{C}$) in the sense of [4]. Thus, for a linear subspace $\mathcal{S} \subseteq B(\mathcal{X},\mathcal{Y})$ we define the k-reflexive cover of $\mathcal{S}$ as $\mathrm{Ref}_k\mathcal{S} := \big(\mathcal{S}^\perp \cap F_k(\mathcal{Y},\mathcal{X})\big)_\perp$. The sets $\mathrm{Ref}_k\mathcal{S}$ are linear subspaces of $B(\mathcal{X},\mathcal{Y})$ closed in the weak operator topology. Of course, $\mathcal{S} \subseteq \mathrm{Ref}_k\mathcal{S}$ and $\mathcal{S}$ is said to be k-reflexive if $\mathcal{S} = \mathrm{Ref}_k\mathcal{S}$. Clearly, the 1-reflexivity coincides with the notion of reflexivity. The reader is referred to [4] for details; especially for the relation to the classical notion of a reflexive algebra.

Let $\mathcal{S} \subseteq B(\mathcal{X},\mathcal{Y})$ be a weakly closed subspace such that $\mathcal{S} = \mathcal{W}_\perp$ with $\mathcal{W} \subseteq F_k(\mathcal{Y},\mathcal{X})$. Then $\mathcal{S}^\perp \cap F(\mathcal{Y},\mathcal{X}) = (\mathcal{W}_\perp)^\perp \cap F(\mathcal{Y},\mathcal{X}) \supseteq \mathcal{W}$ and consequently $\mathrm{Ref}_k\mathcal{S} = \big(\mathcal{S}^\perp \cap F(\mathcal{Y},\mathcal{X})\big)_\perp \subseteq \mathcal{W}_\perp = \mathcal{S}$. It follows that $\mathcal{S}$ is k-reflexive. On the other hand, if $\mathcal{S}$ is k-reflexive, then $\mathcal{S} = \mathcal{W}_\perp$ with $\mathcal{W} = \mathcal{S}^\perp \cap F_k(\mathcal{Y},\mathcal{X}) \subseteq F_k(\mathcal{Y},\mathcal{X})$. Thus, $\mathcal{S}$ is k-reflexive if and only if there is a subset $\mathcal{W} \subseteq F_k(\mathcal{Y},\mathcal{X})$ such that $\mathcal{S} = \mathcal{W}_\perp$.

Proposition 4.1. *For arbitrary $A \in B(\mathcal{X})$, $B \in B(\mathcal{Y})$, and $e \in \mathcal{X}$, the subspace* $\mathrm{I}(A,B;e) \subseteq B(\mathcal{X},\mathcal{Y})$ *is 2-reflexive.*

Proof. It is obvious that an operator $S \in B(\mathcal{X},\mathcal{Y})$ satisfies $SAe = BSe$ if and only if $\langle S, Ae \otimes \eta - e \otimes B^*\eta\rangle = 0$ holds for all $\eta \in \mathcal{Y}^*$. Thus, $\mathrm{I}(A,B;e) = G(A,B;e)_\perp$, where $G(A,B;e) := \{Ae \otimes \eta - e \otimes B^*\eta;\quad \eta \in \mathcal{Y}^*\} \subseteq F_2(\mathcal{Y},\mathcal{X})$. ☐

Let $\mathcal{S} \subseteq B(\mathcal{X},\mathcal{Y})$ be a subspace and $T \in B(\mathcal{X},\mathcal{Y})$. For a positive integer k, define

$$\alpha_k(T,\mathcal{S}) = \sup\{\inf_{A\in\mathcal{S}} \sum_{i=1}^k \|Tx_i - Ax_i\|;\ x_1,\dots,x_k \in \mathcal{X},\ \|x_1\| + \cdots + \|x_k\| = 1\}.$$

In particular, for $k = 1$, we have $\alpha_1(T,\mathcal{S}) = \alpha(T,\mathcal{S})$. The space $\mathcal{S}$ is said to be k-hyperreflexive if the seminorms $\mathrm{dist}(\cdot,\mathcal{S})$ and $\alpha_k(\cdot,\mathcal{S})$ are equivalent.

Again, the notion of 1-hyperreflexivity coincides with that of hyperreflexivity.

Denote by dist_1 the distance in the space $\mathcal{Y}^k$ (the ℓ_1-direct sum of k copies of $\mathcal{Y}$). We have

$$\begin{aligned}
\alpha_k(T,\mathcal{S}) &= \sup_{\substack{x_1,\dots,x_k\in\mathcal{X}\\ \|x_1\|+\cdots+\|x_k\|=1}} \mathrm{dist}_1\big((Tx_1,\dots,Tx_k),\{(Ax_1,\dots,Ax_k);\ A\in\mathcal{S}\}\big)\\
&= \sup_{\substack{x_1,\dots,x_k\in\mathcal{X}\\ \|x_1\|+\cdots+\|x_k\|=1}} \sup_{\substack{y_1^*,\dots,y_k^*\in\mathcal{Y}^*\\ \|y_1^*\|\le1,\dots,\|y_k^*\|\le1}} \left\{\left|\sum_{i=1}^k \langle Tx_i, y_i^*\rangle\right|;\quad \sum_{i=1}^k \langle Ax_i, y_i^*\rangle = 0 \quad \text{for all} \quad A\in\mathcal{S}\right\}\\
&= \sup_{\substack{F\in F_k(\mathcal{Y},\mathcal{X})\\ \|F\|_1\le1}} |\langle T, F\rangle|.
\end{aligned}$$

Thus, this definition agrees with that given by Kliś and Ptak in [6] for Hilbert spaces.

Theorem 4.2. *For arbitrary* $A \in B(\mathcal{X})$, $B \in B(\mathcal{Y})$, *and* $e \in \mathcal{X}$, *the subspace* $\mathrm{I}(A, B; e) \subseteq B(\mathcal{X}, \mathcal{Y})$ *is* 2*-hyperreflexive.*

Proof. If e is an eigenvector of A, then the space $\mathrm{I}(A, B; e)$ is even hyperreflexive, by Proposition 3.1.

Assume that the vectors e and Ae are linearly independent and let $T \in B(\mathcal{X}, \mathcal{Y})$ be arbitrary. By Lemma 3.2, there is a constant $k > 0$ such that $\mathrm{dist}\big(T, \mathrm{I}(A, B; e)\big) \le k\|TAe - BTe\|$. On the other hand, let $y^* \in \mathcal{Y}^*$ satisfy $\|y^*\| = 1$ and $\langle TAe - BTe, y^*\rangle = \|TAe - BTe\|$. We have

$$\begin{aligned}\alpha_2\big(T, \mathrm{I}(A, B, e)\big) &\ge \|Ae \otimes y^* - e \otimes B^* y^*\|_1^{-1} |\langle T, Ae \otimes y^* - e \otimes B^* y^*\rangle| \\ &\ge \big((\|A\| + \|B\|)\|e\|\big)^{-1} |\langle TAe - BTe, y^*\rangle| = \big((\|A\| + \|B\|)\|e\|\big)^{-1} \|TAe - BTe\|.\end{aligned}$$

Hence $I(A, B; e)$ is 2-hyperreflexive. □

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