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The effective boundary conditions for vector fields on domains with rough boundaries: Applications to fluid mechanics

Eduard Feireisl* Šárka Nečasová†

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic

1 Introduction

Consider a vicous incompressible fluid occupying a bounded domain $\Omega \subset R^3$. In the Eulerean reference system, the motion of the fluid is completely determined by the velocity field $\mathbf{u} = \mathbf{u}(t, x)$ - a vector valued function of the time t and the spatial position $x \in \Omega$. Under the hypothesis of impermeability of the boundary, the velocity satisfies

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1.1}$$

where the symbol $\mathbf{n}$ stands for the outer normal vector. In addition to (1.1), the widely accepted hypothesis asserts there is no relative motion between a *viscous* fluid and the rigid wall represented by $\partial\Omega$, meaning

$$[\mathbf{u}]_\tau|_{\partial\Omega} = 0, \tag{1.2}$$

where $[\mathbf{u}]_\tau$ denotes the tangential component of $\mathbf{u}$. The *no-slip* boundary conditions (1.1), (1.2) are the mostly accepted because of their enormous success in reproducing the velocity profiles for macroscopic flows.

On the other hand, recent developments in micro and nanofluidic technologies have renewed interest in the influence of wall roughness on the slip behavior of viscous fluids (see the survey by Priezjev and Troian [12]). As a matter of fact, correctness of the no-slip hypothesis (1.2) has been subjected to discussion for over two centuries by many distinguished scientists. Navier suggested to replace (1.2) by a general relation

$$[\mathbb{S}\mathbf{n}]_\tau + \beta[u]_\tau = 0 \text{ on } \partial\Omega, \tag{1.3}$$

where $\mathbb{S}$ is the viscous stress tensor and β represents a friction coefficient. The case $\beta = 0$ is termed *complete slip*, while (1.3) reduces to (1.2) in the asymptotic limit

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$\beta \to \infty$. Although intuitively more accurate, the Navier slip conditions have been often replaced by (1.2) as the slip length for most fluid motions is likely to be too small to influence the motion on the macroscopic scale. However, numerous experiments as well as theoretical studies have recently shown that the no-slip hypothesis may not be correct when the walls are sufficiently smooth (see Priezjev et al [11], Qiang and Wang [13], among others).

There have been several attempts to justify the no-slip boundary conditions as an inevitable consequence of fluid trapping by surface rougness (see Richardson [14], Janson [7], and, more recently, Amirat et al. [1], Casado-Díaz et al. [5]). On the other hand, in order to simplify a complicated description of the fluid behavior in a boundary layer, the Navier boundary conditions or other so-called wall laws are used instead of (1.2) to facilitate numerical computations (see Jaeger and Mikelic [6], Mohammadi et al. [9]).

Following the programme originated in [2], [3] we consider a family of bounded domains $\{\Omega_\varepsilon\}_{\varepsilon>0}$,

$$\Omega_\varepsilon = \{(x_1, x_2, x_3) \mid (x_1, x_2) \in \mathcal{T}^2,\ 0 < x_3 < 1 + \Phi_\varepsilon(x_1, x_2)\}, \tag{1.4}$$

where the symbol $\mathcal{T}^2 = \Big([0,1]|_{\{0,1\}}\Big)^2$ denotes the two-dimensional torus. In other words, all quantities defined on Ω_ε are periodic with respect to the "horizontal" variables (x_1, x_2). Motivated by physical experiments reported in [11], [13], we assume that the functions Φ_ε depend only on a single spatial, say, $\Phi_\varepsilon = \Phi_\varepsilon(x_1)$ mimicking a ribbed surface, where the amplitude as well as a typical wavelength of oscillations are small for ε approaching zero.

We assume that the time evolution of the fluid velocity is governed by the Navier-Stokes system:

$$\mathrm{div}_x \mathbf{u} = 0 \text{ in } (0,T) \times \Omega_\varepsilon, \tag{1.5}$$

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x P = \mathrm{div}_x \mathbb{S} \text{ in } (0,T) \times \Omega_\varepsilon, \tag{1.6}$$

where P is the pressure and the viscous stress tensor $\mathbb{S}$ is given by the classical Newton's rheological law

$$\mathbb{S} = \mu(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}), \tag{1.7}$$

with the constant viscosity coefficient $\mu > 0$. System (1.5 - 1.7) is supplemented with the complete slip boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\{x_3=0\}} = 0,\ [\mathbb{S}\mathbf{n}]_\tau|_{\{x_3=0\}} = 0, \tag{1.8}$$

$$\mathbf{u} \cdot \mathbf{n}|_{\{x_3=1+\Phi_\varepsilon(x_1,x_2)\}} = 0,\ [\mathbb{S}\mathbf{n}]_\tau|_{\{x_3=1+\Phi_\varepsilon(x_1,x_2)\}} = 0. \tag{1.9}$$

Following the approach developed in [3] we introduce a parametrized rugosity measure generated by the family of upper boundaries $\{x_3 = 1 + \Phi_\varepsilon(x_1, x_2)\}_{\varepsilon>0}$ and identify the limit problem associated to (1.5 - 1.9) for ε tending to zero. In particular, any accumulation point $\mathbf{u}$ of a family of solutions $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ of problem (1.5 - 1.9) satisfies (1.5 - 1.7) on the limit domain

$$\Omega = \mathcal{T}^2 \times (0,1),$$

together with the complete slip boundary condition (1.8) on the bottom part of the boundary $\{x_3 = 0\}$. In addition, the limit velocity $\mathbf{u}$ on the upper boundary is parallel to the riblets, specifically,

$$\mathbf{u}|_{\{x_3=1\}} = (0, u_2, 0),\ \text{and } S_{2,3}|_{\{x_3=1\}} = 0. \tag{1.10}$$

The main result obtained in this paper can be viewed as an extension of the theory developed in [2] to the time-dependent case. Similarly to [3], the main difficulty is to handle possible oscillations in time of the sequence $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ resulting in the lack of compactness of the convective terms $\{\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\}_{\varepsilon>0}$. In order to overcome this stumbling block, we introduce a local pressure in the spirit of Wolf [16] (cf. also Koch and Solonnikov [8]). Although strongly motivated by [16], our construction of the local pressure is different, based on the Riesz transform rather than on the biharmonic decomposition introduced in [16]. The main advantage of our approach lies in the fact that the norm of the local pressure is independent of the parameter ε.

2 Main result

To begin, let us recall the concept of *weak solution* to problem (1.5 - 1.9).

Definition 2.1 *A function $\mathbf{u}_\varepsilon$ is termed a weak solution to problem (1.5 - 1.9) if*

$$\mathbf{u}_\varepsilon \in L^\infty(0,T;L^2(\Omega_\varepsilon;R^3)) \cap L^2(0,T;W^{1,2}(\Omega_\varepsilon;R^3)); \tag{2.1}$$

$$\mathrm{div}_x \mathbf{u}_\varepsilon(t,\cdot) = 0 \ , \ \mathbf{u}_\varepsilon(t,\cdot)\cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0 \ \textit{for a.a. } t \in (0,T); \tag{2.2}$$

the integral identity

$$\int_0^T \int_{\Omega_\varepsilon} \Big(\mathbf{u}_\varepsilon \cdot \partial_t \varphi + \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \varphi + P_\varepsilon \mathrm{div}_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \tag{2.3}$$

$$= \int_0^T \int_{\Omega_\varepsilon} \mu \Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon \Big) : \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t$$

holds for a certain $P_\varepsilon \in L^q((0,T)\times\Omega_\varepsilon)$, $q>1$, and any test function $\varphi \in \mathcal{D}((0,T)\times \overline{\Omega}_\varepsilon;R^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0$;
the energy inequality

$$\int_{\Omega_\varepsilon} \frac{1}{2}|\mathbf{u}_\varepsilon|^2(\tau) \ \mathrm{d}x + \int_0^\tau \int_{\Omega_\varepsilon} \frac{\mu}{2} |\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon|^2 \ \mathrm{d}x \ \mathrm{d}t \le E_{0,\varepsilon} \tag{2.4}$$

is satisfied for a.a. $\tau \in (0,T)$.

Remark: Note that Definition 2.1 anticipates the existence of the pressure P_ε as an integrable function. On the other hand, the *existence* of weak solutions belonging to the class specified in Definition 2.1 can be established for a fairly general set of initial data by the method developed by Bulíček et al. [4].

Similarly, we introduce the concept of *weak solution* of the limit problem as follows.

Definition 2.2 *We shall say that a function $\mathbf{u}$ is a weak solution of problem (1.5 - 1.8), and (1.10) if*

$$\mathbf{u} \in L^\infty(0,T;L^2(\Omega;R^3)) \cap L^2(0,T;W^{1,2}(\Omega;R^3)); \tag{2.5}$$

$$\mathrm{div}_x \mathbf{u}(t,\cdot) = 0, \ \mathbf{u}(t,\cdot)\cdot\mathbf{n}|_{\{x_3=0\}} = 0, \tag{2.6}$$

$$u_1|_{\{x_3=1\}} = u_3|_{\{x_3=1\}} = 0; \tag{2.7}$$

the integral identity

$$\int_0^T \int_\Omega \Big(\mathbf{u} \cdot \partial_t \varphi + (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \mu \Big(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} \Big) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \tag{2.8}$$

holds for any test function $\varphi \in \mathcal{D}((0,T) \times \overline{\Omega}; R^3)$,

$$\mathrm{div}_x \varphi = 0, \ \varphi \cdot \mathbf{n}|_{\{x_3=0\}}, \ \varphi_1|_{\{x_3=1\}} = \varphi_3|_{\{x_3=1\}} = 0. \tag{2.9}$$

At this stage, we are ready to state our main result.

Theorem 2.1 *Let a family of domains* $\{\Omega_\varepsilon\}_{\varepsilon>0}$ *be given by (1.4), with* $\Phi_\varepsilon = \Phi_\varepsilon(x_1)$ *such that*

$$\Phi_\varepsilon \in W^{1,\infty}(\mathcal{T}^1), \ \mathcal{T}^1 = [0,1]|_{\{0,1\}}, \ 0 \le \Phi_\varepsilon \le \varepsilon, \ |\Phi_\varepsilon'| \le L, \tag{2.10}$$

$$\liminf_{\varepsilon \to 0} \int_a^b |\Phi_\varepsilon'(z)| \, \mathrm{d}z \ge \lambda |a-b| \ \textit{for arbitrary} \ a \le b, \ a, b \in \mathcal{T}^1, \tag{2.11}$$

for a certain $\lambda > 0$.

Let $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ *be a family of weak solutions of problem (1.5 - 1.9) in the sense of Definition 2.1 such that*

$$\sup_{\varepsilon>0} E_{0,\varepsilon} = \overline{E} < \infty. \tag{2.12}$$

Then, passing to a subsequence as the case may be, we have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \ \textit{weakly-(*) in} \ L^\infty(0,T; L^2(\Omega; R^3)) \ \textit{and weakly in} \ L^2(0,T; W^{1,2}(\Omega; R^3)), \tag{2.13}$$

where $\mathbf{u}$ *is a weak solution of problem (1.5 - 1.8, 1.10) in the sense specified in Definition 2.2 .*

Remark: The non-degeneracy condition (2.11) is satisfied in a number of interesting particular cases discussed in [2].

The rest of the paper is devoted to the proof of Theorem 2.1.

3 Identifying the limit velocity field

In accordance with the energy inequality (2.4) and hypothesis (2.12), we have

$$\mathrm{ess} \sup_{t \in (0,T)} \|\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon; R^3)} \le c \tag{3.1}$$

and

$$\int_0^T \int_{\Omega_\varepsilon} |\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \le c \tag{3.2}$$

uniformly for $\varepsilon \to 0$.

Estimates (3.1), (3.2), together with Korn's inequality, give rise to

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega_\varepsilon;R^3)} \, \mathrm{d}t \leq c. \tag{3.3}$$

Note that, by virtue of the result of Nitsche [10] and hypothesis (2.10), the bound established in (3.3) is independent of ε.

Consequently, in accordance with (3.1), (3.3), we can assume

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly-(*) in } L^\infty(0,T;L^2(\Omega;R^3)) \text{ and weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)) \tag{3.4}$$

passing to suitable subsequences as the case may be. Moreover, it is easy to check that

$$\mathrm{div}_x \mathbf{u} = 0 \text{ a.a. in } (0,T)\times\Omega,$$

and

$$\mathbf{u}\cdot\mathbf{n}|_{\{x_3=0\}} = u_3|_{\{x_3=0\}} = 0.$$

Finally, exactly as in [2, Section 3], we can show that hypotheses (2.10), (2.11) imply that the limit velocity field $\mathbf{u}$ satisfies

$$u_1|_{\{x_3=1\}} = u_3|_{\{x_3=1\}} = 0.$$

4 Identifying the limit equations

4.1 Pressure

Our ultimate goal is to identify the limit system of equations satisfied by $\mathbf{u}$. Here, the major problem is to control the pressure term P_ε in (2.3). In general, we do not expect to obtain any uniform bound on $\{P_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon \to 0$, however, we claim the following result.

Lemma 4.1 *Under the hypotheses of Theorem 2.1, there exists a pair of functions* $p_{\mathrm{reg},\varepsilon}$, $p_{\mathrm{harm},\varepsilon}$ *such that*

$$\int_0^T\int_{\Omega_\varepsilon} P_\varepsilon \mathrm{div}_x\varphi \, \mathrm{d}\mathbf{x}\, \mathrm{d}t = \int_0^T\int_{\Omega_\varepsilon}\Big(p_{\mathrm{reg},\varepsilon}\mathrm{div}_x\varphi + p_{\mathrm{harm},\varepsilon}\partial_t\mathrm{div}_x\varphi\Big)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t \tag{4.1}$$

for any $\varphi \in \mathcal{D}((0,T)\times\Omega_\varepsilon;R^3)$, *where*

$$\|p_{\mathrm{reg},\varepsilon}\|_{L^{3/2}((0,T)\times\Omega_\varepsilon)} \leq c_1(\overline{E}), \tag{4.2}$$

$$\Delta_x p_{\mathrm{harm},\varepsilon} = 0 \text{ in } \mathcal{D}'((0,T)\times\Omega_\varepsilon),\ \ \|p_{\mathrm{harm},\varepsilon}\|_{L^\infty(0,T;L^2(\Omega_\varepsilon;R^3))} \leq c_2(\overline{E}), \tag{4.3}$$

with the quantities c_1, c_2 *independent of the parameter* ε.

Proof:
The "regular" component of the pressure $p_{\mathrm{reg},\varepsilon}$ is uniquely determined as

$$p_{\mathrm{reg},\varepsilon} = -\sum_{i,j=1}^3 \mathcal{R}_i\mathcal{R}_j[1_{\Omega_\varepsilon}T^\varepsilon_{i,j}], \tag{4.4}$$

where we have set

$$\mathbb{T}^\varepsilon = \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - \mu\Big(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon\Big),$$

and where the symbol $\mathcal{R}$ stands for the standard Riesz transform in the $x-$variable:

$$\mathcal{R}_j[v](x) = \mathcal{F}^{-1}_{\xi\to x}\Big[\mathrm{i}\frac{\xi_j}{|\xi|}\mathcal{F}_{x\to\xi}[v]\Big],\ j=1,\dots,2,$$

with $\mathcal{F}$ denoting the Fourier transform.

Using the uniform bounds (3.1), (3.3), together with continuity of the Riesz transform in the Lebesgue spaces $L^p(R^3)$, $1<p<\infty$, we deduce that $p_{\mathrm{reg},\varepsilon}$ satisfies

$$\|p_{\mathrm{reg},\varepsilon}\|_{L^1(0,T;L^3(\Omega_\varepsilon))} + \|p_{\mathrm{reg},\varepsilon}\|_{L^\infty(0,T;L^1(\Omega_\varepsilon))} \le c_1(E);$$

whence (4.2) follows by interpolation. Note that we have used the Sobolev embedding relation $W^{1,2}(\Omega_\varepsilon) \hookrightarrow L^p(\Omega_\varepsilon)$, $1\le p\le 6$, the norm of which is independent of ε.

As $\mathbf{u}_\varepsilon$ satisfy (2.3), we have

$$\mathbf{u}_\varepsilon \in C_{\mathrm{weak}}([0,T];L^2(\Omega_\varepsilon;R^3)),$$

in particular, it follows from (2.3) that

$$\int_{\Omega_\varepsilon}\Big(\mathbf{u}_\varepsilon(\tau,\cdot)-\mathbf{u}_\varepsilon(0,\cdot)\Big)\cdot\psi\ \mathrm{d}\mathbf{x} - \int_{\Omega_\varepsilon}\Big(\int_0^\tau \mathbb{T}^\varepsilon\ \mathrm{d}t\Big):\nabla_x\psi\ \mathrm{d}\mathbf{x} = 0$$

for all $\psi\in\mathcal{D}(\Omega_\varepsilon;R^3)$, $\mathrm{div}_x\psi=0$ and all $\tau\in[0,T]$.

Thus, by virtue of Lemma 2.2.1 in Sohr [15], there exists a pressure p_ε such that

$$\int_{\Omega_\varepsilon} p_\varepsilon(\tau,\cdot)\ \mathrm{d}\mathbf{x} = 0,$$

and

$$\int_{\Omega_\varepsilon}\Big(\mathbf{u}_\varepsilon(\tau,\cdot)-\mathbf{u}_\varepsilon(0,\cdot)\Big)\cdot\psi\ \mathrm{d}\mathbf{x} - \int_{\Omega_\varepsilon}\Big(\int_0^\tau \mathbb{T}^\varepsilon\ \mathrm{d}t\Big):\nabla_x\psi\ \mathrm{d}\mathbf{x} + \int_{\Omega_\varepsilon} p_\varepsilon(\tau,\cdot)\mathrm{div}_x\psi\ \mathrm{d}\mathbf{x} = 0 \tag{4.5}$$

for all $\psi\in\mathcal{D}(\Omega_\varepsilon;R^3)$, and all $\tau\in[0,T]$. Exactly as in Sections 4, 5 in [2], we can deduce from (4.5) that

$$\sup_{\tau\in[0,T]}\|p_\varepsilon(\tau,\cdot)\|_{L^2(\Omega_\varepsilon)} \le c_2(\overline{E})$$

uniformly with respect to ε.

It follows from (4.5) that

$$\int_0^T\int_{\Omega_\varepsilon}\Big(\mathbf{u}_\varepsilon\cdot\partial_t\varphi + (\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon):\nabla_x\varphi + p_\varepsilon\partial_t\mathrm{div}_x\varphi\Big)\ \mathrm{d}\mathbf{x}\ \mathrm{d}t$$

$$= \int_0^T\int_{\Omega_\varepsilon}\mu\Big(\nabla_x\mathbf{u}_\varepsilon + \nabla_x^t\mathbf{u}_\varepsilon\Big):\nabla_x\varphi\ \mathrm{d}\mathbf{x}\ \mathrm{d}t$$

for all $\varphi\in\mathcal{D}((0,T)\times\Omega_\varepsilon;R^3)$; whence, in accordance with (2.3),

$$\int_0^T\int_{\Omega_\varepsilon} P_\varepsilon\mathrm{div}_x\varphi\ \mathrm{d}\mathbf{x}\ \mathrm{d}t = \int_0^T\int_{\Omega_\varepsilon} p_\varepsilon\partial_t\mathrm{div}_x\varphi\ \mathrm{d}\mathbf{x}\ \mathrm{d}t. \tag{4.6}$$

for all $\varphi \in \mathcal{D}((0,T)\times\Omega_\varepsilon; R^3)$.

Finally, we set

$$p_{\mathrm{harm},\varepsilon}(\tau,\cdot) = p_\varepsilon(\tau,\cdot) - \int_0^\tau \Big(p_{\mathrm{reg},\varepsilon} - \frac{1}{|\Omega_\varepsilon|}\int_{\Omega_\varepsilon} p_{\mathrm{reg},\varepsilon}\ \mathrm{d}\mathbf{x} \Big)\ \mathrm{d}t. \tag{4.7}$$

As relation (4.1) follows from (4.6), it remains to show that $p_{\mathrm{harm},\varepsilon}$ is a harmonic function in the $x-$variable. In order to see this, we use (4.4) to obtain

$$\int_{\Omega_\varepsilon} p_{\mathrm{reg},\varepsilon}\Delta\varphi\ \mathrm{d}\mathbf{x} = \int_{\Omega_\varepsilon} \Big[\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon - \mu\Big(\nabla_x\mathbf{u}_\varepsilon + \nabla_x^t\mathbf{u}_\varepsilon\Big)\Big] : \nabla_x^2\varphi\ \mathrm{d}\mathbf{x} \text{ a.a. in } (0,T) \tag{4.8}$$

for any $\varphi \in \mathcal{D}(\Omega_\varepsilon)$. Consequently, taking $\psi = \nabla_x\varphi$ in (4.5) and comparing the resulting expression with (4.7), (4.8) we deduce the desired conclusion

$$\int_{\Omega_\varepsilon} p_{\mathrm{harm},\varepsilon}(\tau,\cdot)\Delta\varphi\ \mathrm{d}\mathbf{x} = 0 \text{ for all } \varphi\in\mathcal{D}(\Omega_\varepsilon) \text{ and a.a. } \tau\in(0,T).$$

q.e.d.

4.2 Limit equations

It follows from (4.1) that the quantities P_ε and $p_{\mathrm{reg},\varepsilon} - \partial_t p_{\mathrm{harm},\varepsilon}$ differ only by a spatially homogenous time dependent function, in particular, the integral identity (2.3) can be replaced by

$$\int_0^T\int_{\Omega_\varepsilon}\Big(\mathbf{u}_\varepsilon\cdot\partial_t\varphi + \mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla_x\varphi + p_{\mathrm{reg},\varepsilon}\mathrm{div}_x\varphi + p_{\mathrm{harm},\varepsilon}\partial_t\mathrm{div}_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{4.9}$$

$$= \int_0^T\int_{\Omega_\varepsilon}\mu\Big(\nabla_x\mathbf{u}_\varepsilon + \nabla_x^t\mathbf{u}_\varepsilon\Big) : \nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t$$

to be satisfied for any test function $\varphi\in W_0^{1,\infty}((0,T)\times\overline{\Omega}_\varepsilon; R^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0$.

Any test function φ for the limit problem in the sense specified in (2.9) can be extended on $(0,T)\times\Omega_\varepsilon$ to be admissible in (4.9), specifically, we can take φ_1, φ_3 to be zero outside Ω_ε. In particular, taking relation (3.4) together with the uniform pressure estimates (4.2), (4.3) into account, we can let $\varepsilon\to 0$ in (4.9) in order to conclude that

$$\int_0^T\int_\Omega\Big(\mathbf{u}\cdot\partial_t\varphi + (\overline{\mathbf{u}\otimes\mathbf{u}}) : \nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t = \int_0^T\int_\Omega\mu\Big(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u}\Big) : \nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t$$

for any test function $\varphi\in\mathcal{D}((0,T)\times\overline{\Omega}; R^3)$,

$$\mathrm{div}_x\varphi = 0,\ \varphi\cdot\mathbf{n}|_{\{x_3=0\}},\ \varphi_1|_{\{x_3=1\}} = \varphi_3|_{\{x_3=1\}} = 0,$$

where the symbol $\overline{\mathbf{u}\otimes\mathbf{u}}$ stands for a weak limit of the sequence $\{\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ in the Lebesgue space $L^{5/3}((0,T)\times\Omega; R^{3\times 3})$. Consequently, it remains to identify the quantity $\overline{\mathbf{u}\otimes\mathbf{u}}$. This will be done in the last section.

5 Convergence of the convective terms

In order to complete the proof of Theorem 2.1, we have to show that

$$\int_0^T \int_\Omega (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \text{ as } \varepsilon \to 0 \tag{5.1}$$

for any $\varphi \in \mathcal{D}((0,T) \times \overline{\Omega}; R^3)$,

$$\mathrm{div}_x \varphi = 0, \ \varphi \cdot \mathbf{n}|_{\{x_3=0\}}, \ \varphi_1|_{\{x_3=1\}} = \varphi_3|_{\{x_3=1\}} = 0.$$

To begin, it is easy to observe that it is enough to show (5.1) for any $\varphi \in \mathcal{D}((0,T) \times \Omega; R^3)$, $\mathrm{div}_x \varphi = 0$. Indeed we have

$$\int_0^T \int_\Omega (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t = - \int_0^T \int_\Omega \nabla_x \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t;$$

whence (5.1) implies

$$\int_0^T \int_\Omega \nabla_x \mathbf{u}_\varepsilon \mathbf{u}_\varepsilon \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega \nabla_x \mathbf{u}\mathbf{u}\varphi \, \mathrm{d}x \, \mathrm{d}t \tag{5.2}$$

as soon as $\varphi \in \mathcal{D}((0,T) \times \Omega; R^3)$, $\mathrm{div}_x \varphi = 0$. On the other hand, relation (5.2) is easily extended to $\varphi \in \mathcal{D}((0,T) \times \overline{\Omega}; R^3)$, $\mathrm{div}_x \varphi = 0$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$.

In order to see that (5.1) holds for any $\varphi \in \mathcal{D}((0,T) \times \Omega; R^3)$, $\mathrm{div}_x \varphi = 0$, we evoke the method developed in [3] based on the pressure decomposition established in Lemma 4.1. The reader may consult [3] for details.

It follows from (4.9) that

$$\mathbf{u}_\varepsilon + \nabla_x p_{\mathrm{harm},\varepsilon} \to \mathbf{u} + \nabla_x p_{\mathrm{harm}} \text{ in } C_{\mathrm{weak}}([0,T]; L^2(V; R^3)), \ V \subset \overline{V} \subset \Omega,$$

where p_{harm} denotes a weak limit of $\{p_{\mathrm{harm},\varepsilon}\}_{\varepsilon>0}$. Here, we have used the fact that the harmonic part of the pressure is smooth in the $x-$variable on any set $V \subset \overline{V} \subset \Omega$. Consequently, a simple Lions-Aubin type argument yields

$$\mathbf{u}_\varepsilon + \nabla_x p_{\mathrm{harm},\varepsilon} \to \mathbf{u} + \nabla_x p_{\mathrm{harm}} \text{ in } L^2(0,T; L^2(V; R^3)).$$

Finally, we get

$$\int_0^T \int_\Omega \overline{\mathbf{u} \otimes \mathbf{u}} : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t = \lim_{\varepsilon \to 0} \int_0^T \int_\Omega (\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$

$$= \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \Big(\mathbf{u}_\varepsilon + \nabla_x p_{\mathrm{harm},\varepsilon}) \otimes \mathbf{u}_\varepsilon \Big) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$

$$- \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \Big(\nabla_x p_{\mathrm{harm},\varepsilon} \otimes (\mathbf{u}_\varepsilon + \nabla_x p_{\mathrm{harm},\varepsilon}) \Big) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \lim_{\varepsilon \to 0} \int_0^T \int_\Omega (\nabla_x p_{\mathrm{harm},\varepsilon} \otimes \nabla_x p_{\mathrm{harm},\varepsilon}) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$

whenever $\varphi \in \mathcal{D}((0,T) \times \Omega; R^3)$, $\mathrm{div}_x \varphi = 0$. Indeed

$$\int_0^T \int_\Omega (\nabla_x p \otimes \nabla_x p) : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$

$$= - \int_0^T \int_\Omega \Big(\frac{1}{2} \nabla_x |\nabla_x p|^2 \cdot \varphi + \Delta_x p \nabla_x p \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t = 0$$

for $p = p_{\mathrm{harm},\varepsilon}, p_{\mathrm{harm}}$ as both $p_{\mathrm{harm},\varepsilon}$ and p_{harm} are harmonic functions with respect to the $x-$variable.

Thus we have shown relation (5.1). The proof of Theorem 2.1 is complete.

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