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Universal spaces for manifolds equipped with a closed integral k-form

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Abstract

In this note we prove that any integral closed k-form ϕ^k, $k \ge 3$, on a m-dimensional manifold M^m, $m \ge k$, is the restriction of a universal closed k-form h^k on a universal manifold $U^{d(m,k)}$ as a result of an embedding of M^m to $U^{d(m,k)}$.

MSC: 53C10, 53C42
Key words: closed k-form, universal space, H-principle.

1 Introduction.

Let a manifold M^m be equipped with a tensor of degree α^k and a manifold N^n equipped with a tensor β^k. Suppose that $n > m$. We want to know if there is an immersion $f : M^m \to N^n$ such that $f^*(\beta^k) = \alpha^k$. This problem has a long history, the Nash embedding theorem for (M^m, α^2) being a Riemannian manifold and (N^n, β^2) being the standard Euclidean space is one of most spectacular results in this field. Gromov in his seminal book [Gromov1986] developed many methods for solving this problem.

In this note we apply the Gromov theory to obtain the existence of a universal space $U^{d(m,k)}$, equipped with an integral closed k-form h^k, $k \ge 3$, for any m-dimensional manifolds M^m equipped with an integral closed k-form ϕ^k (Theorem 3.6). Theorem 3.6 is a generalization of Tischler's theorem [Tischler1977] on the existence of a symplectic embedding from an integral symplectic manifold (M^{2n}, ω) to $\mathbb{C}P^n$ equipped with the standard Kaehler symplectic form.

This note also contains an Appendix written in communication with Kaoru Ono which contains a new "soft" proof of a version Theorem 3.6 on the existence of a universal space for manifolds equipped with an integral closed k-form. Our soft proof does not use the

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Nash-Gromov implicit function theorem, but we do not get a C^0-perturbation result as in Theorem 3.6.

2 H-principle and Nash-Gromov implicit function theorem.

In this section we briefly recall some important notions and results in the Gromov theory [Gromov1986] which we shall use for our proof of Theorem 3.6.

Let V and W be smooth manifolds. We denote by $(V, W)^{(r)}$, $r \ge 0$, the space of r-jets of smooth mappings from V to W. We shall think of each map $f : V \to W$ as a section of the fibration $V \times W = (V, W)^{(0)}$ over V. Thus $(V, W)^{(r)}$ is a fibration over V, and we shall denote by p^r the canonical projection $(V, W)^{(r)}$ to V, and by p^s_r the canonical projection $(V, W)^{(s)} \to (V, W)^{(r)}$, for any $s > r$.

A section $s : V \to (V, W)^r$ is called **holonomic**, if s is the r-jet of some section $f : V \to (V, W)$.

We say that a differential relation $\mathcal{R} \subset (V, W)^{(r)}$ satisfies the **H-principle**, if every continuous section $\phi_0 : V \to \mathcal{R}$ can be brought to a holonomic section ϕ_1 by a homotopy of sections $\phi_t : V \to \mathcal{R}$, $t \in [0, 1]$.

We say that a differential relation $\mathcal{R} \subset (V, W)^{(r)}$ satisfies the **H-principle C^0-near a map** $f_0 : V \to W$, if every continuous section $\phi_0 : V \to \mathcal{R}$ which lies over f_0, (i.e. $p^r_0 \circ \phi_0 = f_0$) can be brought to a holonomic section ϕ_1 by a homotopy of sections $\phi_t : V \to \mathcal{R}_U$, $t \in [0, 1]$, for an arbitrary small neighborhood U of $f_0(V)$ in $V \times W$ [Gromov1986, 1.2.2]. Here for an open set $U \subset V \times W$, we write

$$\mathcal{R}_U := (p^r_0)^{-1}(U) \cap \mathcal{R} \subset (V, W)^r.$$

The H-principle is called **C^0-dense**, if it holds true C^0-near every map $f : V \to W$.

We also define the **fine C^0-topology** on the space $C^0(X)$ of continuous sections of a smooth fibration $X \to V$ by taking the sets $C^0(U) \subset C^0(X)$, U is open in X, as the basis for this topology. The **fine C^r-topology on $C^r(X)$** is induced by the fine C^0-topology on $C^0(X^{(r)})$ using the embedding $C^r(X) \to C^0(X^{(r)})$.

Suppose we are given a differential relation $\mathcal{R} \subset (V, W)^{(r)}$. We define the **prolongation** $\mathcal{R}^k \subset (V, W)^{r+k}$ inductively. Let $\mathcal{R}' \subset (\mathcal{R}^{(r)})^{(1)}$ consist of the 1-jets of germs of C^1-sections $V \to \mathcal{R}$. We put $\mathcal{R}^1 := \mathcal{R}' \cap X^{(r+1)} \subset (X^{(r)})^{(1)}$. Then repeat this and define $\mathcal{R}^k := (V, W)^{r+k} \cap (\mathcal{R}^{k-1})^1 \subset ((V, W)^{r+k-1})^1$. A C^{r+k}-solution of $\mathcal{R}$ is a holonomic section of $\mathcal{R}^k$.

Fix an integer $k \geq r$ and denote by $\Phi(U)$ the space of C^k-solutions of $\mathcal{R}$ over U for all open $U \subset V$. This set equipped with the natural restriction $\Phi(U) \to \Phi(U')$ for all $U' \subset U$ makes Φ a sheaf which we call the **solution sheaf** of $\mathcal{R}$ over V. We shall say that Φ satisfies the H-principle, if $\mathcal{R}$ satisfies the H-principle.

A sheaf Φ is called **flexible (microflexible)**, if the restriction map $\Phi(C) \to \Phi(C')$ is a fibration (microfibration) for all pair of compact subsets C and $C' \subset C$ in M. We recall that the map $\alpha : A \to A'$ is called **microfibration**, if the homotopy lifting property for a homotopy $\psi : P \times [0,1] \to A'$ is valid only "micro", i.e. there exists $\varepsilon > 0$ such that ψ can lift to a homotopy $\bar{\psi} : P \times [0, \varepsilon] \to A$.

2.1. H-principle and flexibility [Gromov1986, 2.2.1.B]. *If V is a locally compact countable polyhedron (e.g. manifold), then every flexible sheaf over V satisfies the H-principle.*

2.2. A criterion on flexibility. [Gromov1986, 2.2.3.C"] *Let Φ be a microflexible sheaf over V and let a submanifold $V_0 \subset V$ be sharply movable by acting diffeotopies. Then the sheaf $\Phi_0 = \Phi_{|V_0}$ is flexible and hence it satisfies the H-principle.*

One of Gromov's method to get the microflexibility of some sheaf (and then to get the H-principle) is to exploit the Nash-Gromov implicit function theorem.

Let $X \to V$ be a smooth fibration and $G \to V$ be a smooth vector bundle over a manifold V. We denote by $\mathcal{X}^\alpha$ and $\mathcal{G}^\alpha$ respectively the spaces of C^α-sections of the fibrations X and G for all $\alpha = 0, 1, \cdots, \infty$. Let $\mathcal{D} : \mathcal{X}^r \to \mathcal{G}^0$ be a differential operator of order r. In other words the operator $\mathcal{D}$ is given by a bundle map $\triangle : X^{(r)} \to G$, namely $\mathcal{D}(x) = \triangle \circ J^r_x$, where $J^r_x(v)$ denotes the r-jet of x at $v \in V$. We assume below that $\mathcal{D}$ is a C^∞-operator and so we have continuous maps $\mathcal{D} : \mathcal{X}^{\alpha+r} \to \mathcal{G}^\alpha$ for all $\alpha = 0, 1, \cdots, \infty$.

Now we shall define the linearization of a differential operator $\mathcal{D}$. Let x be a C^α-section of a smooth vector bundle $X \to V$. Denote by Y_x the induced vector bundle $x^*(T_{vert}(X))$. For each $\beta \leq \alpha$ we denote by $\mathcal{Y}^\beta_x$ the space of C^β-section $V \to Y_x$. The space $\mathcal{Y}^\alpha_x$ can be considered as the tangent space $T_x(\mathcal{X}^\alpha)$. Now we suppose that the fibration $X \to V$ does not have boundary. For $x \in \mathcal{X}^r$ the **linearization** $L_x : \mathcal{Y}^r_x \to \mathcal{G}^0$ of the operators $\mathcal{D}$ at x is defined as follows. Let $y = \partial x_t / \partial t_{|t=0}$. Then

$$L_x(y) = L(x, y) = \frac{\partial}{\partial t}_{|t=0} \mathcal{D}(x_t).$$

We say that the operator $\mathcal{D}$ is **infinitesimal invertible over a subset** $\mathcal{A}$ in the space of sections $x : V \to X$ if there exists a family of linear differential operators of certain order s, namely $M_x : \mathcal{G}^s \to \mathcal{Y}^0_x$, for $x \in \mathcal{A}$, such that the following three properties are satisfied.

1. There is an integer $d \geq r$, called **the defect of the infinitesimal inversion** M, such that $\mathcal{A}$ is contained in $\mathcal{X}^d$, and furthermore, $\mathcal{A} = \mathcal{A}^d$ consists exactly of C^d-solutions of an open differential relation $A \subset X^{(d)}$. In particular, the sets $\mathcal{A}^{\alpha+d} = \mathcal{A} \cap \mathcal{X}^{\alpha+d}$ are open in $\mathcal{X}^{\alpha+d}$ in the respective fine $C^{\alpha+d}$-topology for all $\alpha = 0, 1, \cdots, \infty$.

2. The operator $M_x(g) = M(x, g)$ is a (non-linear) differential operator in x of order s. Moreover the global operator

$$M : \mathcal{A}^d \times \mathcal{G}^s \to \mathcal{J}^0 = T(\mathcal{X}^0)$$

is a differential operator, that is given by a C^∞-map $A \oplus G^{(s)} \to T_{vert}(X)$.

3. $L_x \circ M_x = Id$ that is

$$L(x, M(x, g)) = g \text{ for all } x \in \mathcal{A}^{d+r} \text{ and } g \in \mathcal{G}^{r+s}.$$

Now let $\mathcal{D}$ admit over an open set $\mathcal{A} = \mathcal{A}^d \subset \mathcal{X}^d$ an infinitesimal inversion M of order s and of defect d. For a subset $\mathcal{B} \subset \mathcal{X}^0 \times \mathcal{G}^0$ we put $\mathcal{B}^{\alpha,\beta} := \mathcal{B} \cap (\mathcal{X}^\alpha \times \mathcal{G}^\beta)$. Let us fix an integer σ_0 which satisfies the following inequality

$$(*) \qquad \sigma_0 > \bar{s} = \max(d, 2r + s).$$

Finally we fix an arbitrary Riemannian metric in the underlying manifold V.

2.3. Nash-Gromov implicit function theorem. [Gromov1986, 2.3.2]. *There exists a family of sets $\mathcal{B}_x \subset \mathcal{G}^{\sigma_0+s}$ for all $x \in \mathcal{A}^{\sigma_0+r+s}$, and a family of operators $\mathcal{D}_x^{-1} : \mathcal{B}_x \to \mathcal{A}$ with the following five properties.*

1. *Neighborhood property: Each set $\mathcal{B}_x$ contains a neighborhood of zero in the space $\mathcal{G}^{\sigma_0+s}$. Furthermore, the union $\mathcal{B} = \{x\} \times \mathcal{B}_x$ where x runs over $\mathcal{A}^{\sigma_0+r+s}$, is an open subset in the space $\mathcal{A}^{\sigma_0+r+s} \times \mathcal{G}^{\sigma_0+s}$.*

2. *Normalization Property: $\mathcal{D}_x^{-1}(0) = x$ for all $x \in \mathcal{A}^{\sigma_0+r+s}$.*

3. *Inversion Property: $\mathcal{D} \circ \mathcal{D}_x^{-1} - \mathcal{D}(x) = Id$, for all $x \in \mathcal{A}^{\sigma_0+r+s}$, that is*

$$\mathcal{D}(\mathcal{D}_x^{-1}(g)) = \mathcal{D}(x) + g,$$

for all pairs $(x, g) \in \mathcal{B}$.

4. *Regularity and Continuity: If the section $x \in \mathcal{A}$ is C^{η_1+r+s}-smooth and if $g \in \mathcal{B}_x$ is C^{σ_1+s}-smooth for $\sigma_0 \leq \sigma_1 \leq \eta_1$, then the section $\mathcal{D}_x^{-1}(g)$ is C^σ-smooth for all $\sigma < \sigma_1$. Moreover the operator $\mathcal{D}^{-1} : \mathcal{B}^{\eta_1+r+s,\sigma_1+s} \to \mathcal{A}^\sigma$, $\mathcal{D}^{-1}(x, g) = \mathcal{D}_x^{-1}(g)$, is jointly continuous in the variables x and g. Furthermore, for $\eta_1 > \sigma_1$, the section $\mathcal{D}^{-1} : \mathcal{B}^{\eta_1+r+s,\sigma_1+s} \to \mathcal{A}^{\sigma_1}$ is continuous.*

5. *Locality: The value of the section $\mathcal{D}_x^{-1}(g) : V \to X$ at any given point $v \in V$ does not depend on the behavior of x and g outside the unit ball $B_v(1)$ in V with center v, and so the equality $(x, g)_{|B_v(1)} = (x', g')_{|B_v(1)}$ implies $\mathcal{D}_x^{-1}(g))(v) = (\mathcal{D}_{x'}^{-1}(g'))(v)$.*

2.4. Corollary. Implicit Funtion Theorem. *For every $x_0 \in \mathcal{A}^\infty$ there exists fine $C^{\bar{s}+s+1}$-neighborhood $\mathcal{B}_0$ of zero in the space of $\mathcal{G}_{\bar{s}+s+1}$, where $\bar{s} = \max(d, 2r+s)$, such that for each $C^{\sigma+s}$-section $g \in \mathcal{B}_0$, $\sigma \geq \bar{s}+1$, the equation $\mathcal{D}(x) = \mathcal{D}(x_0) + g$ has a C^σ-solution.*

Finally we shall show a large class of microflexible solution sheafs Φ by using the Nash-Gromov implicit function theorem.

Let us fix a C^∞-section $g : V \to G$ and we call a C^∞-germ $x : \mathcal{O}p(v) \to X$, $v \in V$, **an infinitesimal solution of order α of the equation** $\mathcal{D}(x) = g$, if at the point v the germ $g' = g - \mathcal{D}(x)$ has zero α-jet , i.e. $J^\alpha_{g'}(v) = 0$. We denote by $\mathcal{R}^\alpha(\mathcal{D}, g) \subset X^{(r+\alpha)}$ the set of all jets represented by these infinitesimal solutions of order α over all points $v \in V$. Now we recall the open set $A \subset X^{(d)}$ defining the set $\mathcal{A} \subset X^{(d)}$, and for $\alpha \geq d - r$ we put

$$\mathcal{R}_\alpha = \mathcal{R}_\alpha(A, \mathcal{D}, g) = A^{r+\alpha-d} \cap \mathcal{R}^\alpha(\mathcal{D}, g) \subset X^{(r+\alpha)},$$

where $A^{r+\alpha-d} = (p_d^{r+\alpha})^{-1}(A)$ for $p_d^{r+\alpha} : X^{r+\alpha} \to X^d$.

A $C^{r+\alpha}$-section $x : V \to X$ satisfies $\mathcal{R}_\alpha$, iff $\mathcal{D}(x) = g$ and $x \in \mathcal{A}$.

Now we set $\mathcal{R} = \mathcal{R}_{d-r}$ and denote by $\Phi = \Phi(\mathcal{R}) = \Phi(A, \mathcal{D}, g)$ the sheaf of C^∞-solutions of $\mathcal{R}$.

2.5. Microflexibility of the sheaf of solutions and the Nash-Gromov implicit functions.[Gromov1986 2.3.2.D”] *The sheaf Φ is microflexible.*

3 Universal space for integral closed k-forms on m-dimensional manifolds.

Suppose that $m \geq k \geq 3$. In this section we shall show that any integral closed k-form ϕ^k on a m-dimensional smooth manifold M^m can be induced from a universal closed k-form h^k on a universal manifold $U^{d(m,k)}$ by an embedding M^m to a universal space $(U^{d(m,k)}, h^k)$, see Theorem 3.6.

Our definition of the universal space $(U^{d(m,k)}, h^k)$ is based on the work of Dold and Thom [D-T1958] as well as the idea of Gromov [Gromov2006] to reduce this problem to the case that ϕ is an exact k-form.

Let $SP^q(X)$ be the q-fold symmetric product of a locally compact, paracompact Hausdorff pointed space $(X,0)$, i.e. $SP^q(X)$ is the quotient space of the q-fold Cartesian $(X^q,0)$ over the permutation group σ_q. We shall denote by $SP(X,0)$ the inductive limit of $SP^q(X)$ with the inclusion

$$X = SP^1(X) \stackrel{i_1}{\to} SP^2(X) \stackrel{i_2}{\to} \cdots \to SP^q(X) \stackrel{i_q}{\to} \cdots ,$$

where

$$SP^q(X) \stackrel{i_q}{\to} SP^{q+1}(X) : \ [x_1, x_2, \cdots, x_q] \mapsto [0, x_1, x_2, \ldots, x_q].$$

Equivalently we can write

$$SP(X,0) = \sum_q SP^q(X)/([x_1, x_2, \cdots, x_q] \sim [0, x_1, x_2, \cdots, x_q]).$$

So we shall also denote by i_q the canonical inclusion $SP^q(X) \to SP(X,0)$.

3.1. Theorem. (see [D-T1958, Satz 6.10]) *There exist natural isomorphisms $j : H_q(X,\mathbb{Z}) \to \pi_q(SP(X,0))$ for $q > 0$.*

3.2. Corollary. ([D-T1958]) *The space $SP(S^n,0)$ is the Eilenberg-McLane complex $K(\mathbb{Z},n)$.*

Now let τ^k be the generator of $H^k(SP(S^k,0),\mathbb{Z})$ and by abusing notations we also denote by τ^k the restriction of the generator τ^k to any subspace $i_q(SP^q(S^k)) \subset SP(S^k,0)$. The following lemma shows that we can replace a classifying map from $(M^m,[\phi^k])$ to $(SP(S^k,0),\tau^k)$ by a map from $(M^m,[\phi^k])$ to $(SP^{[\frac{m-k}{2}]+1}(S^k),\tau^k)$.

3.3. Lemma. *Let $[\phi^k] \in H^k(M^m,\mathbb{Z})$. Then there exists a continuous map f from M^m to $(SP^{[\frac{m-k}{2}]+1}(S^k))$ such that $f^*(\tau^k) = [\phi^k]$.*

Proof. Let f_0 be a classifying map from M^m to $SP(S^k,0)$ such that $f_0^*(\tau^k) = \alpha$. Denote by K^i the i-dimensional skeleton of $SP(S^k,0)$ and by $\bar\tau^k$ the restriction of τ^k to K^m. Then we know that f_0 is homotopic equivalent to a continuous map $f_1 : M^m \to K^m$ such that $f_1^*(\bar\tau^k) = [\phi^k]$. To prove Lemma 3.3 it suffices to find a map $g : K^m \to SP^{[\frac{m-k}{2}]+1}(S^k)$ such that $g^*(\tau^k) = \bar\tau^k$. Then the map $f = g \circ f_1$ satisfies the condition of Lemma 3.3.

We observe that $K^{k+1} = K^k$ consists of the sphere S^k. If $m = k$ or $m = k+1$, then g can be chosen as the identity map. Now suppose that $m \geq k+2$. The following identity [D-P1961, (12.12)]

$$\pi_i(SP^n(X)) = H_i(X) \text{ for } i < k+2n-1,\ n > 1,$$

if X is connected and $H_i(X) = 0$ for $0 < i < k$, $k > 1$, implies that

$$\pi_i(SP^{[\frac{m-k}{2}]+1}(S^k)) = 0, \text{ for } k+1 \le i \le m-1.$$

Using the obstruction theory we obtain a map $g : K^m \to SP^{[\frac{m-k}{2}]+1}(S^k)$ extending the inclusion map $K^k = S^k \to SP^{[\frac{m-k}{2}]+1}(S^k)$. Clearly the map g satisfies the required property that $g^*(\tau^k) = \bar{\tau}^k$. $\square$

Since $SP^{[\frac{m-k}{2}]+1}(S^k)$ has a finite simplicial decomposition we can apply the Thom construction in [Thom1954, III.2] where Thom showed that any finite m-dimensional polyhedron K can be embedded in a compact $(2m+1)$-dimensional manifold M^{2m+1} such that K is a retract of M^{2m+1}. As a result we get the following

3.4. Lemma. *The space $SP^{[\frac{m-k}{2}]+1}(S^k)$ can be embedded into a compact smooth manifold $\mathcal{M}^{s(m,k)}$, $s(m,k) = 2([\frac{m-k}{2}]+3)k+1$, such that (the image of) $SP^{[\frac{m-k}{2}]+1}(S^k)$ is a retract of $\mathcal{M}^{s(m,k)}$.*

Let us denote also by τ^k the pull back of the universal class τ^k from $SP^{[\frac{m-k}{2}]+1}(S^k)$ to $\mathcal{M}^{s(m,k)}$ and let α^k be any differential form representing τ^k on $\mathcal{M}^{s(m,k)}$.

Let β^k_l be the following k-form on $\mathbb{R}^{k \cdot l}$ with coordinates $x^{ij}, 1 \le i \le l$, $1 \le j \le k$

$$\beta^k_l = dx^{11} \wedge dx^{12} \cdots \wedge dx^{1k} + \cdots + dx^{l1} \wedge dx^{l2} \cdots \wedge dx^{lk}.$$

Set $d(m,k) := s(m,k) + 2m + 2 - k + \frac{1}{2}(k-1)[\frac{2m}{k-1}]([\frac{2m}{k-1}] - 1) + k(m+1)\binom{m+1}{k}$.

Now we state the main theorem of this section. Let

$$(U^{d(m,k)}, h^k) = (\mathcal{M}^{s(m,k)} \times \mathbb{R}^{kN}, \alpha^k \oplus \beta^k_N). \tag{3.5}$$

3.6. Theorem. *Suppose that ϕ^k is a closed integral k-form on a smooth manifold M^m. Then there exists an embedding $f : M^m \to (U^{d(m,k)}, h^k)$ such that $f^*(h^k) = \phi$. Moreover for any given map $\tilde{f} : M^m \to (U^{d(m,k)}, h^k)$ such that $\tilde{f}^*[h^k] = [\phi^k]$ there exists a C^0-close to $\tilde{f}$ embedding $f : M^m \to U^{d(m,k)}$ such that $f^*(h^k) = \phi^k$.*

Proof of Theorem 3.6. Using Lemma 3.3 and Lemma 3.4 we see that the first statement of Theorem 3.6 follows from the second statement of Theorem 3.6. Furthermore we shall reduce the second statement to an immersion problem for exact 3-forms as follows. Denote by $\tilde{f}_1 : M^m \to \mathcal{M}^{s(m,k)}$ the projection of $\tilde{f}$ to the first factor. Then we have $\tilde{f}_1^*(\tau^k) = [\phi^k] \in H^k(M^m, \mathbb{Z})$. Let

$$g = \phi - \tilde{f}_1^*(\alpha).$$

Clearly g is an exact k-form on M^m. We can also assume that $\tilde{f}_1$ is an embedding by perturbing this map a little, if necessary. Thus the second statement of Theorem 3.6 is a corollary of the following Proposition.

3.7. Proposition. *For any given map $f_0 : M^m \to \mathbb{R}^{kN}$ there is a C^0-close to f_0 immersion $f_3 : M^m \to (\mathbb{R}^{kN}, \beta^k_N)$ such that $f_3^*(\beta^k_N) = g$ for any exact k-form g.*

We shall apply the Gromov H-principle for immersion of differential forms to prove Proposition 3.7. Gromov extended the Nash idea to add some regularity for an immersion in order to apply the implicit function theorem and then using 2.5 to get the H-principle for the isometric immersion. Finally using the H-principle we shall get immersions required in Proposition 3.7.

Let h be a smooth differential k-form on a manifold W. Denote by $I_{h(w)}$ a linear homomorphism

$$I_{h(w)} : T_wW \to \Lambda^{k-1}(T_wW)^*,\ X \mapsto X \rfloor h.$$

A subspace $T \subset T_wW$ is called $h(w)$**-regular**, if the composition of $I_{h(w)}$ with the restriction homomorphism $r : \Lambda^{k-1}(T_wW)^* \to \Lambda^{k-1}(T)^*$ sends T_wW onto $\Lambda^{k-1}(T)^*$.

An immersion $f : V \to W$ is called **h-regular**, if for all $v \in V$ the subspace $Df(T_vV)$ is $h(f(v))$-regular.

Proof of Proposition 3.7. Roughly speaking, we add the condition of β^k_N-regularity to the isometry property (i.e. $f_3^*(\beta^k_N) = g$) and extend this equation for mappings also denoted by f_3 from the manifold $M^{m+1} = M^m \times (-1, 1)$ provided with a form $g \oplus 0$, denoted from now on also by g, to the space $(\mathbb{R}^{kN}, \beta^k_N)$. Our Proposition 3.10 states that the solution sheaf restricted to $M^m \subset M^{m+1}$ satisfies the H-principle. In fact, this statement is a consequence of Theorem 3.4.1.B' in [Gromov1986]. So essentially we re-expose the Gromov proof of Theorem 3.4.1.B' in our concrete case, and we try to make Gromov's argument more transparent. Now to prove the existence of a β^k_N-regular isometric immersion f_3 which is C^0-close to a given map f_0, it suffices to find a section of this extended differential relation which lies over f_0 (Proposition 3.12). That is only the essential new ingredient in our proof of Proposition 3.7.

Now we are going to define our extended differential relation. Let us denote also by f_0 a map $M^{m+1} \to (\mathbb{R}^{kN}, \beta^k_N)$ extending a given map $f_0 : M^m \to \mathbb{R}^{kN}$. We denote by F_0 the corresponding section of the bundle $M^{m+1} \times \mathbb{R}^{kN} \to M^{m+1}$, i.e. $F_0(v) = (v, f_0(v))$. Denote by $\Gamma_0 \subset M^{m+1} \times \mathbb{R}^{kN}$ the graph of f_0 (i.e. it is the image of F_0), and let $p^*(g)$

and $p^*(\beta_N^k)$ be the pull-back of the forms g and β_N^k to $M^{m+1} \times \mathbb{R}^{kN}$ under the obvious projection. Take a small neighborhood $Y \supset \Gamma_0$ in $M^{m+1} \times \mathbb{R}^{kN}$. Since β_N^k and g are exact forms we get

$$p^*(\beta_N^k) - p^*(g) = d\hat{\beta}_N$$

for some smooth $(k-1)$-form $\hat{\beta}_N$ on Y.

Our next observation is

3.8. Lemma. *Suppose that a map $F : M^{m+1} \to Y$ corresponds to a β_N^k-regular immersion $f : M^{m+1} \to \mathbb{R}^{kN}$. Then F is a $d\hat{\beta}_N^k$-regular immersion.*

Proof. We need to show that for all $y = F(z) \in Y$, $z \in M^{m+1}$, the composition ρ of the maps

$$T_yY \overset{I_{p^*(\beta_N^k)-p^*(g)}}{\longrightarrow} \Lambda^{(k-1)}T_yY \to \Lambda^{(k-1)}(dF(T_{(z)}(M^{m+1}))$$

is onto. This follows from the consideration of the restriction of ρ to the subspace $S \subset T_yY$ which is tangent to the fiber $\mathbb{R}^{kN}$ in $M^{m+1} \times \mathbb{R}^{kN} \supset Y$. $\square$

Now for a map $d\hat{\beta}_N$-regular map $F : M^{m+1} \to Y$ and a $(k-2)$-form ϕ on M^{m+1} we set

$$\mathcal{D}(F, \phi) := F^*(\hat{\beta}_N) + d\phi. \tag{3.9}$$

With this notation the map $f : M^{m+1} \to \mathbb{R}^{kN}$ corresponding to $F : M^{m+1} \to Y$ satisfies

$$f^*(\beta_N^k) = F^*(p^*(\beta_N^k)) = g + F^*(d\hat{\beta}_N) = g + d\mathcal{D}(F, \phi),$$

for any ϕ. Since the space of $(k-2)$-forms ϕ is contractible, it follows that the space of $d\hat{\beta}_N$-regular sections $F : M^{m+1} \to Y$ for which

$$f^*(\beta_N^k) = g + dg_1 \tag{3.9.1}$$

for a given $(k-1)$-form g_1 has the same homotopy type as the space of solutions to the equation

$$\mathcal{D}(F, \phi) = g_1.$$

In particular the equation $f_3^*(\beta_N^k) = g$ reduces to the equation $\mathcal{D}(F, \phi) = 0$ in so far as the unknown map f_3 is C^0-close to f_0 (so that its graph lies inside Y).

We define by $\tilde{\Phi}_{reg}$ the solution sheaf of the equation (3.9) whose component F is $d\hat{\beta}_N$-regular.

3.10. Proposition. *The restriction of the solution sheaf $\tilde{\Phi}_{reg}$ to M^m satisfies the H-principle. Hence the solution sheaf of β^k_N-regular isometric immersions $f : (M^{m+1}, g) \to (\mathbb{R}^{kN}, \beta^k_N)$ such that $F(M^{m+1}) \subset Y$ restricted to M^m also satisfies the H-principle.*

Before proving this Proposition we shall prove the following

3.11. Lemma. *The differential operator $\mathcal{D}$ is infinitesimal invertible at those pairs (F, ϕ) for which the underlying map f is a β^k_N-regular immersion.*

Proof. The linearization $L_{(F,\phi)}\mathcal{D}$ acts on the space of couples $(V, \tilde{\phi})$ where V is a section of $f^*(T_*\mathbb{R}^{kN})$ (a vector field on $\mathbb{R}^{kN}$ along the corresponding map f) and $\tilde{\phi}$ is a $(k-2)$-form on M^{m+1} as follows

$$L_{(F,\phi)}\mathcal{D}(V, \tilde{\phi}) = \mathcal{L}_V \hat{\beta}_N + d\tilde{\phi} = V \rfloor d\hat{\beta}_N + d(V \rfloor \hat{\beta}_N) + d\tilde{\phi}. \tag{3.11.1}$$

By Lemma 3.8 the map F is a $d\hat{\beta}_N$-regular immersion. Hence the equation for V

$$F^*(V \rfloor d\hat{\beta}_N) = \tilde{g}, \tag{3.11.2}$$

is solvable for all $(k-1)$-form $\tilde{g}$ on M^{m+1}. Now we set:

$$\tilde{\phi} := F^*(V \rfloor \hat{\beta}_N) \tag{3.11.3}$$

Clearly every couple $(V, \tilde{\phi})$ satisfying (3.11.2) and (3.11.3) is a solution of the equation $L_{(F,\phi)}\mathcal{D}(V, \tilde{\phi}) = \tilde{g}$ for any given $(k-1)$-form $\tilde{g}$. □

Proof of Proposition 3.10. Taking into account Lemma 3.11 and 2.3 (Nash implicit function theorem), 2.5 (Nash implicit function theorem implies the microflexibility) we get the microflexibility of $\tilde{\Phi}_{reg}$. Next we use the Gromov observation [Gromov1986, 3.4.1.B'] that M^m is a sharply movable submanifold by acting diffeotopies in M^{m+1} which implies that the restriction of $\tilde{\Phi}_{reg}$ to M^m is flexible. Hence we get the first statement of Proposition 3.10 immediately. The second statement follows by a remark above relating (3.9) and (3.9.1). □

Completion of the proof of Proposition 3.7.

Suppose we are given a map $f_0 : M^m \to \mathbb{R}^{kN}$. Since M^m is a deformation retract of M^{m+1} the map f_0 extends to a map $f : M^{m+1} \to \mathbb{R}^{kN}$.

For each $z \in M^m$ we denote by $Mono((T_zM^{m+1}, g), (T_{f(z)}\mathbb{R}^{kN}, \beta^k_N))$ the set of all monomorphisms $\rho : T_zM^{m+1} \to T_{f(z)}\mathbb{R}^{kN}$ such that the restriction of $\beta^k_N(f(z))$ to $Df(T_zM^{m+1})$ is equal to $(Df^{-1})^*g$.

3.12. Proposition. *There exists a section s of the fibration $Mono((TM^{m+1}, g), f^*(T\mathbb{R}^{kN}, \beta^k_N))$ such that $s(z)(T_zM^{m+1})$ is β^k_N-regular subspace for all $z \in M^m$.*

Proof of Proposition 3.12. The proof of Proposition 3.12 consists of 3 steps.

Step 1. We consider TM^{m+1} and $M^m \times \mathbb{R}^{kN}$ as vectors bundles over the same base M^m. We shall show the existence of a section $s_1 \in Mono(TM^{m+1}, M^m \times \mathbb{R}^{kN})$ such that the image $s_1(TM^{m+1})$ is a β^k_N-regular sub-bundle of dimension $(m+1)$ in $M^m \times \mathbb{R}^{kN}$. To save notations we also denote by β^k the following k-form on $\mathbb{R}^{kN} = \oplus^N_{j=1}\mathbb{R}^k_j$

$$\beta^k = \sum_{j=1}^N dx^1_j \wedge \cdots \wedge dx^k_j.$$

Here $(x^i_j), 1 \le i \le k$, are coordinates in $\mathbb{R}^k_j$ for each $j = \overline{1, N}$.

We put for $l \ge k \ge 3$

$$\delta(l,k) := (l-1) + \frac{k-1}{2}(2 + [\frac{l-2}{k-1}])([\frac{l-2}{k-1} - 1]) + [\frac{l-1}{k-1}](1 + ((l-1) \mod (k-1))).$$

Here we set $i \mod (k-1) := i - (k-1) \cdot [i/(k-1)]$.

3.13. Lemma. *For each given $l \ge k \ge 3$ there there exists a l-dimensional subspace V^l in $\mathbb{R}^{kN}$ such that V^l is β^k-regular subspace, provided that $N \ge \delta(l,k)$.*

Proof. We shall construct a linear embedding $f^l : V^l \to \mathbb{R}^{kN}$ whose image satisfies the condition of Lemma 3.13. We work in opposite way, i.e. for each l we shall find a number $\delta(l,k)$ and an embedding $f : V^l \to \mathbb{R}^{k\delta(l,k)} = \oplus^{\delta(l,k)}_{j=1}\mathbb{R}^k_j$ and f can be written as

$$f := f^l = (f^l_1, f^l_2, \cdots, f^l_{\delta(l,k)}),\ f^l_j : V^l \to \mathbb{R}^k_j, j = \overline{1, \delta(l,k)},$$

such that f satisfies Lemma 3.13 with $\delta(l,k) = N$. Clearly the embedding $V^l \to V^{k\delta(l,k)} \to \mathbb{R}^{kN}$ also satisfies the condition of Lemma 3.13 for all $N \ge \delta(l,k)$.

We can assume that $V^k \subset V^{k+1} \subset \cdots \subset V^l$ is a chain of subspaces in V^l which is generated by some vector basis $(e_1, \cdots, e_l)$ in V^l. We denote by $(e^*_1, \cdots, e^*_l)$ the dual basis of $(V^l)^*$. By construction, the restriction of $(e^*_1, \cdots, e^*_i)$ to V^i is the dual basis of $(e_1, \cdots, e_i) \in V^i$.

For the sake of simplicity we shall denote the restriction of any v_j^* to these subspaces also by v_j^* (if the restriction is not zero). We shall construct f_i^l inductively on the dimension l of V^l such that the following condition holds for all $k \le i \le l$

(3.14)
$$< (f_1^l)^*(\Lambda^{k-1}(\mathbb{R}_1^k)), (f_2^l)^*(\Lambda^{k-1}(\mathbb{R}_2^k)), \cdots, (f_{\delta(i,k)}^l)^*(\Lambda^{k-1}(\mathbb{R}^{k-1}_{\delta(i,k)})) >_{\otimes\mathbb{R}} = \Lambda^{k-1}(V^i)^*.$$

The condition (3.14) implies that

$$I_{\beta^k}(\mathbb{R}^{k\delta(l,k)}) = \Lambda^{k-1}(V^i)^*,$$

so $f^i(V^i)$ is β^k-regular. For $l = k$ we can take $f_1^k = Id$, and $\delta(k,k) = 1$. Suppose that $(f_1^i, \cdots, f_{\delta(i,k)}^i)$ are already constructed for our map

$$f^i = (f_1^i, \cdots, f_{\delta(i,k)}^i) : V^i \to \mathbb{R}_1^k \times \cdots \times \mathbb{R}^k_{\delta(i,k)}.$$

We shall construct map f^{i+1} as follows. We set for $j \le \delta(i,k)$

$$f_j^{i+1}(e_p) = f_j^i(e_p) \text{ if } 1 \le p \le i,$$

$$f_j^{i+1}(e_{i+1}) = 0.$$

To find f_j^{i+1}, $\delta(i,k)+1 \le j \le \delta(i+1,k)$, so that (3.14) holds for the next induction step $(i+1)$, it suffices to find linear maps $f^{i+1}_{\delta(i,k)+1}, \cdots, f^{i+1}_{\delta(i+1,k)}$ with the following property

(3.15)
$$< (f^{i+1}_{\delta(i)+1})^*\Lambda^{k-1}(\mathbb{R}^k_{\delta(i,k)+1})^*, \cdots, (f^{i+1}_{\delta(i+1,k)})^*\Lambda^{k-1}(\mathbb{R}^k_{\delta(i+1,k)})^* >_{\otimes\mathbb{R}} \supset \wedge^{k-1}(V^i)^* \wedge e^*_{i+1}.$$

We shall proceed as follows. Set $\delta(i+1,k) := \delta(i,k) + [\frac{i}{(k-1)}] + 1$. Choose for any $1 \le j \le \delta(i+1,k)$ a basic (w_j^i), $1 \le i \le k$, of the space $\mathbb{R}_j^k$. We let

$$f_j^{i+1}(e_{i+1}) = w_j^1 \in \mathbb{R}_j^k, \text{ if } j \ge \delta(i,k)+1,$$

$$f^{i+1}_{\delta(i,k)+1}(e_1) = w^2_{\delta(i,k)+1},\ f^{i+1}_{\delta(i,k)+1}(e_2) = w^3_{\delta(i,k)+1}, \cdots, f^{i+1}(e_{k-1}) = w^k_{\delta(i,k)+1},$$

$$f^{i+1}_{\delta(i,k)+2}(e_k) = w^2_{\delta(i,k)+2},\ f^{i+1}_{\delta(i)+2}(e_{k+1}) = w^3_{\delta(i,k)+2}, \cdots, f^{i+1}(e_{2(k-1)}) = w^k_{\delta(i,k)+2},$$

$$\cdots$$

$$\cdots, f^{i+1}_{\delta(i+1,k)}(e_i) = w^{i \mod (k-1)}_{\delta(i+1,k)}.$$

It is easy to see that the constructed map f^{i+1} satisfies (3.15) and hence also (3.14). Now using the identity the $\delta(i+1) - \delta(i) = [\frac{i}{(k-1)}] + 1$ we get

$$\delta(l,k) = 1 + (l-k) + \sum_{i=k}^{l-1}[\frac{i}{k-1}]$$

$$= (l-1) + 2(k-1) + 3(k-1) + \cdots + [\frac{l-2}{k-1}](k-1) + [\frac{l-1}{k-1}](((l-1) \mod (k-1)) + 1)$$

$$= (l-1) + \frac{k-1}{2}(2 + [\frac{l-2}{k-1}])([\frac{l-2}{k-1} - 1]) + [\frac{l-1}{k-1}](1 + ((l-1) \mod (k-1))).$$

□

We shall consider $M^m \times V^{2m+1}$ as a sub-bundle of $M^m \times \mathbb{R}^{\delta(2m+1,k)}$ over M^m. Next we shall find a section s_1 for the step 1 by requiring that s_1 is a section of the bundle $Mono(TM^{m+1}, M^8 \times V^v)$ of all fiber mono-morphisms from TM^{m+1} to $M^m \times V^v$. This section exists, since the fiber $Mono(T_xM^{m+1}, \mathbb{R}^v)$ is homotopic equivalent to $SO(2m+1)/SO(m)$ which has all homotopy groups π_j vanishing, if $j \le (m-1)$. This completes the step 1.

Step 2. Once a section s_1 in Step 1 is specified we put the following form g_1 on $TM^{m+1}_{|M^m}$:

$$g_1 = g - s_1^*(\beta).$$

In this step we show the existence of a section s_2 of the fibration $Hom((TM^{m+1}, g_1), (M^m \times \mathbb{R}^{k\cdot(m+1)\cdot\binom{m+1}{k}}, \beta^k))$ over M^m. Here we consider $(M^m \times \mathbb{R}^{k\cdot(m+1)\cdot\binom{m+1}{k}}, \beta^k)$ as a fibration over M^m and equipped with the k-form β^k on the fiber $\mathbb{R}^{k\cdot(m+1)\cdot\binom{m+1}{k}}$. We do not require that s_2 is a monomorphism.

Using the Nash trick [Nash1956] (see also the proof of Proof of Theorem B.1 below) we can find a finite number of open coverings U_i^j, $j = \overline{1,(m+1)}$, of M^m which satisfy the following properties:

$$U_i^j \cap U_k^j = \emptyset, \forall j = \overline{1,(m+1)} \text{ and } i \neq k, \tag{3.16}$$

and moreover U_i^j is diffeomorphic to an open ball for all i, j. Since U_i^j satisfy the condition (3.16), for a fixed j we can embed the union $\hat{U}^j = \cup_i U_i^j$ into $\mathbb{R}^m$. Thus for each j on the union $\hat{U}^j$ we have local coordinates x_j^r, $r = \overline{1,(m+1)}$, $j = \overline{1,(m+1)}$. Using partition of unity functions $f_j(z)$ corresponding to $\hat{U}^j$ we can write

$$g_1(z) = \sum_{j=1}^{m+1} f_j(z) \cdot \sum_{1 \le r_1 < r_2 < \cdots < r_k \le m+1} \mu_j^{r_1 r_2 \cdots r_k}(z) \cdot dx_j^{r_1} \wedge dx_j^{r_2} \wedge \cdots \wedge dx_j^{r_k},$$

where the last coordinate x_j^{m+1} corresponds to the direction which is transveral to $T_z\hat{U}_j$ in $\hat{U}_j \times (-1,1) \subset M^{m+1}$. We numerate (i.e. find a function θ with values in $\mathbb{N}^+$) on the set $\{(j, r_1 r_2 \cdots r_k)\}$ of $N_1 = (m+1)\cdot\binom{m+1}{k}$ elements. Next we find a section s_2 of the form

$$s_2(z) = (\tilde{s}_1(z), \cdots, \tilde{s}_{N_1}(z)),\ \tilde{s}_q(z) \in Hom(T_zM^{m+1}, \mathbb{R}^k_q)$$

such that

$$\tilde{s}_{\theta(j,r_1r_2\cdots r_k)}(z) = f_j(z) \cdot \mu_j^{r_1r_2\cdots r_k}(z) \cdot A_{r_1,r_2,\cdots,r_k},$$

$$\text{where } A_{r_1,r_2,\cdots,r_k}(\partial x_{r_l} \in T_zM^{m+1}) := \delta_l^i e_i \in \mathbb{R}^k_q.$$

Here $(e_1, e_2, \cdots, e_k)$ is a vector basis in $\mathbb{R}^k_q$ for $q = \theta(j, r_1r_2\cdots r_k)$ and (∂x_{r_l}) a basic in T_zM^{m+1} defined via embedding $\hat{U}_j \to \mathbb{R}^m$ as above. Clearly the section s_2 satisfies the condition $s_2^*(\beta^k(z)) = g_1(z)$ for all $z \in M^m$. This completes the second step.

Step 3. We put

$$s = (s_1, s_2),$$

where s_1 is the constructed section in Step 1 and s_2 is the constructed section in Step 2. Clearly s satisfies the condition of Lemma 3.13. □

Proposition 3.7 now follows from Proposition 3.10 and Proposition 3.12. □

3.17. Final remark. We conjecture that the isometric embedding map $\tilde{f}$ in Theorem 3.6 is unique up to homotopy. It is the case, if ϕ^k is a closed stable form on M^m (i.e. the orbit of $GL_x(T_xM^m)(\phi^k)$ is dense in the space $\Lambda^k_x(T^*M)$ for all $x \in M^m$, see [LPV2007] for more information).

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Appendix.

in communication with Kaoru Ono[1]

A soft proof of the existence of a universal space.

For readers' convenience, we present here an elementary proof of the following

A.1. Theorem. *For any given positive integers n, k there exists a smooth manifold $\mathcal{M}$ of dimension $N(n,k)$ and a closed differential k-form α on it with the following property. For any closed differential k-form ω on a smooth manifold M^n such that $[\omega] \in H^k(M^n, \mathbb{Z})$ there is a smooth immersion $f : M^n \to \mathcal{M}^{N(n,k)}$ such that $f^*(\alpha) = \omega$.*

Proof. As in the proof of Theorem 3.6 we reduce this problem to the existence of an embedding of M^n to the space $\mathbb{R}^{\overline{N_1}}$ with the constant k-form $\beta_{\overline{N_1}}$ such that the pull-back of $\beta_{\overline{N_1}}$ is equal to a given exact k-form g. Since g is an exact form there exists a $(k-1)$-form ϕ on M^n such that $d\phi = \omega$.

Next we use the Nash trick of a construction of an open covering A_i on M^n

$$M^n = \cup_{i=0}^n A_i, \tag{A.2}$$

such that each A_i is the union of disjoint open balls $D_{i,j}$, $j = 1, \ldots, J(i)$ on M^n. (Pick a simplicial decomposition of M^n and construct A_i by the induction on i. Let $D_{0,j}$ be a small coordinate neighborhood of the j-th vertex. We may assume that they are mutually disjoint. Set $A_0 = \cup_{j=1}^{J(0)} D_{0,j}$. Suppose that $A_0, \ldots, A_i$ are defined. Let $D_{i+1,j}$ be a small coordinate neighborhood, which contains $S_j^{i+1} \setminus \cup_{\ell=0}^i A_\ell$, where S_j^{i+1} is the j-th $i+1$-dimensional simplex. We may assume that they are mutually disjoint. Set $A_{i+1} = \cup_{j=1}^{J(i+1)} D_{i+1,j}$. Hence we obtain desired open sets $A_0, \ldots, A_n$.)

Let $\{\rho_i\}$ be the partition of unity on M subordinate to the covering $\{A_i\}$. We write $\phi(x) = \sum_{i=0}^n \rho_i(x) \cdot \phi$. Note that $\omega = d\phi = \sum d\phi_i$. Clearly the form $\phi_i = \rho_i(x) \cdot \phi$ has support on A_i.

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Let $N_1 = \binom{n}{k-1}$ and

$$\gamma = \sum_{j=1}^{N_1} x_j^1 dx_j^2 \wedge \cdots \wedge dx_j^k.$$

Note that $j = 1, \ldots, \binom{n}{k-1}$ are in one-to-one correspondence with the sequences $1 \le i_2 < \cdots < i_k \le n$.

A.3. Proposition. *There is an immersion $f_i : A_i \to (\mathbb{R}^{N_1 k}, \gamma)$ such that $f_i^*(\gamma) = \phi_i$. In particular, $f_i^* d\gamma = d\phi_i$.*

Proof of Proposition A.3. Since A_i is a union of the disjoint balls $D_{i,j}$ it suffices to prove the existence of immersion f_i on each ball $D = D_{i,j}$. Take some coordinate $(x_1, \cdots, x_n)$ on the ball D. We can write the restriction of the $(k-1)$-form ϕ_i to D as ϕ, where

$$\phi(x) = \sum_{1 \le i_2 < \cdots i_k \le n} \lambda_{i_2 \cdots i_k} dx^{i_2} \wedge \cdots \wedge dx^{i_k}.$$

We construct map f_i as follows

$$f_i(x) = (\ldots, f_{i;i_2\cdots i_k}(x), \ldots)_{1 \le i_2 < \cdots < i_k \le n} \tag{A.4}$$

where for $x = (x_1, x_2, \cdots, x_n)$ we put

$$f_{i;i_2\cdots i_k}(x) : D \to \mathbb{R}^k_{i_2\cdots i_k}(x^1, x^2, \cdots, x^k),$$

$$(x_1, \cdots, x_n) \mapsto (x^1 = \lambda_{i_2\cdots i_k}(x), x^2 = x^{i_2}, \cdots, x^k = x^{i_k}).$$

Clearly we have $f_i^*(\omega) = \phi$. It is easy to check that f_i is an immersion on D. $\square$

We shall use cut-off functions χ_i with support contained in $A_i \subset M$ such that $\chi_i = 1$ on the support of ρ_i. Then $\widetilde{f_i} = \chi_i \cdot f_i$ can be extended to the whole M^n.

Now we construct an immersion $f : M^n \to \mathbb{R}^{\overline{N_1}} = \mathbb{R}^{N_1 k(n+1)}$ by setting

$$f(x) = (\tilde{f}_0, \cdots, \tilde{f}_n). \tag{B.5}$$

Clearly f is an immersion such that $f^*\alpha = \omega$.

Finally we note that we can choose $f : M^n \to \mathbb{R}^{\overline{N_1}}$ such that its image is contained in an arbitrary small neighborhood of the origin. It suffices to construct immersion $\tilde{f}_i$ in (A.5) such that the image of $\tilde{f}_i$ is contained in an arbitrary small neighborhood of the origin. Since $\tilde{f}_i$ is constructed from immersion of ball D_{ij} with help of cut-off function χ_i such that $|\chi_i| \le 1$ we reduce this problem to construct f_i whose image lies in arbitrary small neighborhood of origin. We do it by refining a given covering D_{ij} of M^n and modifying an immersion f satisfying the condition of Theorem A.1.

Choose $R > 0$ such that the image of f is contained in the R-ball centered at the origin $O \in \overline{\mathbb{R}^{N_1}}$. For a given integer m, we pick a refinement $\{V_p\}$ of the covering $\{D_{i,j}\}_{i,j}$ such that $f_{i(p)}(V_p)$ is contained in a ball of radius $1/m^2$. (The center of the ball may not be the origin.) Here $i(p)$ is chosen so that $V_p \subset A_{i(p)}$, i.e., there is $j(p)$ such that $V_p \subset D_{i(p),j(p)}$. Applying the Nash trick again to refine $\{V_p\}$ so that there is an open covering $\{A'_\ell\}$ of M such that each of A'_i is a union of some mutually disjoint family of V_p's. On $V_p \subset A'_i$ we modify the construction of the mapping $f_{i;i_2,\ldots,i_k}$ as follows. Using the translation in $x_2, \ldots x_k$-coordinates in each $\mathbb{R}^k_{i_2,\ldots,i_k}$, we may assume that

$$f_{i;i_2,\ldots i_k}(V_p) \subset [-R, R] \times [-1/m^2, 1/m^2] \times \cdots \times [-1/m^2, 1/m^2].$$

Now we consider the mapping

$$\Phi_m : (x_1, x_2, \ldots, x_k) \mapsto (\frac{x_1}{m}, m \cdot x_2, x_3, \ldots, x_k).$$

Then we find that $\Phi_m^* dx^1 \wedge \cdots \wedge dx^k = dx^1 \wedge \cdots \wedge dx^k$ and

$$\Phi_m \circ f_{i;i_2,\ldots,i_k}(V_j) \subset [-R/m, R/m] \times [-1/m, 1/m] \times [-1/m^2, 1/m^2] \times \cdots \times [-1/m^2, 1/m^2].$$

Clearly the modified map $\tilde{f}_i$ with components $\tilde{f}_{i;i_2,\ldots,i_k}(V_j) := \Phi_m \circ f_{i;i_2,\ldots,i_k}(V_j)$ (see (A.4) for definition of f_i) together with the new refined partition of A_i as above has its image contained in an arbitrary small neighborhood of origin by taking m arbitrary large. □