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Free Boolean algebras over unions of two well orderings

Dedicated to the memory of Jan Pelant (1950-2005)

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Abstract

Given a partially ordered set P there exists the most general Boolean algebra $\widehat{F}(P)$ which contains P as a generating set, called the *free Boolean algebra* over P. We study free Boolean algebras over posets of the form $P = P_0 \cup P_1$, where P_0, P_1 are well orderings. We call them *nearly ordinal algebras*.

Answering a question of Maurice Pouzet, we show that for every uncountable cardinal κ there are 2^κ pairwise non-isomorphic nearly ordinal algebras of cardinality κ.

Topologically, free Boolean algebras over posets correspond to compact 0-dimensional distributive lattices. In this context, we classify all closed sublattices of the product $(\omega_1+1)\times(\omega_1+1)$, thus showing that there are only $\aleph_1$ many of them. In contrast with the last result, we show that there are $2^{\aleph_1}$ topological types of closed subsets of the Tikhonov plank $(\omega_1+1)\times(\omega+1)$.

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1 Introduction

Assume that a given Boolean algebra $\mathbb{B}$ is generated by a well ordered subset P. Then $\mathbb{B}$ is an interval algebra whose Stone space is homeomorphic to $\alpha+1$, where α is the order type of P. We call $\mathbb{B}$ an *ordinal algebra* and we call its Stone space an *ordinal space*. It is clear that for a given infinite cardinal κ, there are only κ^+ many types of ordinal algebras of cardinality κ.

Given a poset P, we denote by $\widehat{F}(P)$ the *free Boolean algebra* over P (the precise definitions are given in the next section). We shall call $\widehat{F}(P)$ a *nearly ordinal algebra* if $P = P_0 \cup P_1$, where P_0, P_1 are well ordered sets. We show that for every uncountable cardinal κ there exist 2^κ isomorphic types of nearly ordinal algebras of cardinality κ. This answers a question of Maurice Pouzet [Po], originally asked in a more general context of posets which are well founded and narrow (the so called *well quasi-ordered* posets).

One of the simplest Boolean algebras which are not generated by a chain is $\mathbb{B} = \widehat{F}(\omega_1 \uplus \omega)$, where $P_0 \uplus P_1$ denotes the disjoint union of posets P_0, P_1 with no extra relations. Topologically, $\mathbb{B}$ corresponds to the *Tikhonov plank* $\mathbb{TP} = (\omega_1+1)\times(\omega+1)$. We show that $\mathbb{TP}$ has $2^{\aleph_1}$ many isomorphic types of closed sets. On the other hand, we show that $\mathbb{TP}$ has only countably many topological types of uncountable closed sets which are at the same time sublattices of $\mathbb{TP}$. Finally, we classify topological types of compact sublattices of $(\omega_1+1)^2$. We show that an uncountable closed sublattice of $(\omega_1+1)^2$ is homeomorphic to one of the following spaces: $(\omega_1+1)\times(\alpha+1)$, where $\alpha \le \omega_1$, and $\triangle = \{\langle x, y\rangle \in (\omega_1+1)^2 : x \le y\}$. This implies that there are only $\aleph_1$ isomorphic types of Boolean algebras of the form $\widehat{F}(P_0 \cup P_1)$, where P_0, P_1 are well ordered chains of order type $\le \omega_1$.

2 Terminology and basic facts

All topological spaces considered in this paper are Hausdorff and zero-dimensional.

Given a compact space X, we denote by $\mathrm{D}^{\alpha}(X)$ the α-th Cantor-Bendixson derivative of X. Specifically, $\mathrm{D}^{1}(X)$ is the set of all non-isolated points of X, $\mathrm{D}^{\alpha+1}(X) = \mathrm{D}^{1}(\mathrm{D}^{\alpha}(X))$ and $\mathrm{D}^{\delta}(X) = \bigcap_{\xi<\delta} \mathrm{D}^{\xi}(X)$ for any limit δ. If X is scattered, then the *Cantor-Bendixson rank* of X, denoted by $\mathrm{rk}(X)$, is the first ordinal α such that $\mathrm{D}^{\alpha+1}(X)$ is empty. A compact scattered space X is *unitary* whenever $\mathrm{D}^{\mathrm{rk}(X)}(X)$ has a unique element, called the *end-point* of X and denoted by $e(X)$. For $x \in X$, we denote by $\mathrm{rk}^{X}(x)$ the unique ordinal θ such that $x \in \mathrm{D}^{\theta}(X) \setminus \mathrm{D}^{\theta+1}(X)$ and $\mathrm{rk}^{X}(x)$ is called the *Cantor-Bendixson rank of x in X*.

Let X and Y be topological spaces. We denote by $X \cong^{\mathrm{top}} Y$ the fact that X and Y are homeomorphic.

A compact space X is *retractive* if every closed subset of X is a retract of X, i.e. for every closed $Y \subseteq X$ there is a continuous map $f\colon X \to Y$ such that $f(x) = x$ for every $x \in Y$. Let us mention a theorem of M. Rubin [R, Theorem 5.1] (see also [K, §15, Theorem 15.18]): every Boolean space which is a continuous image of a linearly ordered compact is retractive.

The *Tikhonov plank* is the space $\mathbb{TP} := (\omega_1 + 1) \times (\omega + 1)$, where both factors are considered as linearly ordered compacta. It is straight to see that $\mathbb{TP}$ is not retractive, namely the closed set

$$A = \Big((\omega_1 + 1) \times \{\omega\}\Big) \cup \Big(\{\omega_1\} \times (\omega + 1)\Big)$$

is not a retract of $\mathbb{TP}$. In particular, $\mathbb{TP}$ is not a continuous image of any linearly ordered compact space.

Given a partially ordered set P, we denote by $\widehat{F}(P)$ the *free Boolean algebra over P*. More precisely, $\widehat{F}(P)$ is a Boolean algebra together with an order preserving embedding $i_P\colon P \to \widehat{F}(P)$ such that for every order preserving map $f\colon P \to \mathbb{B}$ into an arbitrary Boolean algebra $\mathbb{B}$ there exists a unique homomorphism $h\colon \widehat{F}(P) \to \mathbb{B}$ such that $h \circ i_P = f$. These two properties define $\widehat{F}(P)$ uniquely. Topologically, $\widehat{F}(P)$ can be defined to be the clopen algebra of the space

$$\mathrm{FS}(P) := \{F \subseteq P \,:\, (\forall\, x, y \in P)\ (x \in F \text{ and } x \leq y \text{ imply } y \in F)\},$$

with the topology inherited from the Cantor cube $\mathcal{P}(P)$ (the powerset of P). The embedding i_P is defined by the formula $i_P(p) = \{F \in \mathrm{FS}(P) \,:\, p \in F\}$.

Elements of $\mathrm{FS}(P)$ are called *final segments* of P. Note that $\mathrm{FS}(P)$ has a natural structure of a lattice of sets: the union as well as the intersection of any family of final segments is again a final segment. It turns out that the original poset can always be reconstructed from the lattice structure as the set of all clopen prime filters with inclusion. On the other hand, every compact 0-dimensional distributive lattice $\langle L, 0, 1, \vee, \wedge, \tau\rangle$ is isomorphic (in the category of topological lattices) to $\langle \mathrm{FS}(P), \emptyset, P, \cup, \cap, \tau_P\rangle$ for some poset P, where τ_P is the topology of $\mathrm{FS}(P)$. In fact, $P \mapsto \mathrm{FS}(P)$ is a contravariant functor from the category of posets onto the category of compact 0-dimensional distributive lattices, whose inverse is defined by assigning to a lattice the family of all its clopen prime filters. We refer to [ABKR, §2] for more details.

Given a compact lattice L, we shall denote by 1_L and 0_L the maximal and the minimal element of L respectively. For our purposes, it is more convenient to consider the lattice $\mathrm{FS}(P)$ with reversed inclusion, i.e. the join of $x, y \in \mathrm{FS}(P)$ is $x \cap y$ and their meet is $x \cup y$. In particular, $1_{\mathrm{FS}(P)}$ is the empty final segment and $0_{\mathrm{FS}(P)}$ is the whole poset $P \in \mathrm{FS}(P)$. For example, if P is a well ordered chain then $\langle \mathrm{FS}(P), \supseteq\rangle$ is a well ordered compact chain. Note that a basic neighborhood of a point $x \in \mathrm{FS}(P)$ is of the form $s^+ \cap t^-$, where $s, t \subseteq P$ are finite sets, $s^+ := \{y \in \mathrm{FS}(P) : s \subseteq y\}$ and $t^- := \{y \in \mathrm{FS}(P) : t \cap y = \emptyset\}$. Observe that $0_P = P$ is an isolated point of $\mathrm{FS}(P)$, whenever there is a finite set $s \subseteq P$ such that the final segment generated by s is P. In that case $\{P\} = s^+$. This happens, for example, in case where P is the union of finitely many well ordered chains.

Given posets P_0, P_1 we denote by $P_0 \uplus P_1$ the disjoint (incomparable) sum of P_0, P_1, i.e. $P_0 \uplus P_1$ is the disjoint union of P_0, P_1 and $x \le y$ in $P_0 \uplus P_1$ iff $x, y \in P_i$ and $x \le y$ in P_i for some $i < 2$. We denote by $P_0 \cup P_1$ a poset wich is covered by two posets P_0 and P_1. Recall that a *nearly ordinal algebra* is a Boolean algebra of the form $\widehat{F}(P_0 \cup P_1)$, where P_0, P_1 are well ordered chains. We shall call $\mathrm{FS}(P_0 \cup P_1)$ a *nearly ordinal space.*

3 The number of nearly ordinal Boolean algebras

In this section we prove the announced result on the number of nearly ordinal Boolean algebras of a given cardinality. Before we state the precise statement, we shall introduce some necessary notions.

Given unitary scattered compact spaces X and Y, we say that X and Y are *almost homeomorphic* if there are clopen neighborhoods U, V of the end-points of X and Y respectively such that U and V are homeomorphic.

We now define for each uncountable cardinal κ an ordinal $r(\kappa)$ as follows. If $\kappa > \aleph_1$ is a successor cardinal then we set $r(\kappa) = \kappa$. If κ is a limit cardinal which is not a strong limit (i.e. $2^\lambda > \kappa$ for some $\lambda < \kappa$) then again we set $r(\kappa) = \kappa$. If κ is a strong limit (i.e. κ is a limit ordinal and $2^\lambda < \kappa$ for $\lambda < \kappa$) then we set $r(\kappa) = \kappa + \kappa$. It remains to define $r(\aleph_1)$. Let

$$E = \{\theta < \omega_2 \,:\, \omega^\theta = \theta \text{ and } \operatorname{cf}\theta = \omega_1\}.$$

Then $|E| = \aleph_2$. Let E_0 be the initial segment of E whose order type is ω_1. Finally, let $\varpi_1 = \sup(E_0)$ and set $r(\aleph_1) = \varpi_1 + \omega_1$.

Note that always $\kappa \le r(\kappa) < \kappa^+$ and in fact $\kappa < r(\kappa)$ only if $\kappa = \aleph_1$ or κ is a strong limit cardinal.

Theorem 3.1. *Let κ be an uncountable cardinal. There exists a family $\langle X_\alpha \,:\, \alpha < 2^\kappa\rangle$ of nearly ordinal spaces with the following properties:*

(1) *Each X_α is unitary of cardinality κ and of Cantor-Bendixson rank $r(\kappa)$.*

(2) *For each α, the space X_α is not retractive. Specifically: every neighborhood of $e(X_\alpha)$ contains a copy of the Tikhonov plank.*

(3) *If $\alpha \ne \beta$ then X_α and X_β are not almost homeomorphic.*

In particular, for each uncountable cardinal κ there exist 2^κ pairwise non-isomorphic nearly ordinal Boolean algebras of size κ.

The rest of this section is devoted to the proof of Theorem 3.1. We start with some notation and definitions.

We shall denote by Ind the class of all indecomposable ordinals. Recall that an ordinal $\rho > 0$ is *indecomposable* if $\alpha + \beta < \rho$ whenever $\alpha, \beta < \rho$. In this case the linearly ordered compact space $\rho + 1$ is unitary and $e(\rho + 1) = \rho$. The Cantor-Bendixson rank of $\rho + 1$ will be denoted by $\ln(\rho)$.

We shall denote by $\mathcal{W}_2^*$ the class of all lattices isomorphic to $\mathrm{FS}\,(P)$, where P is covered by two well ordered chains and $\emptyset = 1_{\mathrm{FS}(P)}$ is the unique element of maximal Cantor-Bendixson rank in $\mathrm{FS}\,(P)$. So every lattice in $\mathcal{W}_2^*$ is a unitary scattered compact space.

Let $\rho > 0$ be an ordinal and let $\langle L_\xi : \xi < \rho \rangle$ be a collection of compact lattices, where $L_\xi = \mathrm{FS}\,(P_\xi)$ for some poset P_ξ. Let $P = \sum_{\xi<\rho} P_\xi$ by the lexicographic sum of $\{P_\xi : \xi < \rho\}$. We shall write $\vec{\sum}_{\xi<\rho} L_\xi$ for the lattice $\mathrm{FS}\,(P)$. In case where $L_\xi = L$ for every ξ, we shall write $L \vec{\cdot} \rho$ instead of $\vec{\sum}_{\xi<\rho} L_\xi$.

Let $K = \vec{\sum}_{\xi<\rho} L_\xi$, where each L_ξ is a well founded distributive lattice isomorphic to $\mathrm{FS}\,(P_\xi)$, where P_ξ is a union of two well ordered chains. Observe that K is a nearly ordinal space, because $\sum_{\xi<\rho} P_\xi$ is again a union of two well ordered chains. Recall that we consider $\mathrm{FS}\,(P)$ with the reversed ordering. Observe that, given $\eta < \rho$ we can identify $x \in \mathrm{FS}\,(P_\eta)$ with

$$\widehat{x} := \{p \in \sum_{\xi<\rho} P_\xi \;:\; (\exists\, q \in x)\; (p \geq q)\} \;\in\; \mathrm{FS}\,(P).$$

By this way we get a natural embedding of L_η into K. Notice that the empty final segment of P_η is actually identified with the full final segment of $P_{\eta+1}$. In other words, 1_{L_η} is identified with $0_{\eta+1}$. Further, observe that $L_\eta \setminus \{0_{L_\eta}\}$ is clopen in K. Indeed, denoting by s_α the set of minimal elements of P_α, we have

$$L_\eta^- := L_\eta \setminus \{0_{L_\eta}\} = \{x \in \mathrm{FS}\,(P) \;:\; s_\eta \not\subseteq x \text{ and } s_{\eta+1} \subseteq x\}.$$

The set on the righ-hand side is evidently clopen, because each s_α is a finite set. Finally, observe that for a limit ordinal $\delta < \rho$, a basic neighborhood of 0_{L_δ} is of the form $V(\alpha, \delta) := \{0_{L_\delta}\} \cup L_\alpha^- \cup \bigcup_{\xi\in(\alpha,\delta)} L_\xi$ where $\alpha < \delta$. Indeed, $V(\alpha,\delta) = s_\delta^+ \setminus s_\alpha^+$ and thus $V(\alpha,\delta)$ is clopen in K; and a typical neighborhood of 0_{L_δ} is of the form $s^+ \cap t^-$, where s, t are finite subsets of P. Necessarily $s \subseteq \bigcup_{\xi\geq\delta} P_\xi$. Thus $0_{L_\delta} \in V(\alpha,\delta) \subseteq s^+ \cap t^-$, where α is such that $t \subseteq \bigcup_{\xi<\alpha} P_\xi$. Similarly, sets of the form $V(\alpha, \rho) = \{1_L\} \cup L_\alpha^- \cup \bigcup_{\xi>\alpha} L_\xi$, where $\alpha < \rho$, form an open base at $1_L = \emptyset$.

Lemma 3.2. *Let $\rho, \theta > 0$ be ordinals and assume ρ is indecomposable. Further, let $\{Y_\alpha : \alpha < \rho\} \subseteq \mathcal{W}_2^*$ be such that $\mathrm{rk}(Y_\alpha) = \theta$ for each $\alpha < \rho$. Let $Y = \vec{\sum}_{\alpha<\rho} Y_\alpha$. Then $Y \in \mathcal{W}_2^*$, $\mathrm{D}^\theta(Y) \setminus \mathrm{D}^{\theta+1}(Y) = \{1_{Y_\xi} : \xi < \rho\}$ and the Cantor-Bendixson rank of Y equals $\theta + \ln \rho$.*

Proof. Recall that $Y_\xi \setminus \{0_{Y_\xi}\}$ is a clopen subset of Y and $e(Y_\xi) = 1_{Y_\xi}$, because the minimal element of a well founded distributive lattice is isolated. Thus the Cantor-Bendixson rank of $e(Y_\xi) = 1_{Y_\xi}$ in Y is the same as its rank in Y_ξ.

It follows that $\{1_{Y_\xi} : \xi < \rho\} \subseteq \mathrm{D}^\theta(Y)$. By the same reason, $\mathrm{D}^\theta(Y) \cap Y_\xi \subseteq \{0_{Y_\xi}, 1_{Y_\xi}\}$. Thus

$$\mathrm{D}^\theta(Y) = \{1_{Y_\xi} : \xi < \rho\} \cup \{0_{Y_\eta} : \eta \text{ is a limit ordinal } < \rho\} \cup \{1_Y\},$$

because each 1_{Y_ξ} is identified with $0_{Y_{\xi+1}}$. Hence $\mathrm{D}^\theta(Y)$ is homeomorphic to $\rho+1$. Finally, $\mathrm{rk}(\rho+1) = \ln \rho$ and $e(Y) = 1_Y$, which shows that $Y \in \mathcal{W}_2^*$. □

Lemma 3.3. *Let $\rho, \theta > 0$ be ordinals and assume ρ is indecomposable. Further, let $\{Y_\alpha : \alpha < \rho\} \subseteq \mathcal{W}_2^*$ and $\{Z_\alpha : \alpha < \rho\} \subseteq \mathcal{W}_2^*$ be such that $\mathrm{rk}(Y_\alpha) = \mathrm{rk}(Z_\alpha) = \theta$ for each $\alpha < \rho$. Let $Y = \vec{\sum}_{\alpha<\rho} Y_\alpha$, $Z = \vec{\sum}_{\alpha<\rho} Z_\alpha$ and assume that Y and Z are almost homeomorphic. Then there exists $\alpha < \rho$ such that for every $\beta \geq \alpha$ there exists ξ with the property that Y_β is almost homeomorphic to Z_ξ.*

Proof. Let $h \colon U \to V$ be a homeomorphism between clopen neighborhoods of 1_Y and 1_Z respectively. Let $\alpha < \rho$ be such that $Y_\beta \subseteq U$ for every $\beta \geq \alpha$. Fix $\beta \geq \alpha$. By assumption and by Lemma 3.2, we know that 1_{Y_ξ} are precisely the elements of Y whose Cantor-Bendixson rank is θ. The same applies to Z. Thus $h(1_{Y_\beta}) = 1_{Z_\xi} \in V$ for some ξ. Let $U' = U \cap (Y_\beta \setminus \{0_{Y_\beta}\}$. Then U' is a clopen neighborhood of 1_{Y_β} in Y_β. Let $V' = h[U'] \cap (Z_\xi \setminus \{0_{Z_\xi}\}$. Then V' is a clopen neighborhood of 1_{Z_ξ} in Z_ξ, homeomorphic to $h^{-1}[V'] \subseteq U' \subseteq Y_\beta$. Thus Y_β is almost homeomorphic to Z_ξ, which completes the proof. □

Lemma 3.4. *Let κ be an uncountable cardinal and let γ be an ordinal such that $\gamma + \kappa = r(\kappa)$. Assume further that $\{Y^i : i < \kappa\}$ is a family of spaces satisfying the following conditions:*

$\star_1(\kappa)$ *Y^i is a member of $\mathcal{W}_2^*$.*

$\star_2(\kappa)$ $|Y^i| \leq \kappa$.

$\star_3(\kappa)$ $\mathrm{rk}(Y^i) = \gamma$.

$\star_4(\kappa)$ *Y^i is unitary.*

$\star_5(\kappa)$ *Y^i and Y^j are not almost homeomorphic for distinct $i, j < \kappa$.*

$\star_6(\kappa)$ *If U is a neighborhood of $e(Y^i)$ in Y^i then U contains a closed subspace homeomorphic to the Tikhonov plank.*

Then there exists a family of spaces $\{X_\alpha : \alpha < 2^\kappa\} \subseteq \mathcal{W}_2^*$ *satisfying the assertions of Theorem 3.1.*

Proof. We construct the family $\{X_\alpha : \alpha < 2^\kappa\}$ as follows. Let $\{A_\alpha : \alpha < 2^\kappa\}$ be an enumeration of all subsets $A \subseteq \kappa$ such that $|A| \geq 2$. For each $\alpha < 2^\kappa$, let $f_\alpha : \kappa \to A_\alpha$ be such that ${f_\alpha}^{-1}[\xi]$ is cofinal in κ for every $\xi \in A_\alpha$. Let $X_\alpha := \vec{\sum}_{\zeta<\kappa} Y^{f_\alpha(\zeta)}$. We prove that the family $\{X_\alpha : \alpha < 2^\kappa\}$ is as required.

It is clear that X_α is a unitary scattered compact of cardinality κ. By Lemma 3.2, the Cantor-Bendixson rank of X_α equals $\gamma + \kappa = r(\kappa)$. Now let U be a clopen neighborhood of $e(X_\alpha)$ in X_α. There is $\zeta < \kappa$ such that $Y^{f_\alpha(\zeta)} \subseteq X_\alpha$. By $\star_6(\kappa)$, $Y^{f_\alpha(\zeta)}$ contains a copy of $\mathbb{TP}$. This shows condition (2) of Theorem 3.1. It remains to prove (3).

Fix $\alpha \neq \beta$. We need to show that X_α and X_β are not almost homeomorphic. Assume for instance that $\xi \in A_\beta \setminus A_\alpha$. We show that the assertion of Lemma 3.3 is not fulfilled. For fix any $\delta < \kappa$. Since ${f_\beta}^{-1}[\xi]$ is cofinal in κ, let $\zeta > \delta$ be such that $f_\beta(\zeta) = \xi$. On the other hand, $\xi \notin A_\alpha$, which means that $f_\alpha(\eta) \neq \xi$ for every $\eta < \kappa$. Hence, by $\star_5(\kappa)$, $Y^{f_\beta(\zeta)}$ is not homeomorphic to $Y^{f_\alpha(\eta)}$ for every $\eta < \kappa$. Since $\zeta > \delta$, by Lemma 3.3, we conclude that X_α and X_β are not almost homeomorphic. □

Theorem 3.1 is a consequence of the following statement combined with Lemma 3.4.

Lemma 3.5. *For every uncountable cardinal* κ *there exist a family* $\langle Y^i : i < \kappa\rangle$ *and an ordinal* γ *with* $\gamma + \kappa = r(\kappa)$, *satisfying conditions* $\star_1(\kappa)$ – $\star_6(\kappa)$ *of Lemma 3.4.*

Proof. We use induction on the cardinal κ. Assume the statement has been proved for all uncountable cardinals $\lambda < \kappa$. We consider three cases.

Case 0. $\kappa = \aleph_1$.

We set $\gamma = \varpi_1$. Given a limit ordinal θ, let $K_\theta = (\theta + 1)^2$ and let T_θ be the space obtained from two copies of $\theta + 1$ by identifying their last elements. In other words, T_θ can be described as the linearly ordered space $\theta + 1 + \theta^*$, where θ^* is the set θ with the reversed ordering. The following claim is trivial, after noting that $\mathrm{rk}(\theta + 1) = \theta$ and $e(\theta + 1) = \theta$, for any ε-ordinal θ.

Claim 3.6. *Let* $\theta > 0$ *be an ordinal such that* $\omega^\theta = \theta$. *Then*

(a) $\mathrm{rk}(T_\theta) = \theta$ *and* T_θ *is unitary.*

(b) *T_θ is not an ordinal space.*

(c) $\mathrm{D}^{(\theta)}(K_\theta) \cong^{\mathrm{top}} T_\theta$.

(d) $\mathrm{rk}(K_\theta) = \theta \cdot 2$ *and* $e(K_\theta) = e(T_\theta) = 1_{K_\theta}$.

Recall that we have denoted by E_0 the set consisting of the first ω_1 many ε-ordinals of uncountable cofinality. In particular, $\omega^\theta = \theta$ for $\theta \in E_0$. Given $\theta \in E_0$, define $X_\theta = K_\theta \mathbin{\vec{\cdot}} \varpi_1$. Finally, fix a one-to-one enumeration $\langle \theta_i : i < \omega_1 \rangle$ of E_0 and define $Y^i = X_{\theta_i}$.

Since $\mathrm{FS}\,(\theta \uplus \theta) = K_\theta$, clearly, $X_\theta \in \mathcal{W}_2^*$ is unitary and of cardinality $\aleph_1$. Further, every neighborhood of $e(X_\theta) = 1_{X_\theta}$ contains some K_θ and $\mathbb{TP}$ is a sublattice of K_θ, because $\omega_1 \le \theta$. This shows that the family $\langle Y^i : i < \omega_1 \rangle$ satisfies $\star_i(\aleph_1)$ for $i = 1, 2, 4, 6$. By Claim 3.6(d) together with Lemma 3.2, we have that

$$\mathrm{rk}(X_\theta) = \theta \cdot 2 + \ln \varpi_1 = \varpi_1 = \gamma.$$

This shows $\star_3(\aleph_1)$. It remains to show $\star_5(\aleph_1)$. For fix $\theta < \theta'$ in E_0. Then $\theta \cdot 2 < \theta' \cdot 2$, so by Claim 3.6(d), K_θ and $K_{\theta'}$ cannot be almost homeomorphic, because of the Cantor-Bendixson rank. Hence X_θ and $X_{\theta'}$ are not almost homeomorphic by Lemma 3.3. Thus $\langle Y_i : i < \omega_1 \rangle$ satisfies $\star_1(\omega_1) - \star_6(\omega_1)$.

Case 1. κ satisfies: $\lambda < \kappa$ and $2^\lambda \ge \kappa$ for some $\lambda > \aleph_0$.

So, either κ is a successor cardinal or else κ is a limit, but not a strong limit cardinal. By the induction hypothesis applied to λ, followed by Lemma 3.4, there is a sequence $\{Y^i : i < 2^\lambda\}$ satisfying the assertions of Theorem 3.1.

Note that $|Y^i| = \lambda$ and $\mathrm{rk}(Y^i) = r(\lambda)$. Let $\gamma = r(\lambda) < \lambda^+ \le \kappa$. It is clear that the family $\{Y^i : i < \kappa\}$ satisfies conditions $\star_1(\kappa) - \star_6(\kappa)$.

Case 2. κ is a strong limit cardinal.

Fix a strictly increasing sequence of infinite cardinals $\langle \kappa_\alpha : \alpha < \mathrm{cf}\,\kappa \rangle$ with $\kappa = \sup_{\alpha < \mathrm{cf}\,\kappa} \kappa_\alpha$. For each $\alpha < \mathrm{cf}\,\kappa$ let $\mathcal{F}_\alpha \subseteq \mathcal{W}_2^*$ be a family satisfying $\star_1(\kappa_\alpha^+) - \star_6(\kappa_\alpha^+)$, obtained by the induction hypothesis. Let ρ_α denote the common Cantor-Bendixson rank of the spaces from $\mathcal{F}_\alpha$. We define families $\mathcal{G}_\alpha \subseteq \mathcal{F}_\alpha$ satisfying the following conditions.

(i) $|\mathcal{G}_\alpha| = \kappa_\alpha^+$.

(ii) Given $X \in \mathcal{G}_\xi$, $Y \in \mathcal{G}_\alpha$ with $\xi < \alpha$, the space $X \mathbin{\vec{\cdot}} \rho_\alpha$ is not almost homeomorphic to Y.

We start with $\mathcal{G}_0 = \mathcal{F}_0$. Fix $\beta > 0$ and suppose $\mathcal{G}_\xi$ has been defined for every $\xi < \beta$. Let

$$\mathcal{A}_\xi = \{X \in \mathcal{F}_\beta \,:\, (\exists\, Y \in \mathcal{F}_\xi)\ (X \text{ is almost homeomorphic to } Y \mathbin{\vec{\cdot}} \rho_\beta\,)\}.$$

Observe that $|\mathcal{A}_\xi| \leq \kappa_\xi^+$, because $|\mathcal{F}_\xi| = \kappa_\xi^+$ and, by Lemma 3.3, no two elements of $\mathcal{F}_\beta$ are almost homeomorphic. Define

$$\mathcal{G}_\beta = \mathcal{F}_\beta \setminus \bigcup_{\xi<\beta} \mathcal{A}_\xi.$$

Then $|\mathcal{G}_\beta| = \kappa_\beta^+$, because $|\bigcup_{\xi<\beta} \mathcal{A}_\xi| \leq \sup_{\xi<\beta} \kappa_\xi^+ < \kappa_\beta^+ = |\mathcal{F}_\beta|$. Hence (i) holds. By the definition of $\mathcal{G}_\beta$ also (ii) holds. Finally set $\mathcal{G} = \bigcup_{\alpha<\operatorname{cf}\kappa} \mathcal{G}_\alpha$. Then $|\mathcal{G}| = \kappa$, by (i). Define $\mathcal{H} = \{X \mathbin{\vec{\cdot}} \kappa \,:\, X \in \mathcal{G}\}$. We claim that $\mathcal{H}$ satisfies $\star_1(\kappa) - \star_6(\kappa)$ with $\gamma = \kappa$. Recall that $\gamma + \kappa = \kappa + \kappa = r(\kappa)$. Observe that for $X \in \mathcal{G}$, $\operatorname{rk}(X \mathbin{\vec{\cdot}} \kappa) = \kappa$ and $e(X \mathbin{\vec{\cdot}} \kappa)$ is the maximal element of $X \mathbin{\vec{\cdot}} \kappa$. Also $X \in \mathcal{W}_2^*$ and $|X \mathbin{\vec{\cdot}} \kappa| = \kappa$. Thus $\star_i(\kappa)$ holds for $i = 1, 2, 3, 4$. Condition $\star_6(\kappa)$ follows from the induction hypothesis, since every neighborhood of $e(X \mathbin{\vec{\cdot}} \kappa)$ contains a copy of X. It remains to show $\star_5(\kappa)$. For fix $X_0, X_1 \in \mathcal{G}$ with $X_0 \neq X_1$. If $X_0, X_1 \in \mathcal{G}_\alpha \subseteq \mathcal{F}_\alpha$ for some α then X_0 and X_1 are not almost homeomorphic by the induction hypothesis, therefore so are $X_0 \mathbin{\vec{\cdot}} \kappa$ and $X_1 \mathbin{\vec{\cdot}} \kappa$, by Lemma 3.3. Now assume $X_0 \in \mathcal{G}_\alpha$ and $X_1 \in \mathcal{G}_\beta$, where $\alpha < \beta$. Observe that $X_0 \mathbin{\vec{\cdot}} \kappa$ is isomorphic (as a lattice) to $(X_0 \mathbin{\vec{\cdot}} \rho_\beta) \mathbin{\vec{\cdot}} \kappa$. This is because $X_0 \mathbin{\vec{\cdot}} \rho = \operatorname{FS}(P \cdot \rho)$, where P is a poset such that $X_0 = \operatorname{FS}(P)$; clearly $P \cdot \kappa$ is order isomorphic to $(P \cdot \rho) \cdot \kappa$ for any $\rho < \kappa$. By (ii), $X_0 \mathbin{\vec{\cdot}} \rho_\beta$ is not almost homeomorphic to X_1. Recall that $\operatorname{rk}(X_1) = \rho_\beta = \operatorname{rk}(X_0 \mathbin{\vec{\cdot}} \rho_\beta)$. Hence, by Lemma 3.3, $X_0 \mathbin{\vec{\cdot}} \kappa \cong^{\text{top}} (X_0 \mathbin{\vec{\cdot}} \rho_\beta) \mathbin{\vec{\cdot}} \kappa$ is not almost homeomorphic to $X_1 \mathbin{\vec{\cdot}} \kappa$. This shows $\star_5(\kappa)$ and completes the proof. □

4 Closed subsets of the Tikhonov plank

It is clear that every uncountable closed subset of the ordinal space $X = \omega_1 + 1$ is homeomorphic (in fact, order isomorphic) to X. Thus, X has only one non-metrizable compact topological type. More generally, given a natural number $n > 0$, the space $(\omega_1 + 1) \times n$ contains only n topological types of non-metrizable compact spaces. The situation is different in case of the Tikhonov plank $\mathbb{TP} := (\omega_1 + 1) \times (\omega + 1)$. Recall that $\mathbb{TP}$ is a unitary scattered space of rank $\omega_1 + 1$. Moreover, $\mathbb{TP}$ is a distributive lattice corresponding to the free Boolean algebra over $\omega_1 \uplus \omega$.

Theorem 4.1. *The Tikhonov plank contains $2^{\aleph_1}$ many pairwise non-homeomorphic closed subsets. Dually, the Boolean algebra $\widehat{F}(\omega_1 \uplus \omega)$ has $2^{\aleph_1}$ many pairwise non-isomorphic quotients.*

Proof. Given a closed cofinal set $C \subseteq \omega_1$, define

$$X(C) := \Big((\omega_1+1)\times\{\omega\}\Big) \cup \bigcup_{\xi\in C\cup\{\omega_1\}} \Big(\{\xi\}\times(\omega+1)\Big).$$

Clearly, $X(C)$ is a closed unitary subspace of $\mathbb{TP}$, with rank ω_1+1 and $e(X(C)) = \langle\omega_1, \omega\rangle$. We shall construct $2^{\aleph_1}$ many pairwise non-homeomorphic spaces of the form $X(C)$.

Let $E(\omega_1) = \{\gamma < \omega_1 : \omega^\gamma = \gamma\}$ be the set of countable ε-ordinals. Notice that $E(\omega_1)$ is a closed subset of ω_1. For every nonempty subset A of $E(\omega_1)$ we choose a function $f_A : \omega_1+1 \to A$ such that ${f_A}^{-1}[\gamma]$ contains uncountably many successor ordinals for each $\gamma \in A$. Given $\beta < \omega_1$ define

$$\lambda_{A,\beta} := \sum_{\mu<\beta} f_A(\mu),$$

where $\sum$ means the ordinal sum. Consider the closed and unbounded subset $\widehat{A} := \{\lambda_{A,\beta} : \beta < \omega_1\}$ of ω_1 and finally define $X_A := X(\widehat{A})$. Fix nonempty sets $A, B \subseteq E(\omega_1)$ such that $A \neq B$. We shall show that X_A is not homeomorphic to X_B. For suppose $g\colon X_A \to X_B$ is a homeomorphism. Then $g(\langle\omega_1,\, \omega\rangle) = g(e(X_A)) = e(X_B) = \langle\omega_1,\, \omega\rangle$. Note that $\langle\omega_1,\, n\rangle$, where $n < \omega$, are the only points of rank ω_1 in both spaces X_A, X_B. Thus there exists a bijection $\theta\colon (\omega+1) \to (\omega+1)$ such that $g(\langle\omega_1,\, n\rangle) = \langle\omega_1,\, \theta(n)\rangle$ for $n \leq \omega$.

By the continuity of both g and g^{-1}, for every $\eta < \omega_1$ and for every $n \leq \omega$, we can find $\eta_{n,0}$, $\eta_{n,1}$ and $\eta_{n,2}$ in the interval (η, ω_1) so that

$$\big([\eta_{n,0},\omega_1]\cap\widehat{A}\big)\times\{\theta(n)\} \subset g\Big[\big([\eta_{n,1},\omega_1]\cap\widehat{B}\big)\times\{n\}\Big] \subset \big([\eta_{n,2},\omega_1]\cap\widehat{A}\big)\times\{\theta(n)\}.$$

We define $\delta(\eta)$ to be the supremum of all ordinals $\eta_{n,i}$, where $n \leq \omega$ and $i < 3$. Clearly, $\eta < \delta(\eta) < \omega_1$. Define a sequence $\langle\varepsilon_n : n \in \omega\rangle$ by $\varepsilon_0 = 0$ and $\varepsilon_{n+1} = \delta(\varepsilon_n)$. Let $\varepsilon = \sup_{n\in\omega}\varepsilon_n$. Then $g\Big[\big([\varepsilon,\omega_1]\cap\widehat{A}\big)\times\{n\}\Big] = \big([\varepsilon,\omega_1]\cap\widehat{B}\big)\times\{\theta(n)\}$. In particular,

$(*)$ $g\Big[\big([\varepsilon,\omega_1)\cap\widehat{A}\big)\times\{\omega\}\Big] = \big([\varepsilon,\omega_1)\cap\widehat{B}\big)\times\{\omega\}$ and
$g\Big[\big([\varepsilon,\omega_1)\cap\widehat{A}\big)\times\omega\Big] = \big([\varepsilon,\omega_1)\cap\widehat{B}\big)\times\omega.$

By $(*)$ we can define $h\colon [\varepsilon, \omega_1] \to [\varepsilon, \omega_1]$ by setting $g(\langle \mu,\, \omega\rangle) = \langle h(\mu),\, \omega\rangle$. Then h is a homeomorphism of $[\varepsilon, \omega_1]$ onto itself.

Now assume $\gamma \in A \setminus B$. Recall that ${f_A}^{-1}[\gamma]$ contains uncountably many successor ordinals. Fix such an ordinal α above ε. So $\alpha = \delta + 1 > \varepsilon$ and $f_A(\alpha) = \gamma$. Recall that $\lambda_{A,\delta}\,,\, \lambda_{A,\delta+1}$ are members of $\widehat{A}$ and $\lambda_{A,\alpha+1} = \lambda_{A,\alpha} + \gamma$. Notice that the interval $[\lambda_{A,\delta}, \lambda_{A,\delta+1}]$ is order isomorphic (in particular, homeomorphic) to $\gamma + 1$. Because γ is an ε-ordinal,

$$\mathrm{rk}^{X_A}(\langle \lambda_{A,\delta+1},\, \omega\rangle) = \mathrm{rk}^{\omega_1+1}(\lambda_{A,\delta+1}) = \gamma.$$

Since $\lambda_{A,\delta+1} \in \widehat{A}$, necessarily $h(\lambda_{A,\delta+1}) \in \widehat{B}$. That is $h(\lambda_{A,\delta+1}) = \lambda_{B,\mu}$ for some $\mu \in \widehat{B}$. Observe that $[\varepsilon, \omega_1{+}1)$ is a neighborhood of both $\lambda_{A,\delta+1}$, $\lambda_{B,\mu}$ and that $\lambda_{A,\delta+1}$ is an isolated point in $\widehat{A}$. Thus the same holds for $\lambda_{B,\mu}$ in $\widehat{B} \cap [\varepsilon, \omega_1{+}1)$. Hence μ is a successor, i.e. $\mu = \nu + 1$. But, since $\gamma \notin B$, there is no successor μ such that $\mathrm{rk}^{X_B}(\langle \lambda_{B,\mu},\, \omega\rangle) = \mathrm{rk}^{\omega_1+1}(\lambda_{B,\mu}) = \gamma$. This is a contradiction. □

Remark 4.2. A slight modification of the above proof shows that for distinct nonempty sets $A, B \subseteq E(\omega_1)$ spaces X_A, X_B are not almost homeomorphic.

Note that each of the spaces X_A contains a copy of $\mathbb{TP}$, therefore it is not retractive. Consequently, the algebra of clopen subsets of X_A is not embeddable into any interval algebra.

5 Closed sublattices of $(\omega_1 + 1)^2$

The purpose of this section is to describe topological types of closed sublattices of $(\omega_1 + 1)^2$. In a natural way, for $\langle a,\, b\rangle, \langle c,\, d\rangle \in (\omega_1 + 1)^2$, we set $\langle a,\, b\rangle \wedge \langle c,\, d\rangle = \langle \min\{a, c\},\, \min\{b, d\}\rangle$ and analogously for $\langle a,\, b\rangle \vee \langle c,\, d\rangle$. In order to simplify some statements, by a *sublattice* of a lattice L we mean a subset which is closed under meet and join, not necessary containing 0_L and 1_L. We set

$$\mathbb{GP}_\alpha := (\omega_1 + 1) \times (\alpha + 1)$$

and

$$\mathbb{GP} := \mathbb{GP}_{\omega_1} = (\omega_1 + 1)^2\ , \qquad \triangle := \{\langle x,\, y\rangle \in \mathbb{GP}\ :\ x \geq y\}\,.$$

We shall prove below that an uncountable closed sublattice of $\mathbb{GP}$ is homeomorphic to one of the above defined spaces. The above list of spaces cannot be essentially reduced: the only homeomorphic pairs are $\mathbb{GP}_\alpha$ and $\mathbb{GP}_\beta$, where $\alpha, \beta < \omega_1$ are such that $(\alpha+1) \cong^{\rm top} (\beta+1)$. Of course $\mathbb{GP}_\alpha$ for $\alpha < \omega_1$ is neither homeomorphic to $\mathbb{GP}$ nor to $\triangle$, because of the Cantor-Bendixson rank. In order to see that $\triangle$ is not homeomorphic to $\mathbb{GP}$ observe that $\mathrm{D}^{\omega_1}(\triangle) \cong^{\rm top} \omega_1 + 1$ while $\mathrm{D}^{\omega_1}(\mathbb{GP}) \cong^{\rm top} \omega_1 + 1 + \omega_1^*$, where ω_1^* denotes ω_1 with the reversed ordering.

Given topological spaces X, Y, we denote by $X \oplus Y$ their topological (disjoint) sum. Notice that $\mathbb{GP}_\alpha \oplus \mathbb{GP}_\beta \cong^{\rm top} \mathbb{GP}_{\alpha+1+\beta}$ and $\mathbb{GP}_\alpha \oplus \triangle \cong^{\rm top} \triangle$ for a countable ordinal α. The latter follows from the fact that $\triangle$ contains clopen sets homeomorphic to $\mathbb{GP}_\theta$ for every $\theta < \omega_1$ and $\mathbb{GP}_\alpha \oplus \mathbb{GP}_\theta \cong^{\rm top} \mathbb{GP}_\theta$, whenever $\theta > \alpha$ is indecomposable. Notice also that if $X = \mathbb{GP}_\alpha$ for some $\alpha \leq \omega_1$ or $X = \triangle$ then $(\rho+1) \oplus X \cong^{\rm top} X$ for every countable ordinal ρ. We shall use these observations below.

Lemma 5.1. *Let K be a sublattice of $\mathbb{GP}$ and assume that $\alpha_0 \leq \alpha_1$, $\beta_0 \leq \beta_1$ are such that $\langle \alpha_0, \beta_1 \rangle, \langle \alpha_1, \beta_0 \rangle \in K$. Let $A = \{\xi \in [\alpha_0, \alpha_1] : \langle \xi, \beta_0 \rangle \in K\}$ and $B = \{\eta \in [\beta_0, \beta_1] : \langle \alpha_0, \eta \rangle \in K\}$. Then*

$$K \cap ([\alpha_0, \alpha_1] \times [\beta_0, \beta_1]) = A \times B.$$

Proof. Fix $\langle a, b \rangle \in A \times B$. Then $\alpha_0 \leq a$, $\beta_0 \leq b$ and $\langle a, \beta_0 \rangle, \langle \alpha_0, b \rangle \in K$, therefore $\langle a, b \rangle = \langle a, \beta_0 \rangle \vee \langle \alpha_0, b \rangle \in K$. Hence $A \times B \subseteq K$.

Now fix $\langle x, y \rangle \in K$ such that $\alpha_0 \leq x \leq \alpha_1$ and $\beta_0 \leq y \leq \beta_1$. Then $\langle x, \beta_0 \rangle = \langle x, y \rangle \wedge \langle \alpha_1, \beta_0 \rangle \in K$, so $x \in A$. Similarly $y \in B$. □

Of course the above lemma is valid for sublattices of products of two arbitrary chains.

Lemma 5.2. *Let K be a closed uncountable sublattice of $\mathbb{GP}_\theta$, where $\theta < \omega_1$. Then K is homeomorphic to $\mathbb{GP}_\alpha$ for some $\alpha \leq \theta$.*

Proof. Define $\phi(\eta) = \sup\{\xi < \omega_1 : \langle \xi, \eta \rangle \in K\}$. Let $\rho < \omega_1$ be such that $\phi(\eta) < \rho$ whenever $\phi(\eta) < \omega_1$. Let β_0, β_1 be the minimal and the maximal ordinals respectively such that $\phi(\beta_0) = \phi(\beta_1) = \omega_1$. The maximal one exists by compactness. Choose $\alpha_0 > \rho$ so that $\langle \alpha_0, \beta_1 \rangle \in K$ and let $\alpha_1 = \omega_1$. Apply Lemma 5.1. We conclude that $K_1 = K \cap [\alpha_0, \omega_1] \times [\beta_0, \beta_1]$ equals $A \times B$, where the order type of A is necessarily $\omega_1 + 1$ and the order type of B is $\alpha + 1$ for some $\alpha \leq \theta$. Now the remaining part, that is $K_0 = K \cap ([0, \alpha_0) \times (\theta+1))$ is countable, therefore homeomorphic to an ordinal. By increasing α_0, we may assume that K_0 is clopen in K. Hence $K_0 \cup K_1$ is homeomorphic to $K_1 = A \times B \cong^{\rm top} \mathbb{GP}_\alpha$. □

Theorem 5.3. *Let K be an uncountable closed sublattice of $\mathbb{GP}$. Then K is homeomorphic to one of the following spaces: $\triangle$, $\mathbb{GP}$ and $\mathbb{GP}_\alpha$, where $\alpha < \omega_1$.*

Proof. Define $T = \{\alpha < \omega_1 : \langle \alpha, \omega_1 \rangle \in K\}$ and $R = \{\beta < \omega_1 : \langle \omega_1, \beta \rangle \in K\}$. We consider the following cases:

(1) Both sets T and R are nonempty.

(2) Both T and R are empty.

(3) Exactly one of the sets T, R is nonempty.

We first deal with case (1). Fix $\alpha \in T$, $\beta \in R$. By Lemma 5.1, $K \cap ([\alpha, \omega_1] \times [\beta, \omega_1]) = A \times B$ for some sets A, B. Let $\delta = \max\{\alpha, \beta\}$. Define

$$U = K \cap ([0, \omega_1] \times [0, \delta]), \quad V = K \cap ([0, \delta] \times (\delta, \omega_1]), \quad W = K \cap (\delta, \omega_1]^2.$$

Then U, V, W form a partition of K into clopen sublattices. By Lemma 5.2, each of the sets U, V is either countable or homeomorphic to $\mathbb{GP}_\xi$ for some $\xi < \omega_1$. The same applies to $U \cup V = U \oplus V$. Further, $W = A' \times B'$, where $A' = A \cap (\delta, \omega_1]$ and $B' = B \cap (\delta, \omega_1]$. Thus W is either countable or homeomorphic to $\mathbb{GP}_\eta$ for some $\eta \leq \omega_1$. We conclude that $K = U \oplus V \oplus W$ is homeomorphic to $\mathbb{GP}_\gamma$ for some $\gamma \leq \omega_1$.

We now proceed to case (2). We claim that (2) implies $K \cong^{\mathrm{top}} \omega_1 + 1$. Define $r(\xi) = \sup\{\beta : \langle \xi, \beta \rangle \in K\}$ and $u(\eta) = \sup\{\alpha : \langle \alpha, \eta \rangle \in K\}$ for $\xi, \eta < \omega_1$. Both $r(\xi)$ and $u(\eta)$ are countable ordinals, because $T = R = \emptyset$. Observe that for every $\delta < \omega_1$ the set $K \cap (\delta, \omega_1]^2$ is uncountable; in particular $\langle \omega_1, \omega_1 \rangle$ is an accumulation point of K. Indeed, $K \setminus (\delta, \omega_1]^2 \subseteq (\gamma + 1)^2$, where

$$\gamma = \max\Big(\sup_{\xi \leq \delta} r(\xi), \sup_{\eta \leq \delta} u(\eta)\Big).$$

Fix a big enough regular cardinal $\chi > \aleph_1$ and fix a continuous chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable elementary substructures of $\langle H(\chi), \in \rangle$ such that $K \in M_0$. Let $\theta_\alpha = \omega_1 \cap M_\alpha$. By elementarity, θ_α is indecomposable and $\ln(\theta_\alpha) = \theta_\alpha$. Let $K_\alpha = K \cap [0, \theta_\alpha]^2$. Also by elementarity we deduce that $\langle \theta_\alpha, \theta_\alpha \rangle \in K_\alpha$. Recall that $\mathrm{rk}^K(x)$ denotes the Cantor-Bendixson rank of x in K. By elementarity $\mathrm{rk}^{K_\alpha}(x) \in M_\alpha$ whenever $x \in K_\alpha$. Observe that $K_\alpha \setminus \{\langle \theta_\alpha, \theta_\alpha \rangle\} \subseteq M_\alpha$. This is because, again by elemetarity, $r(\xi) < \theta_\alpha$ and

$u(\eta) < \theta_\alpha$ for every $\xi, \eta < \theta_\alpha$. It follows that $\mathrm{rk}(K_\alpha) \leq \theta_\alpha$. On the other hand, for every $\xi < \theta_\alpha$ there is an element x of K with $\mathrm{rk}^K(x) \geq \xi$, therefore by elementarity there is also such an element in M_α. That is, for every $\xi < \theta_\alpha$ there is an element x of K_α such that $\mathrm{rk}^{K_\alpha}(x) \in M_\alpha$ and $\mathrm{rk}^{K_\alpha}(x) \geq \xi$. This shows that $\mathrm{rk}(K_\alpha) = \theta_\alpha$ and $e(K_\alpha) = \langle \theta_\alpha, \theta_\alpha \rangle$. Thus $K_\alpha \cong^{\mathrm{top}} \theta_\alpha + 1$.

It is now straight how to find a homeomorphism $h\colon K \to \omega_1 + 1$. We construct inductively a sequence of homeomorphisms $h_\alpha\colon K_\alpha \to \theta_\alpha + 1$ such that h_β extends h_α whenever $\beta > \alpha$. At a successor stage, notice that K_α is clopen in $K_{\alpha+1}$ and h_α can be extended to a homeomorphism $h_{\alpha+1}\colon K_{\alpha+1} \to \theta_{\alpha+1} + 1$, because $\theta_{\alpha+1}$ is indecomposable. Suppose now that δ is a limit ordinal and h_α has been defined for every $\alpha < \delta$. Observe that $K_\delta = \{\langle \theta_\delta, \theta_\delta \rangle\} \cup \bigcup_{\alpha<\delta} K_\alpha$, by the continuity of the chain $\{M_\alpha\}_{\alpha<\omega_1}$ and by the above remarks. Let h_δ be the extension of $\bigcup_{\alpha<\delta} h_\alpha$ defined by $h_\delta(\langle \theta_\delta, \theta_\delta \rangle) = \theta_\delta + 1$. We only need to verify the continuity of h_δ at $\langle \theta_\delta, \theta_\delta \rangle$. Fix a neighborhood $U = (\rho, \theta_\delta]$ of θ_δ in $\theta_\delta + 1$. We may assume that $\rho = \theta_\alpha$ for some $\alpha < \delta$. Recall that $h_\delta\,[K_\alpha] = \theta_\alpha + 1$, so ${h_\delta}^{-1}[U] = K_\delta \setminus K_\alpha$, which is clopen in K_δ. This shows the continuity of h_δ. Finally $h := h_{\omega_1}$ is the desired homeomorphism of K onto $\omega_1 + 1$.

We are left with case (3). We assume that $T = \emptyset$ and $R \neq \emptyset$ (both possibilities are symmetric). Define the function r as before, i.e. $r(\xi) = \sup\{\beta : \langle \xi, \beta \rangle \in K\}$ for $\xi < \omega_1$. Clearly $r(\xi) < \omega_1$, because $T = \emptyset$. Suppose that $\delta = \sup_{\xi<\omega_1} r(\xi)$ is countable. Let $U = K \cap [0, \omega_1] \times [0, \delta]$ and $V = K \cap \big(\{\omega_1\} \times (\delta, \omega_1]\big)$. Clearly, U, V are disjoint clopen sublattices of $K = U \cup V$. Again by Lemma 5.2, U is either countable or homeomorphic to $\mathbb{GP}_\rho$ for some $\rho \leq \delta$. Finally, V is either countable or homeomorphic to $\omega_1 + 1$, so we conclude that $K \cong^{\mathrm{top}} \mathbb{GP}_\gamma$ for some $\gamma \leq \delta + 1$.

Assume now that the function r is unbounded, i.e. $\sup_{\xi<\omega_1} r(\xi) = \omega_1$. We claim that $K \cong^{\mathrm{top}} \triangle$. First we show that R is unbounded in ω_1. Indeed let $\nu \in R$. Choose $\xi < \omega_1$ and $\nu' \in [\nu, \omega_1)$ such that $r(\xi) > \nu$ and $\langle \xi, \nu' \rangle \in K$. Then $\langle \omega_1, \nu' \rangle = \langle \xi, \nu' \rangle \vee \langle \omega_1, \nu \rangle \in K$. That is $\nu' \in R$. Hence the order-type of R is ω_1.

Next, as in case (2), we fix a continuous chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable elementary substructures of a big enough $\langle H(\chi), \in \rangle$, such that $K \in M_0$. Let $\theta_\alpha = \omega_1 \cap M_\alpha$ and let $K_\alpha = K \cap [0, \theta_\alpha]^2$. Further, let $\triangle_\alpha = \triangle \cap [0, \theta_\alpha]^2$ and let $R_\alpha = R \cap \theta_\alpha$. Since the order-type R is ω_1, by elementarity, the order type of R_α is θ_α.

Fix $\alpha < \omega_1$ and fix $\eta \in R_\alpha$. We claim that for every $\xi < \omega_1$ there is

$\xi' \in [\xi, \omega_1)$ such that $\langle \xi', \eta \rangle \in K$. Indeed, since r is unbounded, we may find an arbitrary large countable ordinal ξ' such that $r(\xi') > \eta$. Let $\langle \xi', \rho \rangle \in K$ be such that $\rho \geq \eta$. Then $\langle \xi', \eta \rangle = \langle \xi', \rho \rangle \wedge \langle \omega_1, \eta \rangle \in K$. By elementarity, for every $\xi < \theta_\alpha$ there is $\xi' \in [\xi, \theta_\alpha)$ such that $\langle \xi', \eta \rangle \in K_\alpha$. This shows that $\langle \theta_\alpha, \eta \rangle$ is an accumulation point of $K_\alpha \cap ([0, \theta_\alpha) \times \{\eta\})$. For every $\xi < \theta_\alpha$ there is an element x of $K \cap ([0, \omega_1) \times \{\eta\})$ with $\mathrm{rk}^K(x) \geq \xi$, therefore by elementarity there is also such an element in M_α. Since θ_α is indecomposable and $\ln(\theta_\alpha) = \theta_\alpha$, we conclude that $\mathrm{rk}^{K_\alpha}(\langle \theta_\alpha, \eta \rangle) \geq \theta_\alpha$. It follows that

$$\mathrm{D}^{\theta_\alpha}(K_\alpha) = K_\alpha \cap \big(\{\theta_\alpha\} \times \omega_1\big) = \{\theta_\alpha\} \times \big(R_\alpha \cup \{\theta_\alpha\}\big) .$$

A similar fact applies to $\triangle_\alpha$, that is $\mathrm{D}^{\theta_\alpha}(\triangle_\alpha) = \{\theta_\alpha\} \times (\theta_\alpha + 1)$. Recall that the order type of R_α is θ_α. It follows that K_α is homeomorphic to $\triangle_\alpha$ and every homeomorphism between these spaces maps $\{\theta_\alpha\} \times R_\alpha$ onto $\{\theta_\alpha\} \times \theta_\alpha$. Given a homeomorphism $f \colon K_\alpha \to \triangle_\alpha$, we shall define $f^* \colon R_\alpha \to \theta_\alpha$ to be the unique map satisfying the equation $f(\langle \theta_\alpha, \eta \rangle) = \langle \theta_\alpha, f^*(\eta) \rangle$.

We construct a sequence of homeomorphisms $h_\alpha \colon K_\alpha \to \triangle_\alpha$ satisfying the following conditions:

(i) h_β extends h_α whenever $\beta > \alpha$.

(ii) If $\alpha < \beta$ then $f_\beta(\langle \xi, \eta \rangle) = \langle \xi, f^*_\alpha(\eta) \rangle$ for every $\langle \xi, \eta \rangle \in K_\beta \setminus K_\alpha$ such that $\eta < \theta_\alpha$.

Condition (ii) requires some explanation. Fix $\alpha < \beta < \omega_1$. Define

$$K^\beta_\alpha = K_\beta \cap \big([\theta_\alpha, \theta_\beta] \times (\theta_\alpha, \theta_\beta]\big) \quad \text{and} \quad B^\beta_\alpha = K_\beta \setminus \big(K_\alpha \cup K^\beta_\alpha\big) .$$

Similarly, define

$$\triangle^\beta_\alpha = \triangle_\beta \cap \big([\theta_\alpha, \theta_\beta] \times (\theta_\alpha, \theta_\beta]\big) \quad \text{and} \quad \square^\beta_\alpha = \triangle_\beta \setminus \big(\triangle_\alpha \cup \triangle^\beta_\alpha\big) .$$

Clearly, $\square^\beta_\alpha$ is a rectangle of the form $(\theta_\alpha, \theta_\beta] \times [0, \theta_\alpha]$. We claim that B^β_α is also a rectangle, whose horizontal side has order type θ_β and whose vertical side is $R_\alpha \cup \{\theta_\alpha\}$ (so its order type is $\theta_\beta + 1$ as well). Let $\pi = \min R$. Then $\pi \in M_0 \subseteq M_\beta$, so again by elementarity we deduce that $\langle \theta_\beta, \pi \rangle$ is an accumulation point of $K_\beta \cap (\theta_\beta \times \{\pi\})$. In particular, $\langle \theta_\beta, \pi \rangle \in K_\beta$. Also $\langle \theta_\alpha, \theta_\alpha \rangle \in K_\beta$, so by Lemma 5.1,

$$K_\beta \cap ([\theta_\alpha, \theta_\beta] \times [\pi, \theta_\beta]) = C \times D,$$

where $D := R_\alpha \cup \{\theta_\alpha\}$ and the order type of C is necessarily $\theta_\beta + 1$. Thus $B_\alpha^\beta = \big(C \setminus \{\theta_\alpha\}\big) \times D$ is naturally homeomorphic to $\square_\alpha^\beta$ and condition (ii) makes sense.

We start with any homeomorphism $f_0 \colon K_0 \to \triangle_0$, which exists because both spaces are unitary of rank $\theta_0 + \theta_0$. Fix $\beta > 0$ and suppose that homeomorphisms f_α have been defined for all $\alpha < \beta$. Suppose first that $\beta = \alpha + 1$. By the above remarks, there is a unique way to define $f_{\alpha+1}$ on $B_\alpha^{\alpha+1}$ so that (ii) holds. Also, $K_\alpha^{\alpha+1}$ is homeomorphic to $\triangle_\alpha^{\alpha+1}$ so it is possible to extend f_α to $f_{\alpha+1}$ and any such extension maps $\langle \theta_\beta,\, \theta_\beta \rangle$ onto $\langle \theta_\beta,\, \theta_\beta \rangle$.

Suppose now that β is a limit ordinal. The set $D = \bigcup_{\xi<\beta} K_\xi$ is dense in K_β and $K_\beta \setminus D = \big(R_\beta \cup \{\theta_\beta\}\big) \times \{\theta_\beta\}$. We need to define $f_\beta \supseteq \bigcup_{\alpha<\beta} f_\alpha$. Fix $\eta < \theta_\beta$. Then $\eta < \theta_\alpha$ for some $\alpha < \beta$. Let $f_\beta(\langle \theta_\beta,\, \eta \rangle) = \langle \theta_\beta,\, f_\alpha^*(\eta) \rangle$. Condition (ii) says that $f_\xi^*(\eta) = f_\alpha^*(\eta)$ for every $\xi \in [\alpha, \beta)$, thus (ii) holds also for β. Since f_α^* is continuous, we deduce that f_β is continuous at $\langle \theta_\beta,\, \eta \rangle$. Finally set $f_\beta(\langle \theta_\beta,\, \theta_\beta \rangle) = \langle \theta_\beta,\, \theta_\beta \rangle$. It is clear that f_β is continuous at $\langle \theta_\beta,\, \theta_\beta \rangle$.

Finally, the map $f_{\omega_1} \colon K \to \triangle$ which is the unique extension of $\bigcup_{\alpha<\omega_1} f_\alpha$, is a homeomorphism. Its continuity follows by the same argument as in the limit stage of the construction. This completes the proof. $\square$

Corollary 5.4. *Let $P = P_0 \cup P_1$ be an uncountable poset such that both P_0, P_1 are well ordered chains of order type $\leq \omega_1$. Then the Boolean algebra $\widehat{F}\,(P)$ is isomorphic to one of the following algebras: $\widehat{F}\,(\omega_1)$, $\widehat{F}\,(\omega_1 \uplus \omega_1)$, $\widehat{F}\,(\omega_1 \uplus \alpha)$, where α is a countable ordinal, and $\widehat{F}\,(\omega_1 \times 2)$, where $\omega_1 \times 2$ is endowed with the coordinatewise order.*

Proof. Since P is uncountable, we may assume that the order type of P_0 is ω_1. Let α be the order type of P_1 and let $f \colon \omega_1 \uplus \alpha \to P$ be the natural order preserving surjection. By duality, f corresponds to a topological lattice embedding of $\mathrm{FS}\,(P)$ into $\mathrm{FS}\,(\omega_1 \uplus \alpha)$. Now the corollary follows from Theorem 5.3, noticing that $\triangle$ is isomorphic to $\mathrm{FS}\,(\omega_1 \times 2)$. $\square$

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