Preprint, Institute of Mathematics, AS CR, Prague. 2010-3-11

THE RATE OF CONVERGENCE IN THE METHOD OF ALTERNATING PROJECTIONS

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Abstract. A generalization of the cosine of the Friedrichs angle between two subspaces to several closed subspaces in a Hilbert space is given. This is used to analyze the rate of convergence in the von Neumann-Halperin method of cyclic alternating projections. General dichotomy theorems are proved, in the Hilbert or Banach space situation, providing conditions under which the alternative QUC/ASC (quick uniform convergence versus arbitrarily slow convergence) holds. Several meanings for ASC are proposed.

1. Introduction

In what follows H will always denote a complex Hilbert space. For a closed linear subspace S of H we denote by P_S the orthogonal projection onto S, and by $S^\perp$ its orthogonal complement in H.

The method of alternating projections. It was proved by J. von Neumann [26] that for two closed subspaces M_1 and M_2 of H, with intersection $M = M_1 \cap M_2$, the following convergence result holds:

$$\lim_{n\to\infty} \|(P_{M_2}P_{M_1})^n(x) - P_M(x)\| = 0 \quad (x \in H). \tag{1.1}$$

Using the notation $T = P_{M_2}P_{M_1}$, von Neumann's result says that the iterates T^n of T are strongly convergent to $T^\infty = P_M$. The method of constructing the iterates of T by alternately projecting onto one subspace and then the other is called the *method of alternating projections*. This algorithm, and its variations, occur in several fields, pure or applied. We refer to [10, Chapter 9] as a source for more information.

A generalization of von Neumann's result to $N \geq 2$ closed subspaces $M_1, \ldots, M_N$ with intersection $M = M_1 \cap M_2 \cdots \cap M_N$ was proved by Halperin [15]: for each $x \in H$ we have

$$\lim_{n\to\infty} \|(P_{M_N} \cdots P_{M_2}P_{M_1})^n(x) - P_M(x)\| = 0. \tag{1.2}$$

The algorithm provided by Halperin's result will be called in this paper the *method of cyclic alternating N projections.*

A Banach space extension of Halperin's result was proved by Bruck and Reich [9]: if X is a *uniformly convex* Banach space and P_j, $1 \leq j \leq N$, are $N \geq 2$ *norm one projections* in $\mathcal{B}(X)$, then the iterates of $T = P_N \cdots P_2P_1$ are strongly convergent. The strong limit T^∞ is a projection of norm one onto the intersection of the ranges of P_j. The same result holds [3] if X is *uniformly smooth* and each projection P_j is of norm one. It also holds [3]

The first two named authors have been partially supported by ANR Projet Blanc DYNOP. The third named author was supported by grant No. 201/06/0128 of GA ČR and Institutional Research Plan AV 0Z 10190503.

if X is a reflexive (complex) Banach space and each projection P_j is hermitian (that is, with real numerical range). We refer to [3] and the references therein for other results of this type.

An interesting extension of the method of cyclic alternating projections is the *method of random alternating projections*. Let P_j, $1 \le j \le N$, be $N \ge 2$ orthogonal projections in $\mathcal{B}(H)$, $M = \cap_{j=1}^N \operatorname{Ran}(P_j)$, and let $(i_k)_{k\ge1}$ be a sequence from $\{1, 2, \ldots, N\}$ (random samples). The method of random alternating projections asks about the convergence of the sequence $(x_n)_{n\ge0}$ given by $x_0 = x$, $x_n = P_{i_n} x_{n-1}$. It is an open problem whenever $(x_n)_{n\ge0}$ converges in the topology of H for each $x_0 \in H$. It was proved by Amemiya and Ando [1] that the sequence $(x_n)_{n\ge0}$ is always *weakly* convergent. If each j between 1 and N occurs infinitely many times in the sequence of random samples, then the sequence is weakly convergent to $P_M x$. We refer to [14, 28, 18] for results related to this problem.

The rate of convergence. It is important for applications to know how fast the algorithm given by the method of alternating projections, or its variations, converge. For $N = 2$ a quite complete description of the rate of convergence is now known, it terms of the notion of *angle* of subspaces.

1.3. **Definition.** (Friedrichs angle) Let M_1 and M_2 be two closed subspaces of the Hilbert space H with intersection $M = M_1 \cap M_2$. The *Friedrichs angle* between the subspaces M_1 and M_2 is defined to be the angle in $[0, \pi/2]$ whose cosine is given by

$$c(M_1, M_2) := \sup\{|\langle x, y\rangle| : x \in M_1 \cap M^\perp \cap B_H, y \in M_2 \cap M^\perp \cap B_H\},$$

where $B_H := \{h \in H : \|h\| \le 1\}$ is the unit ball of H. The *minimal angle* (or Dixmier angle) between the subspaces M_1 and M_2 is defined to be the angle in $[0, \pi/2]$ whose cosine is given by

$$c_0(M_1, M_2) := \sup\{|\langle x, y\rangle| : x \in M_1 \cap B_H, y \in M_2 \cap B_H\}.$$

We note that $c(M_1, M_2) = c_0(M_1 \cap M^\perp, M_2 \cap M^\perp)$, and that $c_0(M_1, M_2) = 1$ if $M \neq \{0\}$. We also have $c(M_1, M_2) = c(M_1^\perp, M_2^\perp)$. We refer to the survey paper [11] for more information about the different notions of angle between subspaces of infinite dimensional Hilbert spaces and their properties.

It was proved by Aronszajn [2] (upper bound) and by Kayalar and Weinert [16] (equality) that

$$\|(P_{M_2} P_{M_1})^n - P_M\| = c(M_1, M_2)^{2n-1} \quad (n \ge 1).$$

This formula shows that the iterates of $T = P_{M_2} P_{M_1}$ converges *uniformly* to $T^\infty = P_M$ if and only if $c(M_1, M_2) < 1$, i.e., if the Friedrichs angle between M_1 and M_2 is positive. When this happens, the iterates of $T = P_{M_2} P_{M_1}$ converges "quickly" (i.e. at the rate of a geometrical progression) to $T^\infty = P_M$, in the following sense:

(QUC) (quick uniform convergence) there exist $C > 0$ and $\alpha \in]0, 1[$ such that

$$\|T^n - T^\infty\| \le C\alpha^n \quad (n \ge 1).$$

It is also known [11] that $c(M_1, M_2) < 1$ if and only if $M_1 + M_2$ is closed, if and only if $M_1^\perp + M_2^\perp$ is closed, if and only if $(M_1 \cap M^\perp) + (M_2 \cap M^\perp)$ is closed.

When $M_1 + M_2$ is not closed, we have strong, but not uniform convergence. It was recently proved by Bauschke, Deutsch and Hundal (see [5] for the history of this result) that given any sequence of reals decreasing to zero, there exists a point in the space with the property that the convergence in the method of alternating projections (von Neumann theorem) is at least as slow as this sequence of reals. Thus the iterates of the product of two projections converge quickly, or arbitrarily slow. We call this alternative the (QUC)/(ASC) dichotomy : one has quick uniform convergence or arbitrarily slow convergence. We shall consider several meanings of (ASC) in this paper.

The results concerning the rate of convergence in Halperin's theorem for $N \geq 3$ are not as complete as the results described above for $N = 2$. We refer to [11], [12], [10, Chapter 9], [29] and their references for several results concerning the rate of convergence of the method of cyclic alternating N projections.

What this paper is about. The main goal of the present paper is to discuss the rate of convergence in Halperin's theorem and to generalize some of the previous known results ($N = 2$) to the case of several subspaces ($N \geq 3$). We show by operator-theoretical methods that the (QUC)/(ASC) dichotomy always holds at soon as the iterates of T are strongly convergent. Several interpretations of (ASC) are proposed, and general dichotomy theorems are obtained in the Hilbert or Banach space situation, depending on several spectral properties imposed upon the operator T. This imply at once the dichotomy (QUC)/(ASC) in all above-mentioned generalizations of the method of alternating projections. We also give a generalization of the Friedrichs angle to several subspaces, $c(M_1, \cdots, M_N)$, and prove that condition (QUC) holds in Halperin's theorem if and only if $c(M_1, \cdots, M_N) < 1$. Estimates for $\|(P_{M_N} \cdots P_{M_2} P_{M_1})^n - P_M\|$ are given in this case and several equivalences of the condition $c(M_1, \cdots, M_N) < 1$ are obtained. Some of them are expressed in terms of random products $P_{i_k} \cdots P_{i_1}$ of projections. More specific descriptions of the results, and information about how the paper is organized, are provided below.

Conditions for arbitrarily slow convergence. Several dichotomy theorems of the type quick uniform convergence *versus* arbitrarily slow convergence are proved in this paper. The quick uniform condition is the condition (QUC) presented above. We shall consider in Section 2 the following conditions for (ASC):

(ASC1) (arbitrarily slow convergence, variant 1) for every $\varepsilon > 0$ and every sequence $(a_n)_{n\geq 1}$ of positive numbers such that $\lim_{n\to\infty} a_n = 0$, there exists a vector $x \in X$ such that $\|x\| < \sup_n a_n + \varepsilon$ and $\|T^n x - T^\infty x\| \geq a_n$ for all n.

(ASC2) (arbitrarily slow convergence, variant 2) for every sequence $(a_n)_{n\geq 1}$ of positive numbers such that $\lim_{n\to\infty} a_n = 0$, there exists a *dense* subset of points $x \in X$ such that $\|T^n x - T^\infty x\| \geq a_n$ for all but a *finite* number of n's.

(ASC3) (arbitrarily slow convergence, variant 3) for every sequence $(a_n)_{n\geq 1}$ of positive numbers such that $\lim_{n\to\infty} a_n = 0$, there exist two vectors $x \in X$ and $y \in X^*$ such that $\mathrm{Re}\,\langle T^n x - T^\infty x, y\rangle \geq a_n$ for all $n \geq 1$. Furthermore, if there is a Banach space Y such that X is a (isometrical) subspace of Y^*, then the vector y can be chosen in Y;

(ASCH) (arbitrarily slow convergence, Hilbertian version) for every $\varepsilon > 0$ and every sequence $(a_n)_{n\geq 1}$ of positive numbers such that $\lim_{n\to\infty} a_n = 0$, there exists a vector $x \in X$ such that $\|x\| < \sup_n a_n + \varepsilon$ and $\operatorname{Re}\langle T^n x - T^\infty x, x\rangle \geq a_n$ for all $n \geq 1$.

In Section 2, based on the results from [23, 24], we prove that if the iterates of $T \in \mathcal{B}(X)$ are strongly convergent, then one has (QUC) or (ASC1). Also, if the iterates are strongly convergent, then the dichotomy (QUC)/(ASC2) holds. In the case when $T \in \mathcal{B}(X)$ is a power bounded, mean ergodic operator with spectrum $\sigma(T)$ included in $\mathbb{D}\cup\{1\}$, it is proved using the Katznelson-Tzafriri theorem [19] that the iterates of T are strongly convergent, and thus the previous dichotomies (QUC)/(ASC1) and (QUC)/(ASC2) apply. Moreover, the dichotomy (QUC)/(ASC3) holds whenever the Banach space X contains no isomorphic copy of c_0. If $X = H$ is a Hilbert space, then also the dichotomy (QUC)/(ASCH) holds. We prove that the (QUC) condition holds if and only if $\operatorname{Ran}(I - T)$ is closed. Applications to products of projections of norm one are given. In particular, the dichotomy (QUC)/(ASC) holds, with several variants of (ASC), for the cases covered by the theorems of von Neumann, Halperin, Bruck-Reich and those of [3].

A generalization of the Friedrichs angle. An extension of the cosine of Friedrichs angle to several subspaces $(M_1, \dots, M_N)$ will be given in Section 3. This generalization is a parameter $c(M_1, \dots, M_N)$ which lies between 0 and 1, and is defined as follows:

$$\begin{aligned} c(M_1,\cdots,M_N) &= \sup\left\{\frac{2}{N-1}\frac{\sum_{j<k}\operatorname{Re}\langle m_j,m_k\rangle}{\|m_1\|^2+\cdots+\|m_N\|^2} :\right. \\ &\qquad \left. m_j\in M_j\cap M^\perp, \|m_1\|^2+\cdots+\|m_N\|^2\neq 0\right\} \\ &= \sup\left\{\frac{1}{N-1}\frac{\sum_{j\neq k}\langle m_j,m_k\rangle}{\sum_{j=1}^N\langle m_j,m_j\rangle} :\right. \\ &\qquad \left. m_j\in M_j\cap M^\perp, \|m_1\|^2+\cdots+\|m_N\|^2\neq 0\right\}. \end{aligned}$$

The fact that this definition coincides with the classical one for two subspaces will be proved in Lemma 3.1. The extremal case $c(M_1, \dots, M_N) = 0$ corresponds to the case of pairwise orthogonal N subspaces ("angle" is $\pi/2$), while the other extremal case $c(M_1, \dots, M_N) = 1$ corresponds to the case of arbitrarily slow convergence in the method of cyclic alternating projections with N subspaces ("angle" is zero). Other related quantities are considered: the *configuration constant* $\kappa(M_1, \dots, M_N)$, the *inclination* $\ell(M_1, \dots, M_N)$, and the Friedrichs angle between the cartesian product $\mathbf{C} = M_1 \times \cdots \times M_N \subset H^N$ and the "diagonal subspace" $\mathbf{D} = \operatorname{diag}(H) = \{(y, \dots, y) : y \in H\} \subset H^N$.

In Section 4 we characterize in several ways when the dichotomy (QUC)/(ASC) arises. The characterizations are in terms of geometric properties of $(M_1, \cdots, M_N)$, of spectral properties of T, or of random products $P_{i_k} \cdots P_{i_1}$. We give an estimate for $\|T^n - P_M\|$ when $c(M_1, \dots, M_N) < 1$.

2. General dichotomy theorems

2.1. **Theorem** ((QUC)/(ASC1) and (QUC)/(ASC2))**.** *Let X be a Banach space and let $T \in \mathcal{B}(X)$ be such that the sequence of iterates (T^n) is strongly convergent to $T^\infty \in$*

$\mathcal{B}(X)$. *Then the following dichotomy holds : either (QUC), or (ASC1). The quick uniform convergence (condition (QUC)) holds if and only if*

$$\text{for every } \lambda \in \partial\mathbb{D}, \mathrm{Ran}(\lambda - T) \text{ is closed.} \tag{2.2}$$

In these statements, the condition (ASC1) can be replaced by (ASC2).

Proof. Suppose that the sequence of iterates T^n is strongly convergent to $T^\infty \in \mathcal{B}(X)$. Then T is *mean ergodic*, i.e., the Cesàro means $(I+T+\cdots+T^{n-1}/n$ are strongly convergent. Therefore ([20, page 73]) the space X can be decomposed as the direct sum of the kernel of $T-I$ and the closure of the range of the same operator, $X = \mathrm{Ker}(T-I) \oplus \overline{\mathrm{Ran}(T-I)}$. Moreover, T^∞ is the projection onto $\mathrm{Ker}(T-I)$ along $\overline{\mathrm{Ran}(T-I)}$. Notice also that T^∞ acts on the space $\mathrm{Ker}(T-I)$ as the identity. With respect to the decomposition $X = \mathrm{Ker}(T-I) \oplus \overline{\mathrm{Ran}(T-I)}$ we can write

$$T = \begin{pmatrix} T^\infty & 0 \\ 0 & A \end{pmatrix}$$

for some $A \in \mathcal{B}(\overline{\mathrm{Ran}(T-I)})$. It is not difficult to prove that for every $\lambda \in \mathbb{C}$, the range $\mathrm{Ran}(T-\lambda I)$ is closed if and only if $\mathrm{Ran}(A-\lambda I)$ is. The strong convergence of T^n and the Banach-Steinhaus theorem imply that T is *power bounded*, that is $\sup_{n\geq 1}\|T^n\| < \infty$. Thus the spectrum of T is included in the closed unit disk. As $\sigma(T) = \{1\} \cup \sigma(A)$, the same spectral inclusion holds for $\sigma(A)$. In particular, the spectral radius of A verifies $r(A) \leq 1$.

We distinguish two cases.

Case (1). We have $r(A) < 1$. Notice that we have

$$T^n - T^\infty = \begin{pmatrix} 0 & 0 \\ 0 & A^n \end{pmatrix}. \tag{2.3}$$

Since $r(A) < 1$, there exist $C > 0$ and $\alpha \in]0,1[$ such that

$$\|A^n\| \leq C\alpha^n \quad (n \geq 1).$$

This estimate and (2.3) gives the quick uniform convergence condition (QUC).

Case (2). We have $r(A) = 1$. Recall that $\|A^n y\| \to 0$ as $n \to \infty$, for each $y \in \overline{\mathrm{Ran}(T-I)}$. The conditions (ASC1) and (ASC2) follow now from [23].

Suppose that Case (1) is fulfilled, i.e. $r(A) < 1$. Then $A-\lambda$ is invertible for every $\lambda \in \partial\mathbb{D}$. In particular, $\mathrm{Ran}(A-\lambda) = \overline{\mathrm{Ran}(T-I)}$ is closed for each $\lambda \in \partial\mathbb{D}$. Thus $\mathrm{Ran}(T-\lambda)$ is also closed, for each $\lambda \in \partial\mathbb{D}$.

Suppose now that all subspaces $\mathrm{Ran}(T-\lambda)$, $\lambda \in \partial\mathbb{D}$, and so all $\mathrm{Ran}(A-\lambda)$, $\lambda \in \partial\mathbb{D}$, are closed. Then $r(A) < 1$. Indeed, suppose that $r(A) = 1$ and let $\lambda \in \partial\mathbb{D} \cap \sigma(A)$ be a point in the unimodular spectrum of A. Then the condition $\|A^n y\| \to 0$ as $n \to \infty$ for each y shows that λ cannot be an eigenvalue: if $Ay = \lambda y$, then $y = 0$. Indeed, we have $\|y\| = \|\lambda^{-n}A^n y\| = \|A^n y\| \to 0$ as $n \to \infty$. Thus $\lambda \in \sigma(A) \setminus \sigma_p(A)$ and $\mathrm{Ran}(A-\lambda)$ is closed. Therefore $A-\lambda$ is an upper semi-Fredholm operator. As $A-\lambda I$ is a limit of invertible operators $A - \frac{n+1}{n}\lambda I$, the index $\mathrm{ind}\,(A-\lambda I)$ of $A-\lambda I$ is 0. Hence $A-\lambda I$ is invertible, a contradiction with the assumption that $\lambda \in \sigma(A)$. Thus $r(A) < 1$. □

2.4. **Remark.** The following is a different argument for the last part of the proof, without using Fredholm theory. As $\lambda \in \sigma(A) \setminus \sigma_p(A)$ and $\mathrm{Ran}(A-\lambda)$ is closed, the operator $A-\lambda$ is lower bounded, and thus λ is not in the approximate point spectrum of A. As every point in the boundary of the spectrum is in the approximate point spectrum, we obtain the desired contradiction. We refer the reader to [25] for all the spectral theory notions and facts we are using in the present paper.

2.5. **Theorem** ((QUC)/(ASC3) and (QUC)/(ASCH))**.** *Let X be a Banach space and let $T \in \mathcal{B}(X)$ be a power bounded, mean ergodic operator with spectrum $\sigma(T)$ included in $\mathbb{D} \cup \{1\}$. Then the sequence of iterates T^n is strongly convergent to a certain operator $T^\infty \in \mathcal{B}(X)$, and the dichotomies of Theorem 2.1 apply. Moreover, the dichotomy (QUC)/(ASC3) holds whenever X contains no isomorphic copy of c_0. If $X = H$ is a Hilbert space, then also the dichotomy (QUC)/(ASCH) holds.*

In all these statements, the quick uniform convergence condition (QUC) holds if and only if

$$\mathrm{Ran}(I-T) \text{ is closed.} \tag{2.6}$$

Proof. Again, using the mean ergodicity and [20, page 73], the space X can be decomposed as the direct sum of the kernel of $T-I$ and the closure of the range of the same operator, $X = \mathrm{Ker}(T-I) \oplus \overline{\mathrm{Ran}(T-I)}$. According to the Katznelson-Tzafriri theorem [19], the power boundedness condition and the spectral condition $\sigma(T) \subset \mathbb{D} \cup \{1\}$ imply $\lim_{n\to\infty} \|T^{n+1} - T^n\| = 0$. This shows that the sequence of iterates (T^n) of T converge strongly to 0 on the range of $T-I$. The same holds for the closure $\overline{\mathrm{Ran}(T-I)}$. As T acts like identity on $\mathrm{Ker}(T-I)$, we get that (T^n) converges strongly to T^∞, the projection onto $\mathrm{Ker}(T-I)$ along $\overline{\mathrm{Ran}(T-I)}$. Thus we can apply Theorem 2.1 to obtain the dichotomies (QUC)/(ASC1) and (QUC)/(ASC2).

Let us show that (QUC)/(ASC3) also holds if X contains no isomorphic copy of c_0. Using the notation of the proof of Theorem 2.1, if the condition (QUC) is not satisfied, then $r(A) = 1$ (Case (2) in the proof of Theorem 2.1). As $\sigma(T) \subset \mathbb{D} \cup \{1\}$, the same inclusion holds for the spectrum of A. Therefore $1 \in \sigma(A)$. Remark also that $\|A^n y\| \to 0$ as $n \to \infty$ since (T^n) converges strongly to T^∞. We can now apply [24, Theorem 1]. To obtain the dichotomy (QUC)/(ASCH) if $X = H$ is a Hilbert space, we use [23, Theorem 2] (see also [4, Theorem 1] for the case of weak convergence). $\square$

Applications to the method of alternating projections. We introduce first some notation, and recall for the convenience of the reader some Banach space terminology. Let X be a Banach space and let $P_1, \cdots, P_N$ be $N \geq 2$ fixed projections ($P_j^2 = P_j$) acting on X. We denote by $\mathcal{S} = \mathcal{S}(P_1, \cdots, P_N)$ the convex multiplicative semigroup generated by $P_1, \cdots, P_N$. Recall that this is the convex hull of the set of all products with factors from $P_1, \cdots, P_N$, and that the convex hull of every multiplicative semigroup of operators is a semigroup.

The space X is said to be *uniformly convex* if for every $\varepsilon > 0$ there is $\delta > 0$ such that for any two vectors, x and y, with $\|x\| \leq 1$ and $\|y\| \leq 1$, $\|x+y\| > 2-\delta$ implies $\|x-y\| < \varepsilon$. An (equivalent) definition of a *uniformly smooth* Banach space is the following: X is uniformly smooth if its dual, X^*, is uniformly convex. We refer to [22] for more information.

We call $P \in \mathcal{B}(X)$ an *norm one projection* if $P^2 = P$ and $\|P\| = 1$. A norm one projection is sometimes called an orthoprojection. A self-adjoint projection in a Hilbert space is called, as usual, an *orthogonal projection.* Recall that an operator T on a Banach space X is called *hermitian* if its numerical range is real. This is equivalent to ask that $\|\exp(itT)\| = 1$ for every real t. Hermitian operators on Hilbert spaces coincide with the self-adjoint ones. See for instance [8] and the references therein.

2.7. **Theorem.** *Let X be a Banach space, and let $P_1, \cdots, P_N$ be $N \geq 2$ projections on X. Let T be an operator in $\mathcal{S}(P_1, \cdots, P_N)$. If one of the following conditions below holds true, then the sequence of iterates of T converges strongly and every dichotomy (QUC)/(ASC1), (QUC)/(ASC2), (QUC)/(ASC3) and (QUC)/(ASCH) (if $X = H$ is a Hilbert space) applies:*

(i) *the space X is uniformly convex and each P_j, $1 \leq j \leq N$, is a norm one projection;*
(ii) *the space X is uniformly smooth, and each P_j, $1 \leq j \leq N$, is a norm one projection;*
(iii) *the space X is reflexive and each P_j verifies $\|P_j - \frac{1}{2}\| = \frac{1}{2}$. In particular, this holds if P_j is hermitian, $1 \leq j \leq N$.*

Proof. It was proved in [3] that in all three situations the spectrum of $T \in \mathcal{S}(P_1, \cdots, P_N)$ is included in $\mathbb{D} \cup \{1\}$ and that the iterates of T are strongly convergent. We apply the above dichotomy theorems. $\square$

3. A generalization of Friedrichs angle for N subspaces

As mentioned in Introduction, the rate of convergence in the method of alternating projections (von Neumann theorem for two closed subspaces M_1 and M_2)) is controlled by the Friedrichs angle $c(M_1, M_2)$. In order to introduce the generalization of the cosine of the Friedrichs angle to $N \geq 2$ closed subspaces, we start by giving an equivalent definition of the Friedrichs angle $c(M_1, M_2)$.

3.1. **Lemma.** *(a) Let M_1 and M_2 be two closed subspaces in H. Then*

$$c_0(M_1, M_2) = \sup\left\{\frac{2\operatorname{Re}\langle m_1, m_2\rangle}{\|m_1\|^2 + \|m_2\|^2} : m_1 \in M_1, m_2 \in M_2, (m_1, m_2) \neq (0,0)\right\}.$$

(b) Let M_1 and M_2 be two closed subspaces in H. Then

$$\begin{aligned} c(M_1, M_2) &= \sup\left\{\frac{2\operatorname{Re}\langle m_1, m_2\rangle}{\|m_1\|^2 + \|m_2\|^2} : m_j \in M_j \cap M^{\perp}, (m_1, m_2) \neq (0,0)\right\} \\ &= \sup\left\{\frac{\langle m_1, m_2\rangle + \langle m_2, m_1\rangle}{\langle m_1, m_1\rangle + \langle m_2, m_2\rangle} : m_j \in M_j \cap M^{\perp}, (m_1, m_2) \neq (0,0)\right\}. \end{aligned}$$

Proof. We give the proof only for the first equality of the second part. Denote by s the first supremum from the statement of part (b). For every pair (m_1, m_2) with $(m_1, m_2) \neq (0,0)$ we have

$$\begin{aligned}\frac{2}{\|m_1\|^2+\|m_2\|^2}\operatorname{Re}\langle m_1,m_2\rangle &\le \frac{1}{\|m_1\|\cdot\|m_2\|}\operatorname{Re}\langle m_1,m_2\rangle\\ &\le \frac{|\langle m_1,m_2\rangle|}{\|m_1\|\cdot\|m_2\|}\\ &\le c(M_1,M_2).\end{aligned}$$

Therefore $s \le c(M_1,M_2)$.

For the reverse inequality, let $\varepsilon>0$. Then there exist two elements $x_1\in M_1\cap M^\perp$ and $x_2\in M_2\cap M^\perp$ with $\|x_1\|=1$ and $\|x_2\|=1$ such that $c(M_1,M_2)<|\langle x_1,x_2\rangle|+\varepsilon$. Let $\theta\in\mathbb{R}$ be such that $\langle x_1,x_2\rangle=|\langle x_1,x_2\rangle|e^{i\theta}$, and set $m_1=e^{-i\theta}x_1$ and $m_2=x_2$. Then $m_1\in M_1\cap M^\perp$, $m_2\in M_2\cap M^\perp$ and $\|m_1\|=1$, $\|m_2\|=1$. We obtain

$$s\ge \frac{2\operatorname{Re}\langle m_1,m_2\rangle}{\|m_1\|^2+\|m_2\|^2}=\operatorname{Re}\left\langle e^{-i\theta}x_1,x_2\right\rangle=|\langle x_1,x_2\rangle|>c(M_1,M_2)-\varepsilon.$$

As ε is arbitrary, we obtain $s=c(M_1,M_2)$. □

3.2. **Definition.** Let $M_1,\cdots,M_N$ be $N\ge 2$ closed subspaces of H with intersection $M=M_1\cap\cdots\cap M_N$. The *Dixmier number* associated to $(M_1,\cdots,M_N)$ is defined as

$$\begin{aligned}c_0(M_1,\cdots,M_N) &= \sup\left\{\frac{2}{N-1}\frac{\sum_{j<k}\operatorname{Re}\langle m_j,m_k\rangle}{\|m_1\|^2+\cdots+\|m_N\|^2}:\right.\\ &\qquad \left. m_j\in M_j, \|m_1\|^2+\cdots+\|m_N\|^2\ne 0\right\}.\end{aligned}$$

The *Friedrichs number* $c(M_1,\cdots,M_N)$ associated to $(M_1,\cdots,M_N)$ is defined as

$$\begin{aligned}c(M_1,\cdots,M_N) &= \sup\left\{\frac{2}{N-1}\frac{\sum_{j<k}\operatorname{Re}\langle m_j,m_k\rangle}{\|m_1\|^2+\cdots+\|m_N\|^2}:\right.\\ &\qquad \left. m_j\in M_j\cap M^\perp, \|m_1\|^2+\cdots+\|m_N\|^2\ne 0\right\}\\ &= \sup\left\{\frac{1}{N-1}\frac{\sum_{j\ne k}\langle m_j,m_k\rangle}{\sum_{j=1}^N\langle m_j,m_j\rangle}:\right.\\ &\qquad \left. m_j\in M_j\cap M^\perp, \|m_1\|^2+\cdots+\|m_N\|^2\ne 0\right\}.\end{aligned}$$

We found convenient to introduce the following parameters, called the (reduced or not) configuration constants, although they can be expressed in terms of the Dixmier and Friedrichs numbers (see Proposition 3.6, (f)).

3.3. **Definition.** Let $M_1,\cdots,M_N$ be $N\ge 2$ closed subspaces of H with intersection $M=M_1\cap\cdots\cap M_N$. The number

$$\kappa_0(M_1,\cdots,M_N)=\sup\left\{\frac{1}{N}\frac{\|\sum_{j=1}^N m_j\|^2}{\sum_{j=1}^N\|m_j\|^2}: m_j\in M_j, \|m_1\|^2+\cdots+\|m_N\|^2\ne 0\right\}$$

is called the *non-reduced configuration constant* of $(M_1,\cdots,M_N)$. The number

$$\kappa(M_1,\cdots,M_N)=\sup\left\{\frac{1}{N}\frac{\|\sum_{j=1}^N m_j\|^2}{\sum_{j=1}^N\|m_j\|^2}: m_j\in M_j\cap M^\perp, \|m_1\|^2+\cdots+\|m_N\|^2\ne 0\right\}$$

is called the *configuration constant* of $(M_1,\cdots,M_N)$.

The configuration constant is related to the maximal possible norms of Gramian matrices. Recall that the *Gramian matrix* of an N-tuple of vectors $(v_1, \cdots, v_N)$ is the $N \times N$ matrix

$$G(v_1, \cdots, v_N) = [\langle v_i, v_j \rangle]_{1 \le i,j \le N}.$$

3.4. **Proposition.** *Let $M_1, \cdots, M_N$ be $N \ge 2$ closed subspaces of H with intersection $M = M_1 \cap \cdots \cap M_N$. Then*

$$\kappa(M_1, \cdots, M_N) = \sup \left\{ \frac{1}{N} \|G(v_1, \cdots, v_N)\| : v_j \in M_j \cap M^\perp, \|v_j\| = 1, j = 1, \cdots, N \right\}.$$

Proof. Let $m_j \in M_j \cap M^\perp$, $j = 1, \cdots, N$, with $\|m_1\|^2 + \cdots + \|m_N\|^2 \ne 0$. Set $v_j = \frac{m_j}{\|m_j\|}$ if $\|m_j\| \ne 0$, or $v_j = 0$ if $m_j = 0$. Denote $\mathbf{x} = (\|m_1\|, \cdots, \|m_N\|) \in \mathbb{C}^N \setminus \{0\}$. We have $\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^t \mathbf{x} \ne 0$ and

$$\begin{aligned} \frac{1}{N} \frac{\| \sum_{j=1}^N m_j \|^2}{\sum_{j=1}^N \|m_j\|^2} &= \frac{1}{N} \frac{\left\langle \sum_i^N \|m_i\| v_i, \sum_j^N \|m_j\| v_j \right\rangle}{\sum_{j=1}^N \|m_j\|^2} \\ &= \frac{1}{N} \frac{\mathbf{x}^t G(v_1, \cdots, v_N) \mathbf{x}}{\mathbf{x}^t \mathbf{x}}. \end{aligned}$$

The conclusion follows by taking the supremum and noting that the Gramian matrix $G(v_1, \cdots, v_N)$ is a Hermitian matrix. □

Consider the product Hilbert space H^N which is the Hilbertian direct sum of N copies of H, with scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle (x_1, \cdots, x_N), (y_1, \cdots, y_N) \rangle := \sum_{j=1}^N \langle x_j, y_j \rangle.$$

We denote by $\mathbf{C}$ the Cartesian product $\mathbf{C} = M_1 \times \cdots \times M_N \subset H^N$, and by $\mathbf{D}$ the diagonal subset $\mathbf{D} = \text{ diag } (H) = \{(y, \ldots, y) : y \in H\} \subset H^N$. Recall that $M = M_1 \cap \cdots \cap M_N$.

3.5. **Lemma.** *The projections onto $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{C} \cap \mathbf{D}$ are given by*

$$P_{\mathbf{C}}(x_1, \cdots, x_N) = (P_1 x_1, \cdots, P_N x_N),$$

$$P_{\mathbf{D}}(x_1, \cdots, x_N) = ((x_1 + \cdots + x_N)/N, \cdots, (x_1 + \cdots + x_N)/N),$$

and, respectively, by

$$P_{\mathbf{C} \cap \mathbf{D}}(x_1, \cdots, x_N) = ((P_M x_1 + \cdots + P_M x_N)/N, \cdots, (P_M x_1 + \cdots + P_M x_N)/N),$$

where $(x_1, \cdots, x_N) \in H^N$.

Proof. The formulae for $P_{\mathbf{C}}$ and $P_{\mathbf{D}}$ were proved in [27]. For the third one, note first that

$$\mathbf{C} \cap \mathbf{D} = \text{ diag } (M) = \{(m, \cdots, m) : m \in M\}.$$

Therefore

$$\|\mathbf{x} - P_{\mathbf{C} \cap \mathbf{D}} \mathbf{x}\|^2 = \text{dist}(\mathbf{x}, \text{ diag } (M))^2 = \inf \{ \sum_{j=1}^N \|x_j - m\|^2 : m \in M \}.$$

We obtain

$$\|\mathbf{x} - P_{\mathbf{C}\cap\mathbf{D}}\mathbf{x}\|^2 = \sum_{j=1}^{N} \|x_j - P_M x_j\|^2 + \inf\left\{\sum_{j=1}^{N} \|P_M x_j - m\|^2 : m \in M\right\}.$$

The infimum is realized when the gradient is zero, $\sum_{j=1}^{N}(m - P_M x_j) = 0$, that is when $m = N^{-1}\sum_{j=1}^{N} P_M x_j$. □

3.6. **Proposition.** *Let $M_1, \cdots, M_N$ be $N \geq 2$ closed subspaces of H with intersection $M = M_1 \cap \cdots \cap M_N$. Then*

(a) *$c_0(M_1, \cdots, M_N) = 1$ if $M \neq \{0\}$, while $c_0(M_1, \cdots, M_N) = 0$ if and only if the subspaces $(M_1, \cdots, M_N)$ are pairwise orthogonal;*
(b) *$c(M_1, \cdots, M_N) = c(M_1 \cap M^\perp, \cdots, M_N \cap M^\perp)$, $c(M_1, \cdots, M_N) = c_0(M_1 \cap M^\perp, \cdots, M_N \cap M^\perp)$, and thus $c(M_1, \cdots, M_N) = c_0(M_1, \cdots, M_N)$ if $M = \{0\}$;*
(c) *$0 \leq c_0(M_1, \cdots, M_N) \leq 1$ and $0 \leq c(M_1, \cdots, M_N) \leq 1$;*
(d) *$\frac{1}{N} \leq \kappa_0(M_1, \cdots, M_N) \leq 1$ and $\frac{1}{N} \leq \kappa(M_1, \cdots, M_N) \leq 1$;*
(e) *$\kappa_0(M_1, \cdots, M_N) = c_0(\mathbf{C}, \mathbf{D})^2$ and $\kappa(M_1, \cdots, M_N) = c(\mathbf{C}, \mathbf{D})^2$;*
(f) *$c(M_1, \cdots, M_N) = \frac{N}{N-1}\kappa(M_1, \cdots, M_N) - \frac{1}{N-1} = \frac{N}{N-1}c(\mathbf{C}, \mathbf{D})^2 - \frac{1}{N-1}$ and similar statements for $c_0(M_1, \cdots, M_N)$.*

Proof. We start by giving the proof of part (e). We have

$$\begin{aligned}
c(\mathbf{C}, \mathbf{D})^2 &= \sup\left\{\frac{|\langle (m_1, \ldots, m_N), (y, \ldots, y)\rangle_{H^N}|^2}{N(\|m_1\|^2 + \cdots + \|m_N\|^2)\|y\|^2} :\right. \\
&\qquad \left. y \in H, y \neq 0, m_j \in M_j \cap M^\perp, \|m_1\|^2 + \cdots + \|m_N\|^2 \neq 0\right\} \\
&= \sup\left\{\frac{\left\langle \sum_{j=1}^{N} m_j, y\right\rangle^2}{N(\|m_1\|^2 + \cdots + \|m_N\|^2)\|y\|^2} :\right. \\
&\qquad \left. y \in H, y \neq 0, m_j \in M_j \cap M^\perp, \|m_1\|^2 + \cdots + \|m_N\|^2 \neq 0\right\} \\
&= \sup\left\{\frac{\|\sum_{j=1}^{N} m_j\|^2}{N(\|m_1\|^2 + \cdots + \|m_N\|^2)} :\right. \\
&\qquad \left. y \in H, y \neq 0, m_j \in M_j \cap M^\perp, \|m_1\|^2 + \cdots + \|m_N\|^2 \neq 0\right\} \\
&= \kappa(M_1, \ldots, M_N).
\end{aligned}$$

The proof of the equality $\kappa_0(M_1, \cdots, M_N) = c_0(\mathbf{C}, \mathbf{D})^2$ is similar.

We prove now that $c(M_1,\cdots,M_N)=\frac{N}{N-1}\kappa(M_1,\cdots,M_N)-\frac{1}{N-1}$. Indeed, we have

$$\begin{aligned} c(M_1,\cdots,M_N) &= \sup\Big\{\frac{2}{N-1}\frac{\sum_{j<k}\mathrm{Re}\,\langle m_j,m_k\rangle}{\|m_1\|^2+\cdots+\|m_N\|^2}: \\ &\quad m_j\in M_j\cap M^\perp, \|m_1\|^2+\cdots+\|m_N\|^2\neq 0\Big\} \\ &= \sup\Big\{\frac{1}{N-1}\frac{\|\sum_j m_j\|^2-\sum_j\|m_j\|^2}{\|m_1\|^2+\cdots+\|m_N\|^2}: \\ &\quad m_j\in M_j\cap M^\perp, \|m_1\|^2+\cdots+\|m_N\|^2\neq 0\Big\} \\ &= \frac{N}{N-1}\kappa(M_1,\cdots,M_N)-\frac{1}{N-1}. \end{aligned}$$

The proof of the equality for $c_0(M_1,\cdots,M_N)$ is similar.

The upper bound $\kappa_0(M_1,\cdots,M_N)\le 1$ in (d) follows from the Cauchy-Schwarz inequality:

$$\begin{aligned} \|m_1+\cdots+m_N\|^2 &\le (\|m_1\|+\cdots+\|m_N\|)^2 \\ &\le N(\|m_1\|^2+\cdots+\|m_N\|^2). \end{aligned}$$

For the lower bound $\kappa_0(M_1,\cdots,M_N)\ge 1/N$, note that, for $m_1\in M_1$,

$$\kappa_0(M_1,\cdots,M_N)\ge\frac{1}{N}\frac{\|\|m_1\|^2}{\|m_1\|^2}=\frac{1}{N}.$$

The inequalities for $\kappa(M_1,\cdots,M_N)$ follow from

$$\kappa(M_1,\cdots,M_N)=\kappa_0(M_1\cap M^\perp,\cdots,M_N\cap M^\perp).$$

Now (c) is a consequence of (f) and (d), while (b) and (a) are easy to prove. For the first equality in (b) notice that $\cap_{j=1}^N(M_j\cap M^\perp)=\{0\}$. □

3.7. **Proposition.** *Let $M_1,\cdots,M_N$ be $N\ge 2$ closed subspaces of H with intersection $M=M_1\cap\cdots\cap M_N$. Then*

$$\kappa(M_1,\cdots,M_N)=\left\|\frac{P_1+\cdots+P_N}{N}-P_M\right\|$$

and

$$c(M_1,\cdots,M_N)=\frac{N}{N-1}\left\|\frac{P_1+\cdots+P_N}{N}-P_M\right\|-\frac{1}{N-1}.$$

Proof. We have (see for instance [16])

$$c(\mathbf{C},\mathbf{D})=\|P_{\mathbf{D}}P_{\mathbf{C}}-P_{\mathbf{C}\cap\mathbf{D}}\|.$$

Using Lemma 3.5, $P_{\mathbf{D}}P_{\mathbf{C}}-P_{\mathbf{C}\cap\mathbf{D}}$ can be written as

$$\frac{1}{N}\begin{bmatrix}1&1&\cdots&1\\1&1&\cdots&1\\ \vdots&\vdots&&\vdots\\1&1&\cdots&1\end{bmatrix}\begin{bmatrix}P_1&0&\cdots&0\\&\ddots&&\\&&\ddots&\\0&0&\cdots&P_N\end{bmatrix}-\frac{1}{N}\begin{bmatrix}P_M&P_M&\cdots&P_M\\P_M&\cdots&\cdots&P_M\\ \vdots&&&\vdots\\P_M&P_M&\cdots&P_M\end{bmatrix}.$$

Therefore

$$P_{\mathbf{D}}P_{\mathbf{C}} - P_{\mathbf{C}\cap\mathbf{D}} = \frac{1}{N}\begin{bmatrix} P_1 - P_M & P_2 - P_M & \cdots & P_N - P_M \\ P_1 - P_M & P_2 - P_M & \cdots & P_N - P_M \\ \vdots & & & \vdots \\ P_1 - P_M & P_2 - P_M & \cdots & P_N - P_M \end{bmatrix},$$

and so

$$\kappa := \kappa(M_1, \ldots, M_N) = c(\mathbf{C}, \mathbf{D})^2 = \frac{1}{N^2}\left\| \begin{bmatrix} P_1 - P_M & P_2 - P_M & \cdots & P_N - P_M \\ P_1 - P_M & P_2 - P_M & \cdots & P_N - P_M \\ \vdots & & & \vdots \\ P_1 - P_M & P_2 - P_M & \cdots & P_N - P_M \end{bmatrix} \right\|^2.$$

We obtain that $N^2\kappa$ is equal to

$$\left\| \begin{bmatrix} P_1 - P_M & P_2 - P_M & \cdots & P_N - P_M \\ P_1 - P_M & P_2 - P_M & \cdots & P_N - P_M \\ \vdots & & & \vdots \\ P_1 - P_M & P_2 - P_M & \cdots & P_N - P_M \end{bmatrix} \begin{bmatrix} P_1 - P_M & P_1 - P_M & \cdots & P_1 - P_M \\ P_2 - P_M & P_2 - P_M & \cdots & P_2 - P_M \\ \vdots & & & \vdots \\ P_N - P_M & P_N - P_M & \cdots & P_N - P_M \end{bmatrix} \right\|.$$

Therefore

$$N^2\kappa = \left\| \begin{bmatrix} \Sigma & \Sigma & \cdots & \Sigma \\ \Sigma & \Sigma & \cdots & \Sigma \\ \vdots & & & \vdots \\ \Sigma & \Sigma & \cdots & \Sigma \end{bmatrix} \right\|, \quad \text{where } \Sigma := \sum_{j=1}^{N}(P_j - P_M)^2.$$

Since

$$(P_j - P_M)^2 = P_j - P_jP_M - P_MP_j + P_M = (I - P_M)P_j,$$

we have

$$\Sigma = (I - P_M)\sum_{j=1}^{N} P_j.$$

Let K be the matrix having all entries equal to Σ. One way to compute the norm of K is to note that, like every circulant matrix, K is unitarily equivalent to a diagonal matrix. Indeed, denote by F the $N \times N$ unitary matrix representing the discrete Fourier transform $F = N^{-1/2}[(\omega^{jk})]_{0\le j,k\le N-1}$, where $\omega = \exp(-2i\pi/N)$ is a primitive Nth root of unity. Then

$$F^*KF = \begin{bmatrix} N\Sigma & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Therefore

$$\kappa = \frac{1}{N}\|\Sigma\| = \|(I - P_M)\sum_{j=1}^{N} P_j\| = \|\sum_{j=1}^{N} P_j(I - P_M)\|.$$

Since $P_j P_M = P_M$, this can be written as

$$\kappa = \left\| \frac{\sum_{j=1}^N P_j}{N} - P_M \right\|.$$

The proof is complete. □

The following definition is related to the minimum gap between two subspaces (see [17, p. 219 and Lemma 4.4]). See also the regularity (or boundedly linearly regularity) condition from [6], and the references therein.

3.8. **Definition.** Let $M_1, \cdots, M_N$ be $N \geq 2$ closed subspaces of H with intersection $M = M_1 \cap \cdots \cap M_N$. The number

$$\ell(M_1, \ldots, M_N) = \inf_{x \notin M} \frac{\max_{1 \leq j \leq N} \operatorname{dist}(x, M_j)}{\operatorname{dist}(x, M)}$$

is called the *inclination* of $(M_1, \cdots, M_N)$.

3.9. **Proposition.** *Let $M_1, \cdots, M_N$ be $N \geq 2$ closed subspaces of H with intersection $M = M_1 \cap \cdots \cap M_N$. Then*

$$1 - \ell(M_1, \ldots, M_N) \leq c(\mathbf{C}, \mathbf{D}) = \kappa(M_1, \ldots, M_N)^{1/2} \leq 1 - \frac{\ell(M_1, \ldots, M_N)^2}{2N}$$

and

$$1 - \frac{2N}{N-1} \ell(M_1, \ldots, M_N) \leq c(M_1, \ldots, M_N) \leq 1 - \frac{\ell(M_1, \ldots, M_N)^2}{N-1}.$$

In particular, $\ell(M_1, \ldots, M_N) = 0$ if and only if $c(M_1, \ldots, M_N) = 1$, if and only if $\kappa(M_1, \ldots, M_N) = 1$.

Proof. Denote $\ell = \ell(M_1, \ldots, M_N)$. Let $\varepsilon > 0$. There exists $x \in H$ with $\|x - P_M x\| = \operatorname{dist}(x, M) = 1$ such that $\operatorname{dist}(x, M_j) < \ell + \varepsilon$ for each j. Set $u_j = P_j(x - P_M x) = P_j x - P_M x$, where P_j is the orthogonal projection onto M_j $(1 \leq j \leq N)$. Then $u_j \in M_j$ and $\|u_j\| \leq \|x - P_M x\| = 1$. Recall that $\mathbf{C}$ is the ℓ^2-direct sum $\mathbf{C} = M_1 \times \cdots \times M_N \subset H^N$, while $\mathbf{D}$ is the diagonal $\mathbf{D} = \operatorname{diag}(H) = \{(y, \ldots, y) : y \in H\} \subset H^N$. We have $\mathbf{C} \cap \mathbf{D} = \operatorname{diag}(M)$, and so $\mathbf{y} = (y_1, \cdots, y_N) \in \operatorname{diag}(M)^\perp$ if and only if $\langle y_1 + \cdots + y_N, m \rangle = 0$ for every $m \in M$. Thus $(y_1, \cdots, y_N) \in \operatorname{diag}(M)^\perp$ if and only if $y_1 + \cdots + y_N \in M^\perp$.

Consider

$$\mathbf{d} = \left(\frac{1}{\sqrt{N}}(I - P_M)x, \ldots, \frac{1}{\sqrt{N}}(I - P_M)x\right) \in \mathbf{D} \cap \operatorname{diag}(M)^\perp$$

and

$$\mathbf{c} = \left(\frac{1}{\sqrt{N}} u_1, \ldots, \frac{1}{\sqrt{N}} u_N\right) \in \mathbf{C}.$$

Note that, for each $m \in M$, we have

$$\begin{aligned}\left\langle \sum_{j=1}^{N} u_j, m \right\rangle &= \frac{1}{\sqrt{N}} \left[\sum_{j=1}^{N} \langle P_j x, m\rangle - N \langle P_M x, m\rangle \right] \\ &= \frac{1}{\sqrt{N}} \left[\sum_{j=1}^{N} \langle x, P_j m\rangle - N \langle x, m\rangle \right] \\ &= 0.\end{aligned}$$

Therefore $\mathbf{c} \in \mathbf{C} \cap \operatorname{diag}(M)^{\perp}$. We also have $\|\mathbf{d}\| = 1$ and $\|\mathbf{c}\|^2 = \frac{1}{N}(\|u_1\|^2 + \cdots + \|u_N\|^2) \le 1$. Thus

$$c(\mathbf{C}, \mathbf{D}) \ge |\langle \mathbf{c}, \mathbf{d}\rangle| = \frac{1}{N} \left| \left\langle \sum_{j=1}^{N} u_j, x - P_M x \right\rangle \right| \ge \frac{1}{N} \operatorname{Re} \left\langle \sum_{j=1}^{N} u_j, x - P_M x \right\rangle .$$

For a fixed j we have $\|x - P_M x - u_j\| = \|x - P_j x\| = \operatorname{dist}(x, M_j) < \ell + \varepsilon$. Therefore

$$2 \operatorname{Re} \langle x - P_M x, u_j\rangle = \|x - P_M x\|^2 + \|u_j\|^2 - \|x - P_M x - u_j\|^2 > 1 + \|u_j\|^2 - (\ell + \varepsilon)^2.$$

We also have $\|u_j\| \ge \|x - P_M x\| - \|x - P_M x - u_j\| > 1 - (\ell + \varepsilon)$. We obtain

$$\begin{aligned} c(\mathbf{C}, \mathbf{D}) &\ge \frac{1}{N} \operatorname{Re} \left\langle \sum_{j=1}^{N} u_j, x - P_M x \right\rangle \\ &\ge \frac{1}{2N} \sum_{j=1}^{N} \left(1 + \|u_j\|^2 - (\ell + \varepsilon)^2\right) \\ &\ge \frac{1}{2N} \left(N + N(1 - (\ell + \varepsilon))^2 - N(\ell + \varepsilon)^2\right) \\ &= \frac{1}{2} \left(1 + (1 - (\ell + \varepsilon))^2 - (\ell + \varepsilon)^2\right). \end{aligned}$$

As this inequality is true for every $\varepsilon > 0$ we get

$$c(\mathbf{C}, \mathbf{D}) \ge \frac{1 - \ell^2 + (1 - \ell)^2}{2} = 1 - \ell.$$

Denote $c = c(\mathbf{C}, \mathbf{D})$. Let $\varepsilon > 0$. There exist $\mathbf{c} \in \mathbf{C} \cap \operatorname{diag}(M)^{\perp}$ and $\mathbf{d} \in \mathbf{D} \cap \operatorname{diag}(M)^{\perp}$ with $\|\mathbf{c}\| = 1$, $\|\mathbf{d}\| = 1$ such that $c < |\langle \mathbf{c}, \mathbf{d}\rangle| + \varepsilon$. Let $\theta \in \mathbb{R}$ be such that $\langle \mathbf{c}, \mathbf{d}\rangle = e^{i\theta} |\langle \mathbf{c}, \mathbf{d}\rangle|$. Then

$$\|\mathbf{c} - e^{i\theta}\mathbf{d}\|^2 = 2 - 2 \operatorname{Re}(e^{-i\theta} \langle \mathbf{c}, \mathbf{d}\rangle) = 2 - 2|\langle \mathbf{c}, \mathbf{d}\rangle| \le 2 - 2(c - \varepsilon).$$

Set $\mathbf{c} = (m_1, \ldots, m_N) \in \mathbf{C} \cap \operatorname{diag}(M)^{\perp}$ and $e^{i\theta}\mathbf{d} = (y, \ldots, y) \in \mathbf{D} \cap \operatorname{diag}(M)^{\perp}$. Then $y \in M^{\perp}$ and

$$\|m_1\|^2 + \cdots + \|m_N\|^2 = 1, \quad \|y\| = 1/\sqrt{N}, \quad \text{and} \quad \sum_{j=1}^{N} \|m_j - y\|^2 \le 2 - 2(c - \varepsilon).$$

Let $x = \sqrt{N} y$. Then $x \in M^{\perp}$, $\operatorname{dist}(x, M) = \|x\| = 1$ and we have

$$\operatorname{dist}(x, M_j)^2 \le \|x - \sqrt{N} m_j\|^2 = N \|m_j - y\|^2 \le N(2 - 2(c - \varepsilon)).$$

We finally obtain $\ell^2 \leq 2N(1-c)$, and so

$$1-\ell \leq c(\mathbf{C},\mathbf{D}) \leq 1-\frac{\ell^2}{2N}.$$

Using the equalities

$$c(\mathbf{C},\mathbf{D}) = \kappa(M_1,\dots,M_N)^{1/2} \quad \text{and } \kappa(M_1,\dots,M_N) = \frac{N-1}{N}c(M_1,\cdots,M_N)+\frac{1}{N}$$

we obtain

$$\frac{N(1-\ell)^2-1}{N-1} \leq c(M_1,\cdots,M_N) \leq \frac{N(1-(\ell^2/2N)^2)-1}{N-1}.$$

Since $2-\ell \leq 2$, we can write

$$\frac{N(1-\ell)^2-1}{N-1} = 1-\frac{N}{N-1}\ell(2-\ell) \geq 1-\frac{2N}{N-1}\ell.$$

We also have

$$\frac{N(1-(\ell^2/2N)^2)-1}{N-1} = 1-\frac{\ell^2}{N-1}(1-\frac{\ell^2}{4N}) \leq 1-\frac{\ell^2}{N-1}.$$

□

3.10. **Proposition.** *Let $M_1,\cdots,M_N$ be $N \geq 2$ closed subspaces of H with intersection $M = M_1\cap\cdots\cap M_N$. Denote $c_j = c(M_1\cap\cdots\cap M_{j-1},M_j)$ for j between 2 and N. Then*

$$c(M_1,\dots,M_N) \leq 1-\frac{1}{N-1}\prod_{j=2}^{N}\left(1-\sqrt{\frac{c_j+1}{2}}\right)^2 \leq 1-\frac{1}{(N-1)4^{N-1}}\prod_{j=2}^{N}(1-c_j)^2.$$

In particular, $c(M_1,\dots,M_N) < 1$ if each $c_j < 1$, $2 \leq j \leq N$.

Proof. The estimates are clear if one of the c_j's is one. Suppose $c_j < 1$ for every j. Denote $\ell_j = c(M_1\cap\cdots\cap M_{j-1},M_j) > 0$ for j between 2 and N. It follows from the proof of [6, Theorem 5.11] that $\ell(M_1,\dots,M_N) \geq \ell_2\ell_3\cdots\ell_N$. The proof of Proposition 3.9 for $N=2$, and two given subspaces S_1 and S_2, yields

$$1-\ell(S_1,S_2) \leq \sqrt{\kappa(S_1,S_2)} = \sqrt{\frac{c(S_1,S_2)+1}{2}}.$$

This implies that

$$\ell(S_1,S_2) \geq 1-\sqrt{\frac{c(S_1,S_2)+1}{2}} \geq \frac{1-c(S_1,S_2)}{4}.$$

Using Proposition 3.9 we obtain

$$\begin{aligned} c(M_1,\dots,M_N) &\leq 1-\frac{\ell(M_1,\dots,M_N)^2}{N-1} \\ &\leq 1-\frac{\ell_2^2\ell_3^2\cdots\ell_N^2}{N-1} \\ &\leq 1-\frac{1}{N-1}\prod_{j=2}^{N}\left(1-\sqrt{\frac{c_j+1}{2}}\right)^2 \\ &\leq 1-\frac{1}{(N-1)4^{N-1}}\prod_{j=2}^{N}(1-c_j)^2, \end{aligned}$$

which completes the proof. □

4. Characterising (ASC) for products of projections, and applications

When T is the product of $N \geq 2$ orthogonal projections, we know by Theorem 2.7 that the dichotomy (QUC)/(ASC) holds, and that we have quick uniform convergence if and only if the range of $T - I$ is closed. The following qualitative result gives a characterization of the (ASC) condition in terms of several parameters associated to $(M_1, \cdots, M_N)$, or spectral properties of T, or random products. We denote by $\| \cdot \|_e$ the *essential norm* and by σ_e the *essential spectrum.*

4.1. **Theorem.** *Let $M_1, \cdots, M_N$ be $N \geq 2$ closed subspaces of H with intersection $M = M_1 \cap \cdots \cap M_N$. Denote P_j the orthogonal projection onto M_j, $1 \leq j \leq N$, and by P_M the orthogonal projection onto M. Let $T = P_N P_{N-1} \cdots P_1$. The following assertions are equivalent:*

(1) $\mathrm{Ran}(T - I)$ *is not closed;*

(1′) *for every $k \geq N$, and every sequence of indices $(i_k)_{k \geq 1}$ with $\{i_1, \ldots, i_k\} = \{1, 2, \ldots, N\}$, $\mathrm{Ran}(P_{i_k} \cdots P_{i_1} - I)$ is not closed;*

(2) *one of the conditions (ASC1), (ASC2), (ASC3), (ASCH) holds for T;*

(2′) *(ASCH) for random products: for every $\varepsilon > 0$, every sequence $(a_n)_{n \geq 0}$ of positive reals with $\lim_{n \to \infty} a_n = 0$, and every sequence of indices $(i_k)_{k \geq 1}$ in $\{1, 2, \ldots, N\}$, there exists $x \in H$ with $\|x\| < \sup_n a_n + \varepsilon$ such that $\mathrm{Re} \left\langle P_{i_n} P_{i_{n-1}} \cdots P_{i_1} x, x \right\rangle > a_n$ for each $n \geq 1$;*

(3) $c(M_1, \cdots, M_N) = 1$. *Equivalently, $\kappa(M_1, \cdots, M_N) = 1$, or $\ell(M_1, \cdots, M_N) = 0$;*

(4) *for every $\varepsilon > 0$, every closed subspace $K \subset M^\perp$ of finite codimension (in $M^\perp$), there exists $x \in K$ such that $\|x\| = 1$ and $\max\{\mathrm{dist}(x, M_j) : j = 1, \cdots, N\} < \varepsilon$;*

(5) $1 \in \sigma(T - P_M)$;

(5′) *for every k and every $i_1, \cdots, i_k \in \{1, 2, \cdots, N\}$ we have $1 \in \sigma(P_{i_k} \cdots P_{i_1} - P_M)$;*

(6) $\|T - P_M\| = 1$;

(6′) *for every k and every sequence of indices $(i_k)_{k \geq 1}$, $1 \leq i_k \leq N$, with $\{i_1, \ldots, i_k\} = \{1, 2, \ldots, N\}$ we have $\|P_{i_k} \cdots P_{i_1} - P_M\| = 1$;*

(7) $\|T - P_M\|_e = 1$;

(7′) *for every k and every $i_1, \cdots, i_k \in \{1, 2, \cdots, N\}$ we have $\|P_{i_k} \cdots P_{i_1} - P_M\|_e = 1$;*

(8) $1 \in \sigma_e(T - P_M)$;

(8′) *for every k, every $i_1, \cdots, i_k \in \{1, 2, \cdots, N\}$ we have $1 \in \sigma_e(P_{i_k} \cdots P_{i_1} - P_M)$;*

(9) *for every $\varepsilon > 0$, every closed subspace $K \subset M^\perp$ of finite codimension (in $M^\perp$), there exists $x \in K$ such that $\|Tx - x\| \leq \varepsilon$;*

(9′) *for every $\varepsilon > 0$, every closed subspace $K \subset M^\perp$ of finite codimension (in $M^\perp$), there exists $x \in K$ such that $\|P_{i_k} \cdots P_{i_1} x - x\| \leq \varepsilon$ for every k, every $i_1, \cdots, i_k \in \{1, 2, \cdots, N\}$;*

(10) *the sum of $diag(M_1) \subset H^{N-1}$ and $M_2 \oplus \cdots \oplus M_N \subset H^{N-1}$ is not closed in H^{N-1} (and equivalent statements for $diag(M_j) \subset H^{N-1}$, $2 \leq j \leq N$);*

(11) $M_1^\perp + \cdots + M_N^\perp$ *is not closed in H.*

Proof. "(1) $\Leftrightarrow$ (2)" The equivalence of (1) and (2) follows from Theorem 2.7.

"(1) $\Leftrightarrow$ (5)" The equivalence of (1) and (5) follows from the proof of Theorem 2.1 (see also Remark 2.4). Notice that, with respect to the decomposition $H = M \oplus M^{\perp}$, we have $T = P_M \oplus A$, where $A = T\mid_{M^{\perp}} = T(I - P_M) = T - P_M$.

"(1) $\Rightarrow$ (3)" We prove this implication in a quantitative form. Denote

$$\gamma = \gamma(I - T) = \inf\{\|x - Tx\| : x \in H, \operatorname{dist}(x, \operatorname{Ker}(T - I) = 1\}$$

the *reduced minimum modulus* of $T - I$. Then $\operatorname{Ran}(T - I)$ is closed if and only if $\gamma > 0$. Clearly $Ty = y$ for $y \in M$. If $Tx = x$, then

$$\|x\| = \|P_N \cdots P_1 x\| \le \|P_{N-1} \cdots P_1 x\| \le \|P_1 x\| \le \|x\|.$$

We successively obtain $P_1 x = x$, $P_2 x = x$, $\ldots$, $P_N x = x$, and finally $x \in M$. Thus $\operatorname{Ker}(T - I) = M$.

Let $\varepsilon > 0$. There exists $x \in H$ with $\|x - P_M x\| = \operatorname{dist}(x, M) = 1$ such that $\|x - Tx\| \le \gamma + \varepsilon$. We obtain

$$\begin{aligned} 1 = \|x - P_M x\| &\ge \|P_1(x - P_M x)\| = \|P_1 x - P_M x\| \ge \|P_2 P_1 x - P_M x\| \\ &\ge \cdots \ge \|P_N \cdots P_1 x - P_M x\| = \|Tx - P_M x\| \\ &\ge \|x - P_M x\| - \|x - Tx\| \ge 1 - \gamma - \varepsilon. \end{aligned}$$

We also have

$$\begin{aligned} \|(I - P_1)(x - P_M x)\|^2 &= \|x - P_M x\|^2 - \|P_1 x - P_M x\|^2 \\ &\le 1 - (1 - \gamma - \varepsilon)^2 = -(\gamma + \varepsilon)^2 + 2(\gamma + \varepsilon) \le 2\gamma + 2\varepsilon. \end{aligned}$$

Thus $\operatorname{dist}(x, M_1) = \|x - P_1 x\| = \|(I - P_1)(x - P_M x)\| \le (2\gamma + 2\varepsilon)^{1/2}$.

Let $y = x - P_M x$; then $\|y\| = 1$. For a fixed s between 1 and N we can write

$$\begin{aligned} \|P_s \cdots P_1 y - P_{s+1} \cdots P_1 y\|^2 &= \|P_s \cdots P_1 y\|^2 - \|P_{s+1} \cdots P_1 y\|^2 \\ &\le \|y\|^2 - \|P_{s+1} \cdots P_1 x - P_M x\|^2 \\ &\le 1 - (1 - \gamma - \varepsilon)^2 \le 2\gamma + 2\varepsilon. \end{aligned}$$

Thus

$$\begin{aligned} \operatorname{dist}(x, M_j) &= \operatorname{dist}(y, M_j) \le \|y - P_j \cdots P_1 y\| \\ &\le \|y - P_1 y\| + \|P_1 y - P_2 P_1 y\| + \cdots + \|P_{j-1} \cdots P_1 y - P_j \cdots P_1 y\| \\ &\le j\sqrt{(2\gamma + 2\varepsilon)} \end{aligned}$$

for every j. Hence $\max_{1 \le j \le N} \operatorname{dist}(x, M_j) \le N\sqrt{(2\gamma + 2\varepsilon)}$ and, as ε is arbitrary,

$$\ell := \ell(M_1, \ldots, M_N) \le N\sqrt{2\gamma}.$$

We obtain $\frac{1}{2N^2}\ell(M_1, \ldots, M_N)^2 \le \gamma(T - I)$. Therefore $\operatorname{Ran}(I - T)$ not closed ($\gamma = 0$) implies $\ell(M_1, \ldots, M_N) = 0$.

"(3) $\Rightarrow$ (1′)" Let $k \ge N$ and let $(i_k)_{k \ge 1}$ be a sequence of indices with $\{i_1, \ldots, i_k\} = \{1, 2, \ldots, N\}$. This implies that $\operatorname{Ker}(I - P_{i_k} P_{i_{k-1}} \cdots P_{i_1}) = M$. Let $\ell = \ell(M_1, \ldots, M_N)$

and let $\varepsilon > 0$. There exists $x \in H$ with $\|x - P_M x\| = \operatorname{dist}(x, M) = 1$ such that $\max_j \operatorname{dist}(x, M_j) < \ell + \varepsilon$. We have

$$\|x - P_{i_1} x\| = \operatorname{dist}(x, M_{i_1}) < \ell + \varepsilon$$

and

$$\|P_{i_2} P_{i_1} x - P_{i_1} x\| = \operatorname{dist}(P_{i_1} x, M_{i_2}) \leq \|x - P_{i_1} x\| + \operatorname{dist}(x, M_{i_2}) < 2(\ell + \varepsilon).$$

Set $x_0 = x$ and $x_s = P_{i_s} P_{i_{s-1}} \cdots P_{i_1} x$ for $s \geq 1$. Suppose that

$$\|x_s - x_{s-1}\| \leq 2^{s-1}(\ell + \varepsilon) \tag{4.2}$$

holds for $1 \leq s \leq r$. Then

$$\begin{aligned} \|x_{r+1} - x_r\| &= \operatorname{dist}(P_{i_r} \cdots P_{i_1} x, M_{i_{r+1}}) \\ &\leq \|P_{i_r} \cdots P_{i_1} x - x\| + \operatorname{dist}(x, M_{i_{r+1}}) \\ &\leq \|x_s - x_{s-1}\| + \|x_{s-1} - x_{s-2}\| + \cdots + \|x_1 - x\| \\ &+ \operatorname{dist}(x, M_{r+1}) \\ &\leq (2^{r-1} + 2^{r-2} + \cdots + 2 + 1 + 1)(\ell + \varepsilon) = 2^r(\ell + \varepsilon). \end{aligned}$$

Therefore (4.2) holds for every s, and we obtain

$$\begin{aligned} \|P_{i_k} P_{i_{k-1}} \cdots P_{i_1} x - x\| &= \|x_k - x\| \\ &\leq \|x_k - x_{k-1}\| + \|x_{k-1} - x_{k-2}\| + \cdots + \|x_1 - x\| \\ &\leq (2^{k-1} + 2^{k-2} + \cdots + 1)(\ell + \varepsilon) = (2^k - 1)(\ell + \varepsilon) \end{aligned}$$

Thus $\gamma(P_{i_k} P_{i_{k-1}} \cdots P_{i_1} - I) \leq (2^k - 1)(\ell + \varepsilon)$. Making $\varepsilon \to 0$ we obtain

$$\gamma(P_{i_k} P_{i_{k-1}} \cdots P_{i_1} - I) \leq (2^k - 1)\ell.$$

This shows that if $\ell = 0$ or, equivalently, if $c(M_1, \cdots, M_N) = 1$, then the range of $P_{i_k} P_{i_{k-1}} \cdots P_{i_1} - I$ is not closed.

The implication "$(1') \Rightarrow (1)$" is clear. Note also that the above proof for $k = N$ and $i_s = s$ implies that

$$\frac{1}{2N^2}\ell^2 \leq \gamma(T - I) \leq (2^N - 1)\ell. \tag{4.3}$$

Here $\ell = \ell(M_1, \cdots, M_N)$.

"$(1') \Rightarrow (6')$" Note that $\|P_{i_k} \cdots P_{i_1} - P_M\| \leq 1$ always. Suppose now that $a := \|P_{i_k} \cdots P_{i_1} - P_M\| < 1$. We want to show that the range of $I - P_{i_k} \cdots P_{i_1}$ is closed. Notice first that $\operatorname{Ker}(I - P_{i_k} \cdots P_{i_1}) = M$ since $\{i_1, \ldots, i_k\} = \{1, 2, \ldots, N\}$. Let $x \in H$ be such that $\operatorname{dist}(x, M) = \|x - P_M x\| = 1$. We have

$$\begin{aligned} \|(I - P_{i_k} \cdots P_{i_1})x\| &= \|x - P_M x + P_M x - P_{i_k} \cdots P_{i_1} x\| \\ &\geq 1 - \|P_{i_k} \cdots P_{i_1} x - P_M x\| \\ &= 1 - \|(P_{i_k} \cdots P_{i_1} - P_M)(x - P_M x)\| \\ &\geq 1 - a. \end{aligned}$$

Therefore the reduced minimum modulus of $I - P_{i_k} \cdots P_{i_1})$ verifies $\gamma(I - P_{i_k} \cdots P_{i_1}) \geq 1 - \|P_{i_k} \cdots P_{i_1} - P_M\|$. In particular, $\operatorname{Ran}(I - P_{i_k} \cdots P_{i_1})$ is closed if $a < 1$.

The implication ”(6′) ⇒ (6)“ is easy.

4.4. **Lemma.** *Let $x \in H$, and set $u_j = P_j \cdots P_1 x - P_M x$ for $j \geq 1$ and $u_0 = x - P_M x$. For every j with $1 \leq j \leq N$ we have*

$$\|u_{j-1} - u_j\|^2 \leq \|u_{j-1}\|^2 - \|Tx - P_M x\|^2 \leq \|x - P_M x\|^2 - \|Tx - P_M x\|^2.$$

Proof. Note that

$$\|Tx - P_M x\| = \|u_N\| = \|P_N u_{N-1}\| \leq \|u_{N-1}\| \leq \cdots \leq \|u_0\| = \|x - P_M x\|.$$

We have

$$\begin{aligned} \|u_{j-1} - u_j\|^2 + \|Tx - P_M x\|^2 &= \|u_{j-1} - P_j u_{j-1}\|^2 + \|P_N \cdots P_{j+1} P_j u_{j-1}\|^2 \\ &\leq \|u_{j-1} - P_j u_{j-1}\|^2 + \|P_j u_{j-1}\|^2 \\ &= \|u_{j-1}\|^2 \\ &= \|P_{j-1} \cdots P_1 (x - P_M x)\|^2 \\ &\leq \|x - P_M x\|^2, \end{aligned}$$

completing the proof of the Lemma. □

We continue the proof of Theorem 4.1.

”(6) ⇒ (3)“ Let j between 1 and N. Using the Cauchy-Schwarz inequality and Lemma 4.4 we obtain

$$\begin{aligned} \operatorname{dist}(x, M_j)^2 &\leq \|x - P_j \cdots P_1 x\|^2 \\ &\leq (\|x - P_1 x\| + \|P_1 x - P_2 P_1 x\| + \cdots + \|P_{j-1} \cdots P_1 x - P_j \cdots P_1 x\|)^2 \\ &\leq j \left(\|u_0 - u_1\|^2 + \|u_1 - u_2\|^2 + \cdots + \|u_{j-1} - u_j\|^2\right) \\ &\leq j^2 \left(\|x - P_M x\|^2 - \|Tx - P_M x\|^2\right) \\ &\leq N^2 \left(\|x - P_M x\|^2 - \|Tx - P_M x\|^2\right). \end{aligned}$$

We get

$$N^2 \left(\|x - P_M x\|^2 - \|Tx - P_M x\|^2\right) \geq \max_{1 \leq j \leq N} \operatorname{dist}(x, M_j)^2 \geq \ell^2 \|x - P_M x\|^2,$$

which yields

$$\|Tx - P_M x\|^2 \leq \left(1 - \frac{\ell^2}{N^2}\right) \|x - P_M x\|^2.$$

In particular

$$\|T - P_M\| \leq \sqrt{1 - \frac{\ell^2}{N^2}}. \tag{4.5}$$

Therefore $\|T - P_M\| = 1$ implies $\ell = 0$, i.e., (6) implies (3).

”(1) ⇒ (9)“ Let $\varepsilon > 0$. Let $K \subset M^\perp$ be a closed subspace of finite codimension in $M^\perp$. With respect to the decomposition $H = M \oplus M^\perp$, the operator T has the following matrix decomposition

$$T = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}.$$

Since $\operatorname{Ran}(T-I)$ is not closed, the range of the operator $I-A$, acting on $M^\perp$, is not closed. This means that $I-A \in \mathcal{B}(M^\perp)$ is not an upper semi-Fredholm operator, and therefore there exists $x \in K$ such that $\|x\| = 1$ and $\|x - Ax\| \le \varepsilon$. It follows that $\|x - Tx\| \le \varepsilon$.

"(9) $\Rightarrow$ (4)" Let x be as in (9). Then $x \in K$, $\|x\| = 1$, and $\|x - Tx\| \le \varepsilon$. We have

$$1 = \|x\| \ge \|P_1 x\| \ge \|P_2 P_1 x\| \ge \ldots \ge \|Tx\| \ge 1 - \varepsilon.$$

Set $x_s = P_s P_{s-1} \cdots P_1 x$ for $s \ge 1$ and $x_0 = x$. Then $x_s \in M_s \cap M^\perp$ for each $s \ge 0$ and $x_{s-1} - x_s = (I - P_s)x_{s-1}$ is orthogonal to x_s. Hence

$$\|x_{s-1} - x_s\|^2 = \|x_{s-1}\|^2 - \|x_s\|^2 \le 1 - (1-\varepsilon)^2 < 2\varepsilon$$

and $\|x_{s-1} - x_s\| \le \sqrt{2\varepsilon}$, for each s. We obtain

$$\operatorname{dist}(x, M_1) = \|x - P_1 x\| = \|x_0 - x_1\| \le \sqrt{2\varepsilon}.$$

For $s \ge 1$ we have $\operatorname{dist}(x, M_s) \le \|x - P_s P_{s-1} \cdots P_1 x\|$; hence

$$\operatorname{dist}(x, M_s) \le \|x - P_1 x\| + \|P_1 x - P_2 P_1 x\| + \cdots + \|P_{s-1} \cdots P_1 x - P_s P_{s-1} \cdots P_1 x\| \le s\sqrt{2\varepsilon}.$$

Therefore $\max\{\operatorname{dist}(x, M_j) : j = 1, \cdots, N\} \le N\sqrt{2\varepsilon}$. As $\varepsilon > 0$ is arbitrary, the proof of this implication is over.

"(4) $\Rightarrow$ (9′)" Suppose that (4) holds. Let $\varepsilon > 0$ and let $K \subset M^\perp$ be a closed subspace of finite codimension in $M^\perp$. Then there exists $x \in K$ such that $\operatorname{dist}(x, M) = \|x\| = 1$ and $\max\{\operatorname{dist}(x, M_j) : j = 1, \cdots, N\} \le \varepsilon$. Let $i_1, \cdots, i_k \in \{1, 2, \ldots, N\}$. Set $x_0 = x$, $x_s = P_{i_s} \cdots P_{i_1} x$ for $s \ge 1$. Then $x_0 \in K$ and $x_s \in M^\perp \cap M_{i_s}$ for $s \ge 1$.

We shall prove by induction the following two claims :

$$\text{(*)} \qquad \operatorname{dist}(x_s, M_j) \le 2^s \varepsilon \quad (j \ge 1)$$

and

$$\text{(**)} \qquad \|x_s - x_0\| \le (2^s - 1)\varepsilon \quad (s \ge 1).$$

Both claims are clearly true for $s = 0$. Suppose that both inequalities are true for some $s \ge 0$. Then, using several times the induction hypothesis, we have

$$\begin{aligned} \|x_{s+1} - x_0\| &\le \|x_{s+1} - x_s\| + \|x_s - x_0\| \\ &= \operatorname{dist}(x_s, M_{i_{s+1}}) + \|x_s - x_0\| \\ &\le 2^s \varepsilon + (2^s - 1)\varepsilon = (2^{s+1} - 1)\varepsilon. \end{aligned}$$

For $j \ge 1$ we can write

$$\operatorname{dist}(x_{s+1}, M_j) \le \|x_{s+1} - x_0\| + \operatorname{dist}(x_0, M_j) \le (2^{s+1} - 1)\varepsilon + \varepsilon = 2^{s+1}\varepsilon.$$

Thus both (*) and (**) are true ; in particular we have

$$\|P_{i_k} \cdots P_{i_1} x - x\| = \|x_k - x_0\| \le 2^k \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we obtain (9′).

"(9′) $\Rightarrow$ (8′)" We have $P_s P_{s-1} \cdots P_1 - P_M = P_s P_{s-1} \cdots P_1 (I - P_M)$, so the range of this operator is in $M^\perp$. The assertion (9′) implies that 1 belongs to the essential spectrum of the restriction of $P_s P_{s-1} \cdots P_1 - P_M$ to $M^\perp$. Therefore $1 \in \sigma_e(P_s P_{s-1} \cdots P_1 - P_M)$.

The implication "(8′) $\Rightarrow$ (8)" is clear.

"(8) $\Rightarrow$ (7) $\Rightarrow$ (6)" The statement (8) implies the following sequence of inequalities for the essential spectral radius $r_e(T-P_M)$ and the essential norm of $T-P_M$:

$$1 \le r_e(T-P_M) \le \|T-P_M\|_e \le \|T-P_M\| = \|P_N\cdots P_1(I-P_M)\| \le 1.$$

Thus all inequalities are equalities.

The proofs of implications "(8′) $\Rightarrow$ (7′) $\Rightarrow$ (6′)" are similar. The implications "(8′) $\Rightarrow$ (5′) $\Rightarrow$ (5)" are clear.

"(9′) $\Rightarrow$ (2′)" Denote $A_k = P_{i_k}P_{i_{k-1}}\cdots P_{i_1} - P_M$ restricted to $M^\perp$. The condition (9′)implies that 1 is in the boundary of the essential spectrum of the operator A_k. According to [1], on the space $M^\perp$ the operators A_k converge weakly to 0. The assertion (2′) can be proved exactly as in [4, Theorem 1] by replacing there T^n by A_n.

The implication "(2′) $\Rightarrow$ (2)" is clear.

"(10) $\Leftrightarrow$ (3)" We have $\ell(M_1,\cdots,M_N)=0$ if and only if $c(\operatorname{diag}(M_1), M_2\oplus\cdots\oplus M_N)=1$. The proof of this assertion is analogous to that of the first part of Proposition 3.9. Therefore $\ell(M_1,\cdots,M_N)=0$, or equivalently $c(M_1,\cdots,M_N)=1$, if and only if $\operatorname{diag}(M_1)+M_2\oplus\cdots\oplus M_N$ is not closed in H^{N-1}.

The implication "(11) $\Leftrightarrow$ (3)" follows from [7]. The proof is complete. □

We note that $(1),(2),(5),(6),(7),(8)$ and (9) are conditions, most of them of spectral nature, about $T=P_N\cdots P_1$, the corresponding conditions denoted with primes are analog conditions about random products $P_{i_N}\cdots P_{i_1}$, while the conditions $(3),(4),(10)$ and (11) are about the geometry of subspaces M_j.

Quantitative statements. Some remarks concerning the proof of Theorem 4.1 are in order.

4.6. **Remark.** The proof of Theorem 4.1 gives some quantitative information between the parameters involved. Some other estimates can be proved in a similar way. For instance, we present here the quantitative version of the implication "(6′) $\Rightarrow$ (3)". Let $k\ge 1$. Suppose $i_1,\dots,i_k\in\{1,2,\dots,N\}$ and $\{i_1,\dots,i_k\}=\{1,2,\dots,N\}$. Denote $\ell=\ell(M_1,\dots,M_N)$ and

$$a := \|P_{i_k}\cdots P_{i_1} - P_M\| \le 1.$$

Let $\varepsilon>0$. There exists $x\in H$ with $\|x\|=1$ such that $\|P_{i_k}\cdots P_{i_1}x - P_Mx\| > a-\varepsilon$. Denote $y = x-P_Mx$ and

$$x_s = P_{i_s}\cdots P_{i_1}x - P_Mx, \quad x_0 = x-P_Mx = y \quad (s\ge 1).$$

Clearly

$$1=\|x\|=\sqrt{\|y\|^2+\|P_Mx\|^2}\ge\|y\|=\|x_0\|\ge\|x_1\|\ge\cdots\ge\|x_k\|>a-\varepsilon.$$

Since $x_{s-1}-x_s=(I-P_{i_s})x_{s-1}$ is orthogonal to M_{i_s}, and $x_s\in M_{i_s}$, we have

$$\|x_{s-1}-x_s\|^2=\|x_{s-1}\|^2-\|x_s\|^2\le\|y\|^2-(a-\varepsilon)^2.$$

For each $r\in\{1,\dots,k\}$ we have

$$\begin{aligned}\|x-P_Mx-x_r\| &\le \|x_0-x_1\|+\|x_1-x_2\|+\cdots+\|x_{r-1}-x_r\|\\ &\le k\sqrt{\|y\|^2-(a-\varepsilon)^2}.\end{aligned}$$

Since $\{i_1, \dots, i_k\} = \{1, 2, \dots, N\}$, for each $j \in \{1, 2, \dots, N\}$ we have $\{x_1, \dots, x_k\} \cap M_j \neq \emptyset$. Therefore

$$\operatorname{dist}(x, M_j) \leq \max\{\|x - P_M x - x_r\| : 1 \leq r \leq k\} \leq k\sqrt{\|y\|^2 - (a-\varepsilon)^2}$$

and

$$\|y\|^2 \ell^2 \leq \max\{\operatorname{dist}(x, M_j)^2 : 1 \leq j \leq k\} \leq k^2 \left(\|y\|^2 - (a-\varepsilon)^2\right).$$

Hence $k^2(a-\varepsilon)^2 \leq (k^2 - \ell^2)\|y\|^2 \leq (k^2 - \ell^2)$. As this is satisfied for every $\varepsilon > 0$, we obtain $\|P_{i_k} \cdots P_{i_1} - P_M\| = a \leq (1 - \ell^2/k^2)^{1/2}$.

4.7. **Corollary.** *Let H be a complex Hilbert space. Let $M_1, \cdots, M_N$ be $N \geq 2$ closed subspaces of H with intersection $M = M_1 \cap M_2 \cdots \cap M_N$. Let $P_j = P_{M_j}$, $1 \leq j \leq N$, and P_M be the corresponding orthogonal projections. Denote $T = P_N \cdots P_1$.*

(i) Suppose that $c := c(M_1, \cdots, M_N) < 1$. Then $(T^n)_{n \geq 1}$ is uniformly convergent to P_M, with

$$\|T^n - P_M\| \leq \left(1 - \left(\frac{1-c}{4N}\right)^2\right)^{n/2} \qquad (n \geq 1).$$

(ii) Suppose that $c := c(M_1, \cdots, M_N) = 1$. Then $(T^n)_{n \geq 1}$ is strongly convergent to P_M and we have (ASC), in all possible meanings of this paper.

Proof. We have to prove only the estimate in Part (i). Suppose that $c := c(M_1, \cdots, M_N) < 1$. Denote $\widetilde{M_j} = M_j \cap M^\perp$ and $Q_j = P_{\widetilde{M_j}}$ for $1 \leq j \leq N$. Then the intersection of $\widetilde{M_j}$, $1 \leq j \leq N$, is $\{0\}$ and, according to Proposition 3.6, (b), we have $c(\widetilde{M_1}, \dots, \widetilde{M_N}) = c < 1$. We also have $T^n - P_M = (Q_N Q_{N-1} \cdots Q_1)^n$ for each $n \geq 1$ (see [10, Lemma 9.30]). We apply (4.5), which was proved in the implication "(6) $\Rightarrow$ (3)" of Theorem 4.1, to Q_j. We obtain $\|Q_N Q_{N-1} \cdots Q_1\| \leq \sqrt{1 - \frac{\ell^2}{N^2}}$, where now $\ell = \ell(\widetilde{M_1}, \dots, \widetilde{M_N})$. According to Proposition 3.9, applied for the subspaces $\widetilde{M_j}$, we have $1 - \frac{2N}{N-1}\ell \leq c$. This implies $\ell^2 \geq \left(\frac{N-1}{2N}\right)^2 (1-c)^2 \geq \frac{1}{16}(1-c)^2$. Therefore

$$\begin{aligned} \|T^n - P_M\|^2 &= \|(Q_N Q_{N-1} \cdots Q_1)^n\|^2 \leq \|(Q_N Q_{N-1} \cdots Q_1)\|^{2n} \\ &\leq \left(1 - \frac{\ell^2}{N^2}\right)^n \leq \left(1 - \left(\frac{(1-c)^2}{16N^2}\right)\right)^n, \end{aligned}$$

which implies (i). □

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