*Preprint, Institute of Mathematics, AS CR, Prague. 2010-10-19*

# A NOTE ON J-SETS OF LINEAR OPERATORS

M. R. AZIMI AND V. MÜLLER

Abstract. We construct a Banach space operator $T \in B(X)$ such that the set $J_T(0)$ has a nonempty interior but $J_T(0) \neq X$. This gives a negative answer to a problem raised by G. Costakis and A. Manoussos.

## 1. Introduction and preliminaries

Let $X$ be an infinite dimensional complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on $X$. For $T \in B(X)$ and $x \in X$ let $\mathrm{Orb}(T,x) = \{x, Tx, T^2x, ...\}$ be the orbit of $T$ at $x$.

By a result of Bourdon and Feldman [**?**], if the closure $\overline{\mathrm{Orb}(T,x)}$ has a non-empty interior, then $\overline{\mathrm{Orb}(T,x)} = X$, and so $x$ is a hypercyclic vector for $T$.

In [**?**], a weaker concept to that of the limit set of an orbit was introduced and studied. For $T \in B(X)$ and $x \in X$, let $J_T(x)$ be the set of all vectors $y \in X$ such that there exist a strictly increasing sequence $(k_n) \subseteq \mathbb{N}$ and a sequence $(x_n) \subseteq X$ with $x_n \to x$ and $T^{k_n}x_n \to y$ as $n \to \infty$. It is easy to see that the set $J_T(x)$ is always closed.

In [**?**], Problem 1, it was asked whether there is an analogue of the Bourdon-Feldman theorem in the case of $J$-sets: if the set $J_T(x)$ has a nonempty interior, does it imply that $J_T(x) = X$?

The goal of this paper is to give a negative answer to this question.

Let $X$ be a Banach space, $x \in X$ and $r > 0$. We denote by $B(x,r) = \{y \in X : \|y - x\| \leq r\}$ the closed ball with radius $r$ and center $x$. We denote by $\mathrm{int}\, A$ the interior of any subset $A \subset X$.

## 2. Main result

**Example.** There exist a Banach space $X$ and an operator $T \in B(X)$ such that $\mathrm{int}\, J_T(0) \neq \emptyset$ and $J_T(0) \neq X$.

**Construction.** Let $(k_n)_{n=1}^\infty$ be a fixed fast increasing sequence of positive integers. It is sufficient to assume that $k_{n+1} \geq 5k_n^2$ for all $n \in \mathbb{N}$. Let $X$ be the $\ell_1$ space with the standard basis

$$\{u_i : i = 0, 1, ...\} \cup \{v_{n,j} : n \in \mathbb{N}, 1 \leq j \leq k_n\}.$$

1991 *Mathematics Subject Classification.* 47A16.

*Key words and phrases.* hypercyclic vectors, orbits, J-sets.

The research was supported by grants 201/09/0473 of GA CR and IAA100190903 of GA AV.

More precisely, the elements of $X$ can be expressed as

$$x=\sum_{i=0}^\infty \alpha_i u_i+\sum_{n=1}^\infty\sum_{j=1}^{k_n}\beta_{n,j}v_{n,j}$$

with complex coefficiens $\alpha_i,\beta_{n,j}$ such that

$$\|x\|:=\sum_{i=0}^\infty|\alpha_i|+\sum_{n=1}^\infty\sum_{j=1}^{k_n}|\beta_{n,j}|<\infty.$$

Let $\{w_n: n\in\mathbb{N}\}$ be a countable dense set in $B(0,\frac14)$. Without loss of generality we may assume that each $w_n$ belongs to the space $\bigvee\bigl\{u_0,u_1,...,u_n,v_{m,j}: 1\le m< n,\ 1\le j\le k_m\bigr\}$.

We are going to construct an operator $T$ with $J_T(0)\supset B(u_0,1/4)$. To this end it is sufficient to have $u_0+w_n\in T^{k_n}B(0,1/n)$ for each $n$. The purpose of the finite-dimensional subspace $\bigvee\{v_{n,j}: 1\le j\le k_n\}$ is to achieve this relation. The infinite-dimensional subspace $\bigvee\{u_i: i=0,1,\dots\}$ will ensure that $J_T(0)\ne X$.

Let $T\in B(X)$ be defined by

$$\begin{aligned} Tu_i &= 2u_{i+1} \qquad (i=0,1,...),\\ Tv_{n,j} &= 2v_{n,j+1} \qquad (n\in\mathbb{N},1\le j\le k_n-1),\\ Tv_{n,k_n} &= \frac{n}{2^{k_n-1}}(u_0+w_n) \qquad (n\in\mathbb{N}).\end{aligned}$$

It is easy to see that $\|T\|=2$. For each $n\in\mathbb{N}$ we have

$$T^{k_n}(n^{-1}v_{n,1})=2^{k_n-1}n^{-1}Tv_{n,k_n}=u_0+w_n.$$

This implies that $B(u_0,\frac14)\subset J_T(0)$. Indeed, let $z\in X$ with $\|z\|\le\frac14$ and let $(n_i)$ be an increasing sequence in $\mathbb{N}$ satisfying $w_{n_i}\to z$ as $i\to\infty$. Then $n_i^{-1}v_{n_i,1}\to 0$ and $\lim_{i\to\infty}T^{k_{n_i}}(n_i^{-1}v_{n_i,1})=\lim_{i\to\infty}(u_0+w_{n_i})=u_0+z$. In particular, int $J_T(0)\ne\emptyset$.

It remains to show that $J_T(0)\ne X$. Suppose on the contrary that $J_T(0)=X$. In particular, it means that $v_{1,1}\in J_T(0)$, and so there exist $k\in\mathbb{N}$ and $y\in X, \|y\|\le 1$ with

$$\|T^ky-v_{1,1}\|<\frac14. \tag{1}$$

Moreover, we may assume that $k>k_2+k_1$. Write $m_n=k_n+k_{n-1}+\dots+k_1$. Since $k_{i+1}\ge 5k_i^2\ge 5k_i$, we have $m_n\le\frac{5k_n}{4}$, and so $k_n\le m_n\le\frac54 k_n$.

Let $n\in\mathbb{N}$ satisfy $m_{n-1}\le k<m_n$. By assumption, $n\ge 3$. Write $X_0=\bigvee\{u_i: i=0,1,\dots\}$. For $n\in\mathbb{N}$ let $X_n=\bigvee\{v_{n,i}: 1\le i\le k_n\}$. Let $P_j$ be the natural projection onto $X_j$, i.e., $\ker P_j=\bigvee_{i\ne j}X_i$. Clearly $\|P_j\|=1$ for each $j$.

Write $y=y_0+y_1+x+y_2$, where $y_0=P_0y,\ y_1=\Bigl(\sum_{i=1}^{n-1}P_i\Bigr)y,\ x=P_ny$ and $y_2=\Bigl(\sum_{i=n+1}^\infty P_i\Bigr)y$. We have $\|y_0\|+\|y_1\|+\|x\|+\|y_2\|=\|y\|\le 1$. Obviously $T^ky_0\in X_0$ and

$$T^ky_1\in T^k\Bigl(\bigvee_{i=1}^{n-1}X_i\Bigr)\subset T^{k-k_{n-1}}\Bigl(\bigvee_{i=0}^{n-2}X_i\Bigr)\subset\cdots\subset T^{k-k_{n-1}-\dots-k_1}(X_0)\subset X_0.$$

Finally, $\Bigl\|\Bigl(\sum_{i=0}^{n-1}P_i\Bigr)T^ky_2\Bigr\|\le\frac{2^k(n+1)}{2^{k_{n+1}-1}}\le\frac{n+1}{2^{k_{n+1}-m_n}}\le\frac{n+1}{2^{k_n}}<\frac14.$

If $m_{n-1} \le k < m_n - 2m_{n-1} = k_n - m_{n-1}$, then

$$\|P_1 T^k y\| \le \|P_1 T^k x\| + \|P_1 T^k y_2\| \le \frac{2^k n}{2^{k_n-1}} + \frac{1}{4} \le \frac{n}{2^{m_{n-1}}} + \frac{1}{4} < \frac{1}{2}.$$

So $\|T^k y - v_{1,1}\| \ge \|P_1(T^k y - v_{1,1})\| \ge 1 - \frac{1}{2} = \frac{1}{2}$, a contradiction with (1).

So we may assume that $k_n - m_{n-1} \le k \le k_n + m_{n-1} = m_n$. Write for short $m = m_{n-1}$. For $j = 1, 2, \dots$ let $Y_j = \bigvee\{u_{(j-1)m}, \dots, u_{jm-1}\}$. Write also $Y_0 = \bigcup_{i=1}^{n-1} X_i$. Let $Q_j$ be the natural projection onto $Y_j$ $(j = 0, 1, \dots)$. Note that $k - m \ge k_n - 2m \ge 5k_{n-1}^2 - 2m \ge \frac{16}{5}m^2 - 2m \ge m^2$, and so $T^k(y_0 + y_1) \in \bigvee\{u_i : i \ge m^2\}$. Thus $\left(\sum_{i=0}^m Q_j\right) T^k(y_0 + y_1) = 0$ and

$$\begin{aligned} \left\|\left(\sum_{j=0}^m Q_j\right)(T^k x - v_{1,1})\right\| &= \left\|\left(\sum_{j=0}^m Q_j\right)(T^k(y_0 + y_1 + x) - v_{1,1})\right\| \\ &\le \left\|\left(\sum_{j=0}^m Q_j\right)(T^k y - v_{1,1})\right\| + \left\|\left(\sum_{j=0}^m Q_j\right)T^k y_2\right\| \le \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}. \end{aligned} \tag{2}$$

Let $x = \sum_{i=1}^{k_n} \alpha_i v_{n,i}$. Let $i_0 = k_n - k + 1$ and $x_0 = \sum_{i=1}^{i_0-1} \alpha_i v_{n,i}$ (if $i_0 \le 1$ then $x_0 = 0$). For $j = 1, ..., m$ let

$$x_j = \sum_{i=i_0+(j-1)m}^{i_0+jm-1} \alpha_i v_{n,i}.$$

We have $T^k x_0 \in X_n$, and so $\left(\sum_{j=0}^m Q_j\right) T^k x_0 = 0$. For $j = 1, ..., m$, we have

$$\begin{aligned} T^k x_j &= \sum_{i=i_0+(j-1)m}^{i_0+jm-1} \alpha_i T^k v_{n,i} = \sum_i \alpha_i 2^{k_n - i} T^{k-k_n+i} v_{n,k_n} \\ &= \sum_i \alpha_i \frac{2^{k_n-i} n}{2^{k_n-1}} T^{k-k_n+i-1}(u_0 + w_n) = s_j + q_j, \end{aligned}$$

where

$$s_j = 2^{k-k_n} n \sum_{i=i_0+(j-1)m}^{i_0+jm-1} \alpha_i u_{k-k_n+i-1}$$

and

$$q_j = \sum_{i=i_0+(j-1)m}^{i_0+jm-1} \alpha_i 2^{1-i} n T^{k-k_n+i-1} w_n.$$

Note that

$$\|s_j\| = n 2^{k-k_n} \sum_{i=i_0+(j-1)m}^{i_0+jm-1} |\alpha_i| = n 2^{k-k_n} \|x_j\|$$

and

$$\|q_j\| \le \sum_{i=i_0+(j-1)m}^{i_0+jm-1} |\alpha_i| 2^{1-i} n 2^{k-k_n+i-1} \|w_n\| \le \frac{1}{4}\|s_j\|.$$

Note also that

$$T^k x_j \in Y_{j-1} \vee Y_j \vee Y_{j+1}.$$

Write $t_j = Q_{j-1} q_j, t'_j = Q_j q_j$ and $t''_j = Q_{j+1} q_j$. For $j = 1, \dots, m-1$, we have

$$\left\|\left(\sum_{i=0}^j Q_i\right)(T^k x - v_{1,1})\right\| = \|t_1 - v_{1,1}\| + \|s_1 + t'_1 + t_2\| + \|s_2 + t''_1 + t'_2 + t_3\| + \cdots$$

$$\begin{aligned}
&\quad \cdots + \|s_{j-1} + t''_{j-2} + t'_{j-1} + t_j\| + \|s_j + t''_{j-1} + t'_j + t_{j+1}\| \\
&\geq 1 - \|t_1\| + \|s_1\| - \|t'_1\| - \|t_2\| + \|s_2\| - \|t''_1\| - \|t'_2\| - \|t_3\| + \cdots \\
&\quad \cdots + \|s_j\| - \|t''_j\| - \|t'_j\| - \|t_{j+1}\| \\
&\geq 1 + \big(\|s_1\| - \|t_1\| - \|t'_1\| - \|t''_1\|\big) + \cdots \\
&\quad \cdots + \big(\|s_{j-1}\| - \|t_{j-1}\| - \|t'_{j-1}\| - \|t''_{j-1}\|\big) + \big(\|s_j\| - \|t_j\| - \|t'_j\|\big) - \|t_{j+1}\| \\
&\geq 1 + \frac{3}{4}(\|s_1\| + \|s_2\| + \cdots + \|s_j\|) - \frac{\|s_{j+1}\|}{4}.
\end{aligned}$$

Since $\left\|\left(\sum_{i=0}^{j} Q_i\right)(T^k x - v_{1,1})\right\| \leq \frac{1}{2}$ by (2), we have
$\|s_{j+1}\| \geq 3(\|s_1\| + \|s_2\| + \cdots + \|s_j\|) \geq 3\|s_j\|$. So $\|x_{j+1}\| \geq 3\|x_j\|$.
By induction, $\|x_m\| \geq 3\|x_{m-1}\| \geq \cdots \geq 3^{m-1}\|x_1\|$. Since $\|x_m\| \leq \|x\| \leq 1$, we have $\|x_1\| \leq 3^{1-m}$. Hence

$$\|Q_0 T^k x\| = \|Q_0 T^k x_1\| = \|t_1\| \leq 2^{k-k_n} n \frac{\|x_1\|}{4} \leq 2^{k-k_n-2} n 3^{1-m} \leq \frac{2^m n}{3^m} \leq \frac{1}{2},$$

which is a contradiction with the fact that

$$\|Q_0 T^k x\| \geq \|Q_0 v_{1,1}\| - \|Q_0(T^k x - v_{1,1})\| \geq 1 - \|T^k x - v_{1,1}\| \geq \frac{3}{4}.$$

**Remark.** The construction above can be modified easily so that we obtain an operator $V \in B(Y)$ and a non-zero vector $y \in Y$ such that $\operatorname{int} J_V(y) \neq \emptyset$ and $J_V(y) \neq Y$.

Let $X$ and $T \in B(X)$ be as in the previous example. Let $Y = X \oplus \ell_1$ and let $V = T \oplus 2S$, where $S \in B(\ell_1)$ is the backward shift. Let $y \neq 0$ and $Sy = 0$. Then $V(0 \oplus y) = 0$. It is easy to see that $J_V(0 \oplus y) = J_V(0 \oplus 0)$. Clearly $J_V(0 \oplus 0) \subset J_T(0) \oplus J_{2S}(0)$. Furthermore, it is easy to see that for all $\varepsilon > 0$, $y' \in \ell_1$ and all $n$ sufficiently large there exists $y_n \in \ell_1$ with $\|y_n\| < \varepsilon$ and $(2S)^n y_n = y'$. This implies that $J_V(0 \oplus 0) = J_T(0) \oplus \ell_1$.

Hence $\operatorname{int} J_V(0 \oplus y) \neq \emptyset$ and $J_V(0 \oplus y) \neq Y$.

**Acknowledgment.** The first author is extremely grateful to Professor Vladimir Müller for his constant scientific support and also for his hospitality during the first author's research stay at the Mathematical Institute of the Czech Academy of Sciences, Prague, Czech Republic in 2010.

## References

[BF] P.S. Bourdon, N.S. Feldman, Somewhere dense orbits are everywhere dense, Indiana Univ. Math. J., **52**(2003), 811-819.

[CM] G. Costakis, A. Manoussos, J-class operators and hypercyclicity, J. Operator Theory, to appear.

University of Tabriz, Faculty of Mathematical Sciences, 29 Bahnan st., 51666-16471 Tabriz, Iran

*E-mail address*: mh_azimi@tabrizu.ac.ir

Institute of Mathematics, Czech Academy of Sciences, Zitna 25, 115 67 Prague 1, Czech Republic

*E-mail address*: muller@math.cas.cz