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# THE DIRICHLET BVP FOR THE SECOND ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATION AT RESONANCE

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Abstract. Efficient sufficient conditions are established for the solvability of the Dirichlet problem

$$u''(t) = p(t)u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad a \le t \le\ b,$$
$$u(a) = 0, \quad u(b) = 0,$$

where $h, p \in L([a, b]; R)$ and $f \in K([a, b]; R)$, in the case where the linear problem

$$u''(t) = p(t)u(t), \quad u(a) = 0, \quad u(b) = 0$$

has nontrivial solutions.

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## Introduction

Consider on the set $I = [a, b]$ the second order nonlinear ordinary differential equation

$$u''(t) = p(t)u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad t \in I \tag{0.1}$$

with the boundary conditions

$$u(a) = 0, \quad u(b) = 0, \tag{0.2}$$

where $h, p \in L(I; R)$ and $f \in K(I; R)$.

By a solution of the problem (0.1), (0.2) we understand a function $u \in \widetilde{C}'(I, R)$, which satisfies the equation (0.1) almost everywhere on $I$ and satisfies the conditions (0.2).

Consider also the homogeneous problem

$$w''(t) = p(t)w(t) \quad \text{for} \quad t \in I, \tag{0.3}$$

$$w(a) = 0, \quad w(b) = 0. \tag{0.4}$$

At present, the foundations of the general theory of two-point boundary value problems are already laid and problems of this type are studied by many authors and investigated in detail (see, for instance, [1], [4], [5], [8], [12], [13], [14]- [16], [17] and references therein). On the other hand, in all of these works, only the case when the homogeneous problem (0.3), (0.4) has only a trivial solution is studied. The case where the problem (0.3), (0.4) has also a nontrivial solution is still little investigated and in the majority of articles, the authors study the case with $q$ constant in the equation (0.1), i.e., when the problem (0.1), (0.2) and the equation (0.3) are of type

$$u''(t) = -\lambda^2 u(t) + f(t, u(t)) + h(t) \quad \text{for} \quad t \in [0, \pi], \tag{0.5}$$

$$u(0) = 0, \quad u(\pi) = 0, \tag{0.6}$$

and

$$w''(t) = -\lambda^2 w(t) \quad \text{for} \quad t \in [0, \pi] \tag{0.7}$$

respectively and $\lambda = 1$. (see, for instance, [2], [3], [4], [6]- [11], [14]- [16], and references therein).

In the present paper, we study the problem (0.1), (0.2) in the case when the function $p \in L(I; R)$ is not necessarily constant, under the assumption that the homogeneous problem (0.3), (0.4) has the nontrivial solution with an arbitrary number of zeroes. For the equation (0.7), this is the case when $\lambda$ is not necessarily the first eigenvalue of the problem (0.7), (0.6).

The obtained results are new and generalise some well-known results(see, [2], [3], [4], [6], [10]).

The following notation is used throughout the paper:

$N$ is the set of all natural numbers. $R$ is the set of all real numbers, $R_+ = [0, +\infty[$. $C(I; R)$ is the Banach space of continuous functions $u : I \to R$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$. $\widetilde{C}'(I; R)$ is the set of functions $u : I \to R$ which are absolutely continuous together with their first derivatives. $L(I; R)$ is the Banach space of Lebesgue integrable functions $p : I \to R$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$. $K(I; R)$ is the set of functions $f : I \to R$ satisfying the Carathéodory conditions. i.e., $f(\cdot, x) : I \to R$ is a measurable function for all $x \in R$, $f(t, \cdot) : R \to R$ is a continuous function for almost all $t \in I$, and for every $r > 0$ there exists $q_r \in L(I; R_+)$ such that $|f(t, x)| \le q_r(t)$ for almost all $t \in I$, $|x| \le r$.

For $w : I \to R$ we put: $N_w \overset{def}{=} \{t \in ]a, b[ : w(t) = 0\}$,

$$\Omega_w^+ \overset{def}{=} \{t \in I : w(t) > 0\}, \quad \Omega_w^- \overset{def}{=} \{t \in I : w(t) < 0\},$$

and $[w(t)]_+ = (|w(t)| + w(t))/2$, $[w(t)]_- = (|w(t)| - w(t))/2$ for $t \in I$.

**Definition 0.1.** Let, $A$ be a finite (empty) subset of $I$. We say that $f \in E(A)$, if $f \in K(I; R)$, and for any measurable set $G \subseteq I$ and the constant $r > 0$, we can choose $\varepsilon > 0$ such that if

$$\int_G |f(s, x)|ds \neq 0 \text{ for } x \ge r \ (x \le -r)$$

then

$$\int_{G\setminus U_\varepsilon} |f(s,x)|ds - \int_{U_\varepsilon} |f(s,x)|ds \geq 0 \quad \text{for} \quad x \geq r \ (x \leq -r),$$

where $U_\varepsilon = I \cap \left(\cup_{k=1}^n ]t_k - \varepsilon/2n,\ t_k + \varepsilon/2n[\right)$ if $A = \{t_1, t_2, ..., t_n\}$, and $U_\varepsilon = \emptyset$ if $A = \emptyset$.

**Remark 0.1.** If $f \in K(I;R)$ then $f \in E(\emptyset)$.

**Remark 0.2.** It is clear that if $f(t,x) = f_0(t)g_0(x)$, where $f_0 \in L(I;R)$ and $g_0 \in C(I;R)$, then $f \in E(A)$ for every finite set $A \subset I$.

The example below shows that there exists a function $f \in K(I;R)$ such that $f \notin E(\{t_1, ..., t_k\})$ for some points $t_1, ..., t_k \in I$.

**Example 0.1.** Let $f(t,x) = |t|^{-1/2} g(t,x)$ for $t \in [0,1]$, and $f(0,.) \equiv 0$, where $g(-t,x) = g(t,x)$ for $t \in [-1,1]$ and

$$g(t,x) = \begin{cases} x & \text{for} \quad x \leq 1/t,\ t > 0 \\ 1/t & \text{for} \quad x > 1/t,\ t > 0 \end{cases} .$$

Then $f \in K([-1,1];R)$ and it is clear that $f \notin E(\{0\})$ because, for every $\varepsilon > 0$, if $x \geq 1/\varepsilon$ then $\int_\varepsilon^1 f(s,x)ds - \int_0^\varepsilon f(s,x)ds = 4(\varepsilon^{-1/2} - x^{1/2}) - 2 < 0$.

## 1. Main Results

**Theorem 1.1.** *Let $w$ be an arbitrary nonzero solution of the problem* (0.3), (0.4),

$$N_w = \emptyset, \tag{1.1}$$

*there exist the constant $r > 0$, the functions $f^-, f^+ \in L(I;R_+)$ and $g, h_0 \in L(I;\ ]0,+\infty[\ )$ such that*

$$f(t,x)\text{sgn}x \leq g(t)|x| + h_0(t) \quad \text{for} \quad t \in I,\ |x| \geq r, \tag{1.2}$$

*and*

$$\begin{aligned} f(t,x) &\leq -f^-(t) \quad \text{for} \quad x \leq -r, \\ f^+(t) &\leq f(t,x) \quad \text{for} \quad x \geq r, \end{aligned} \tag{1.3}$$

*on $I$. Let, moreover, there exist $\varepsilon > 0$ such that*

$$-\int_a^b f^-(s)|w(s)|ds + \varepsilon||\gamma_r||_L \leq -\int_a^b h(s)|w(s)|ds \leq$$

$$\leq \int_a^b f^+(s)|w(s)|ds - \varepsilon||\gamma_r||_L, \tag{1.4$_1$}$$

*where*

$$\gamma_r(t) = \sup\{|f(t,x)| : |x| \leq r\}. \tag{1.5}$$

*Then the problem* (0.1), (0.2) *has at least one solution.*

**Example 1.1.** It follows from Theorem 1.1 that the equation

$$u''(t) = -\lambda^2 u(t) + \sigma|u(t)|^\alpha \text{sgn} u(t) + h(t) \quad \text{for} \quad 0 \le t \le \pi \tag{1.6}$$

where $\sigma = 1$, $\lambda = 1$, and $\alpha \in ]0,1]$, under the conditions (0.6) has at least one solution for every $h \in L([0,\pi], R)$.

**Theorem 1.2.** *Let $w$ be an arbitrary nonzero solution of the problem* (0.3), (0.4)*, condition* (1.1) *holds, there exist the constant $r > 0$, the functions $f^-, f^+ \in L(I; R_+)$ and $q \in K(I; R_+)$ such that $q$ is non-decreasing in the second argument,*

$$|f(t,x)| \le q(t,x) \quad for \quad t \in I, \ |x| \ge r, \tag{1.7}$$

$$\begin{aligned} f^-(t) \le f(t,x) \quad for \quad x \le -r, \\ f(t,x) \le -f^+(t) \quad for \quad x \ge r, \end{aligned} \tag{1.8}$$

*on $I$, and*

$$\lim_{|x| \to +\infty} \frac{1}{x} \int_a^b q(s,x) ds = 0. \tag{1.9}$$

*Let, moreover, there exist $\varepsilon > 0$ such that*

$$\begin{aligned} -\int_a^b f^-(s)|w(s)|ds + \varepsilon||\gamma_r||_L \le \int_a^b h(s)|w(s)|ds \le \\ \le \int_a^b f^+(s)|w(s)|ds - \varepsilon||\gamma_r||_L, \end{aligned} \tag{1.4$_2$}$$

*where $\gamma_r$ is defined by* (1.5)*. Then the problem* (0.1), (0.2) *has at least one solution.*

**Example 1.2.** From Theorem 1.2 it follows that the problem (1.6), (0.6) with $\sigma = -1$, $\lambda = 1$, and $\alpha \in ]0,1[$ has at least one solution for any $h \in L([0,\pi]; R)$.

**Remark 1.1.** In the Theorem 1.$i$ ($i = 1,2$), the condition (1.4$_i$) can be replaced by

$$\begin{aligned} -\int_a^b f^-(s)|w(s)|ds < (-1)^i \int_a^b h(s)|w(s)|ds < \\ < \int_a^b f^+(s)|w(s)|ds, \end{aligned} \tag{1.10$_i$}$$

because, from (1.10$_i$) there follows the existence of a constant $\varepsilon > 0$ such that the condition (1.4$_i$) is satisfied.

**Theorem 1.3.** *Let $i \in \{0,1\}$, $w$ be an arbitrary nonzero solution of the problem* (0.3), (0.4)*, $f \in E(N_w)$, there exist the constant $r > 0$ such that the function $(-1)^i f$ is non-decreasing in the second argument for $|x| \ge r$,*

$$(-1)^i f(t,x) \text{sgn} x \ge 0 \quad for \quad t \in I, \ |x| \ge r, \tag{1.11}$$

$$\int_{\Omega_w^+} |f(s,r)|ds + \int_{\Omega_w^-} |f(s,-r)|ds \neq 0, \tag{1.12}$$

*and*

$$\lim_{|x|\to+\infty} \frac{1}{|x|} \int_a^b |f(s,x)|ds = 0. \tag{1.13}$$

*Then there exists* $\delta > 0$ *such that the problem* (0.1), (0.2) *has at least one solution for every* $h$ *satisfying the condition*

$$\left| \int_a^b h(s)w(s)ds \right| < \delta. \tag{1.14}$$

**Corollary 1.1.** *Let the assumptions of Theorem1.3 be satisfied and let*

$$\int_a^b h(s)w(s)ds = 0. \tag{1.15}$$

*Then the problem* (0.1),(0.2) *has at least one solution.*

**Example 1.3.** From Theorem 1.3 it follows that the problem (1.6), (0.6) with $\sigma \in \{-1, 1\}$, $\lambda \in N$, and $\alpha \in ]0, 1[$ has at least one solution if $h \in L([0, \pi], R)$ is such that $\int_0^\pi h(s)\mathrm{sin}\lambda s ds = 0$.

**Theorem 1.4.** *Let* $i \in \{0, 1\}$, $w$ *be an arbitrary nonzero solution of the problem* (0.3),(0.4), $f(t, x) = f_0(t)g_0(x)$ *with* $f_0 \in L(I; R_+)$, $g_0 \in C(R; R)$, *there exist the constant* $r > 0$ *such that* $(-1)^i g_0$ *is nondecreasing for* $|x| \geq r$, *and*

$$(-1)^i g_0(x)\mathrm{sgn}x \geq 0 \qquad \textit{for} \quad |x| \geq r. \tag{1.16}$$

*Let, moreover,*

$$|g_0(r)| \int_{\Omega_w^+} f_0(s)ds + |g_0(-r)| \int_{\Omega_w^-} f_0(s)ds \neq 0 \tag{1.17}$$

*and*

$$\lim_{|x|\to+\infty} |g_0(x)| = +\infty, \qquad \lim_{|x|\to+\infty} \frac{g_0(x)}{x} = 0. \tag{1.18}$$

*Then, for every* $h \in L(I; R)$, *the problem* (0.1), (0.2) *has at least one solution.*

**Example 1.4.** From the Theorem 1.4 it follows that the equation

$$u''(t) = p_0(t)u(t) + p_1(t)|u(t)|^\alpha \mathrm{sgn}u(t) + h(t) \quad \text{for} \quad t \in I, \tag{1.19}$$

where $\alpha \in ]0, 1[$ and $p_0, p_1, h \in L(I; R)$, under the conditions (0.2) has at least one solution provided that $p_1(t) > 0$ for $t \in I$.

**Theorem 1.5.** *Let* $i \in \{0, 1\}$ *and* $w$ *be an arbitrary nonzero solution of the problem* (0.3),(0.4). *Let, moreover, there exist the constants* $r > 0$, $\varepsilon_0 > 0$, *and the functions* $\alpha, f^+, f^- \in L(I; R_+)$ *such that the conditions*

$$\begin{aligned} (-1)^i f(t,x) &\leq -f^-(t) \quad \textit{for} \quad x \leq -r, \\ f^+(t) &\leq (-1)^i f(t,x) \quad \textit{for} \quad x \geq r, \end{aligned} \tag{1.20$_i$}$$

$$\sup\{|f(t,x)| : x \in R\} = \alpha(t) \tag{1.21}$$

*hold on $I$, and let*

$$-\int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds + \varepsilon||\alpha||_L \leq$$
$$\leq (-1)^{i+1}\int_a^b h(s)w(s)ds \leq \tag{$1.22_i$}$$
$$\leq \int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds - \varepsilon||\alpha||_L.$$

*Then the problem* (0.1), (0.2) *has at least one solution.*

**Remark 1.2.** If $f \not\equiv 0$ then the condition ($1.22_i$) ($i = 1, 2$) of Theorem 1.5 can be replaced by

$$-\int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds <$$
$$< (-1)^{i+1}\int_a^b h(s)w(s)ds < \tag{$1.23_i$}$$
$$< \int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds.$$

because from ($1.23_i$) there follows the existence of a constant $\varepsilon > 0$ such that the condition ($1.22_i$) is satisfied.

**Example 1.5.** From Theorem 1.5 it follows that the equation

$$u''(t) = -\lambda^2 u(t) + \frac{|u(t)|^\alpha}{1 + |u(t)|^\alpha}\mathrm{sgn}u(t) + h(t) \quad \text{for} \quad 0 \leq t \leq \pi, \tag{1.24}$$

where $\lambda \in N$ and $\alpha \in ]0, +\infty[$ , under the conditions (0.6) has at least one solution if $h \in L([0, \pi], R)$ is such that $|h(t)| < 1$ for $0 \leq t \leq \pi$.

## 2. Problem (0.5), (0.6).

Throughout this section we will assume that $a = 0$, $b = \pi$, and $I = [0, \pi]$. In view of the fact that the functions $\pm\sin\lambda t$ are the solutions of the problem (0.7), (0.6), from Theorems 1.1–1.5 the following corollaries are true

**Corollary 2.1.** *Let $\lambda = 1$ and all the assumptions of Theorem 1.1 (resp. Theorem 1.2) except* (1.1) *be fulfilled with $w(t) = \sin t$. Then the problem* (0.5), (0.6) *has at least one solution.*

Now, note that

$$N_{\sin\lambda t} = \begin{cases} \emptyset & \text{for} \quad \lambda = 1 \\ \{\pi n/\lambda : n = 1, ..., \lambda - 1\} & \text{for} \quad \lambda \geq 2 \end{cases}.$$

**Corollary 2.2.** *Let $i \in \{0,1\}$, $\lambda \in N$, $f \in E(N_{\sin\lambda t})$, there exist the constant $r > 0$ such that the function $(-1)^i f$ is non-decreasing in the second argument for $|x| \geq r$, and let the conditions* (1.11)–(1.13) *be fulfilled with $w(t) = \sin\lambda t$. Then there exists $\delta > 0$ such that the problem* (0.5), (0.6) *has at least one solution for every $h \in L(I;R)$ satisfying the condition $|\int_0^\pi h(s)\sin\lambda s ds| < \delta$.*

**Corollary 2.3.** *Let $i \in \{0,1\}$, $\lambda \in N$, and let all the assumptions of Theorem 1.4 be fulfilled with $w(t) = \sin\lambda t$. Then, for any $h \in L(I;R)$, the problem* (0.5), (0.6) *has at least one solution.*

**Corollary 2.4.** *Let $i \in \{0,1\}$, $\lambda \in N$ and let there exist the constant $r > 0$ such that* $(1.20_i)$–$(1.22_i)$ *be fulfilled with $w(t) = \sin\lambda t$. Then the problem* (0.5), (0.6) *has at least one solution.*

**Remark 2.1.** In the Corollary 2.1 (resp. Corollary 2.4), the condition $(1.4_i)$ (resp. $(1.22_i)$) can be changed by the condition $(1.10_i)$ (resp. $(1.23_i)$) with $w(t) = \sin t$ (resp. $w(t) = \sin\lambda t$).

## 3. Auxiliary Propositions

Let $u_n \in \widetilde{C}'(I;R)$, $\|u_n\|_C \neq 0$ $(n \in N)$, $w$ be an arbitrary solution of the problem (0.3), (0.4), and $r > 0$. Then, for every $n \in N$, we define: $A_{n,1} \stackrel{def}{=} \{t \in I : |u_n(t)| \leq r\}$, $\quad A_{n,2} \stackrel{def}{=} \{t \in I : |u_n(t)| > r\}$,

$$B_{n,i} \stackrel{def}{=} \{t \in A_{n,2} : \mathrm{sgn} u_n(t) = (-1)^{i-1}\mathrm{sgn} w(t)\} \ (i = 1,2),$$

$$C_{n,1} \stackrel{def}{=} \{t \in A_{n,2} : |w(t)| \geq 1/n\}, \qquad C_{n,2} \stackrel{def}{=} \{t \in A_{n,2} : |w(t)| < 1/n\},$$

$$D_n \stackrel{def}{=} \{t \in I : |w(t)| > r||u_n||_C^{-1} + 1/2n\},$$

$$A^{\pm}_{n,2} \stackrel{def}{=} \{t \in A_{n,2} : \pm u_n(t) > r\},\ B^{\pm}_{n,i} \stackrel{def}{=} A^{\pm}_{n,2} \cap B_{n,i},$$

$$C^{\pm}_{n,i} \stackrel{def}{=} A^{\pm}_{n,2} \cap C_{n,i}\ (i = 1,2),\ D^{\pm}_n \stackrel{def}{=} \{t \in I : \pm w(t) > r||u_n||_C^{-1} + 1/2n\},$$

From these definitions it is clear that, for any $n \in N$, we have

$$A_{n,1} \cap A_{n,2} = \emptyset, A^+_{n,2} \cap A^-_{n,2} = \emptyset,\ B_{n,1} \cap B_{n,2} = \emptyset,\ C_{n,1} \cap C_{n,2} = \emptyset,$$

$$D^+_n \cap D^-_n = \emptyset,\ B^+_{n,2} \cap B^-_{n,2} = \emptyset,\ C^+_{n,i} \cap C^-_{n,i} = \emptyset\ (i = 1,2), \tag{3.1}$$

$$\begin{gathered} A_{n,1} \cup A_{n,2} = I,\ A^+_{n,2} \cup A^-_{n,2} = A_{n,2},\ B_{n,1} \cup B_{n,2} = A_{n,2} \setminus N_w, \\ C_{n,1} \cup C_{n,2} = A_{n,2}, B^+_{n,2} \cup B^-_{n,2} = B_{n,2},\ C^{\pm}_{n,1} \cup C^{\pm}_{n,2} = A^{\pm}_{n,2}, \\ C^+_{n,i} \cup C^-_{n,i} = C_{n,i}\ (i = 1,2),\ D^+_n \cup D^-_n = D_n. \end{gathered} \tag{3.2}$$

**Lemma 3.1.** *Let $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$, $r > 0$, $w$ be an arbitrary solution of the problem* (0.3), (0.4) *and*

$$||u_n||_C \geq 2rn \qquad \textit{for} \qquad n \in N, \tag{3.3}$$

$$||v_n - w||_C \leq 1/2n \quad \textit{for} \quad n \in N, \tag{3.4}$$

*where* $v_n(t) = u_n(t)||u_n||_C^{-1}$. *Then there exists* $n_0 \in N$ *such that*

$$D_{n_0}^+ \subset A_{n,2}^+, \quad D_{n_0}^- \subset A_{n,2}^- \quad for \quad n \geq n_0, \tag{3.5}$$

$$C_{n_0,1}^+ \subset D_n^+ \quad C_{n_0,1}^- \subset D_n^- \quad for \quad n \geq n_0. \tag{3.6}$$

*Moreover*

$$\lim_{n\to+\infty} \text{mes} A_{n,1} = 0, \ \lim_{n\to+\infty} \text{mes} A_{n,2} = \text{mes} I, \tag{3.7}$$

$$C_{n,1} \subset B_{n,1}, \qquad B_{n,2} \subset C_{n,2}, \tag{3.8}$$

$$B_{n,2}^+ \subset C_{n,2}^+, \qquad B_{n,2}^- \subset C_{n,2}^-, \tag{3.9}$$

$$C_{n,1}^+ \subset B_{n,1}^+, \qquad C_{n,1}^- \subset B_{n,1}^-, \tag{3.10}$$

$$\begin{gathered} \lim_{n\to+\infty} \text{mes} C_{n,1} = \lim_{n\to+\infty} \text{mes} B_{n,1} = \text{mes} I, \\ \lim_{n\to+\infty} \text{mes} C_{n,2} = \lim_{n\to+\infty} \text{mes} B_{n,2} = 0, \end{gathered} \tag{3.11}$$

$$|v_n(t)| \leq 1/2n \qquad for \qquad t \in B_{n,2}, \tag{3.12}$$

$$|v_n(t)| \geq 1/2n \qquad for \qquad t \in C_{n,1}, \tag{3.13}$$

$$\lim_{n\to+\infty} \text{mes}\Big(C_{n,1}^{\pm} \cap \Omega_w^{\pm}\Big) = \text{mes}\Omega_w^{\pm}. \tag{3.14}$$

*Proof.* From the unique solvability of Caushy's problem for the equation (0.3) it follows that the set $N_w$ is finite. Consequently we can assume that $N_w = \{t_1, ..., t_k\}$. Let also $t_0 = a$, $t_{k+1} = b$ and $T_n \overset{def}{=} I \cap \Big( \cup_{i=0}^{k+1} [t_i - 1/n, \ t_i + 1/n]\Big)$. We first show that, for every $n_0 \in N$, there exists $n_1 > n_0$ such that

$$A_{n,1} \subseteq T_{n_0} \quad \text{for} \quad n \geq n_1. \tag{3.15}$$

Suppose on the contrary that, for some $n_0 \in N$, there exists the sequence $t'_{n_j} \in A_{n_j,1}$ $(j \in N)$ with $n_j < n_{j+1}$, such that $t'_{n_j} \notin T_{n_0}$ for $j \in N$. Without loss of generality we can assume that $\lim_{j\to+\infty} t'_{n_j} = t'_0$. Then from the conditions (3.3),(3.4), the definition of the set $A_{n,1}$ and the equality $w(t'_0) = (w(t'_0) - w(t'_{n_j})) + (w(t'_{n_j}) - v_{n_j}(t'_{n_j})) + v_{n_j}(t'_{n_j})$, we get $|w(t'_0)| = 0$, i.e., $t'_0 \in \{t_0, t_1, ..., t_{k+1}\}$. But this contradicts the condition $t'_{n_j} \notin T_{n_0}$ and thus (3.15) is true. Since $\lim\limits_{n\to+\infty} \text{mes} T_n = 0$, it follows from (3.2) and (3.15) that (3.7) is valid.

Let $t_0 \in D_{n_0}^+$. Then from (3.4) it follows that $\frac{u_n(t_0)}{||u_n||_C} \geq w(t_0) - |v_n(t_0) - w(t_0)| > \frac{r}{||u_{n_0}||_C} + \frac{1}{2n_0} - \frac{1}{2n} \geq \frac{r}{||u_{n_0}||_C}$ for $n \geq n_0$, and thus $t_0 \in A_{n,2}^+$ for $n \geq n_0$, i.e., $D_{n_0}^+ \subset A_{n,2}^+$ for $n \geq n_0$. The second relation of (3.5) can be proved analogously. Now suppose that $t_0 \in C_{n,1}$ and $t_0 \notin B_{n,1}$. Then, in view of (3.1) and (3.2), it is clear that $t_0 \in B_{n,2}$, and thus

$$|v_n(t_0) - w(t_0)| = |v_n(t_0)| + |w(t_0)| > 1/n \ , \tag{3.16}$$

which contradicts (3.4). Consequently, $C_{n,1} \subset B_{n,1}$ for $n \in N$. This, together with the relations $C_{n,2} = A_{n,2} \backslash C_{n,1}$, $B_{n,2} \subseteq A_{n,2} \backslash B_{n,1}$, implies $B_{n,2} \subset C_{n,2}$, i.e., (3.8) holds. The conditions (3.9) and (3.10) follow immediately from (3.8). In view of the fact that $\lim_{n\to+\infty} \mathrm{mes} C_{n,i} = (2-i)\mathrm{mes} I$, from (3.8) we gat (3.11). Now, let $t_0 \in B_{n,2}$ and suppose that $|v_n(t_0)| > 1/2n$. Then from (3.4) we obtain the following contradiction $1/2n \geq |v_n(t_0) - w(t_0)| = |v_n(t_0)| + |w(t_0)| > 1/2n + |w(t_0)|$, i.e., (3.12) holds. From (3.4) and the definition of the set $C_{n,1}$ we obtain (3.13). Now we will show that

$$C^{\pm}_{n,1} = \{t \in A_{n,2} : \pm w(t) \geq 1/n\} \quad \text{for} \quad n \in N. \tag{3.17}$$

Let there exists $t_0 \in C^+_{n,1}$ such that $t_0 \notin \{t \in A_{n,2} : w(t) \geq 1/n\}$. Then from the definition of the sets $C_{n,1}$ and $C^+_{n,1}$ we get that $w(t) \leq -1/n$ and $t_0 \in A^+_{n,2}$. In this case the inequality (3.16) is fullfild, which contradict (3.4). Therefore $C^+_{n,1} \subset \{t \in A_{n,2} : w(t) \geq 1/n\}$. Let now $t_0 \in \{t \in A_{n,2} : w(t) \geq 1/n\}$ and $t_0 \notin C^+_{n,1}$. Then from the definition of the set $C_{n,1}$ and (3.2) it is clear that $t_0 \in C^-_{n,1}$, i.e. $t_0 \in A^-_{n,2}$, and that the inequality (3.16) is fullfild, which contradicts (3.4).Therefore $\{t \in A_{n,2} : w(t) \geq 1/n\} \subset C^+_{n,1}$. From the last two inclusions it follows that (3.17) is valid for $C^+_{n,1}$. From (3.2) and (3.17) for $C^+_{n,1}$ it is clear that (3.17) is true for $C^-_{n,1}$ too. From (3.17), the definition of the sets $D^{\pm}_n$ and (3.3) we obtain (3.6). From the definition of the set $\Omega^{\pm}_w$ and (3.17) we get $C^{\pm}_{n,1} \cap \Omega^{\pm}_w = \{t \in I : \pm w(t) \geq 1/n\} \setminus (I \setminus A_{n,2})$ and then

$$\mathrm{mes}(C^{\pm}_{n,1} \cap \Omega^{\pm}_w) \geq \mathrm{mes}(\{t \in I : \pm w(t) \geq 1/n\}) - \mathrm{mes}(I \setminus A_{n,2})$$

where in view of (3.7) the equality $\lim\limits_{n\to+\infty} \mathrm{mes}(I \setminus A_{n,2}) = 0$ holds. From the last two relation and the fact that $C^{\pm}_{n,1} \cap \Omega^{\pm}_w \subset \Omega^{\pm}_w$, we obtain (3.14). □

**Lemma 3.2.** *Let $i \in \{1,2\}$, $r > 0$, $k \in N$, $w$ be an arbitrary solution of the problem* (0.3), (0.4), *$N_w = \{t_1, ..., t_k\}$, the function $f_1 \in E(N_w)$ be non-decreasing in second argument for $|x| \geq r$, and let the conditions* (3.3) *and*

$$f_1(t,x)\mathrm{sgn} x \geq 0 \qquad \text{for} \quad t \in I,\ |x| \geq r, \tag{3.18}$$

*hold. Then:*

*a)If $G \subset I$ and*

$$\int_G |f_1(s,(-1)^i r)w(s)|ds \neq 0, \tag{3.19}$$

*then there exist $\delta_0 > 0$ and $\varepsilon_1 > 0$ such that*

$$\mathbb{I}(G, U_\varepsilon, x) \stackrel{def}{\equiv} \int\limits_{G\setminus U_\varepsilon} |f_1(s,x)w(s)|ds - \int\limits_{U_\varepsilon} |f_1(s,x)w(s)|ds > \delta_0, \tag{3.20}$$

*for $(-1)^i x \geq r$, $0 < \varepsilon \leq \varepsilon_1$ where $U_\varepsilon = I \cap \Big(\cup^k_{j=1}[t_j - \varepsilon/2k,\ t_j + \varepsilon/2k]\Big)$.*

*b)For any $r > 0$ and $\delta_1 > 0$ there exist $\varepsilon_2 > 0$ and $n_0 \in N$ such that*

$$\mathbb{I}(D^+_n, U^+_\varepsilon, x) \geq -\delta_1 \quad \text{for} \quad x \geq r, \tag{3.21$_1$}$$

$$\mathbb{I}(D_n^-, U_\varepsilon^-, x) \geq -\delta_1 \quad for \quad x \leq -r, \tag{3.21$_2$}$$

*for* $n \geq n_0$ *and* $0 < \varepsilon \leq \varepsilon_2$, *where* $U_\varepsilon^\pm = \{t \in U_\varepsilon : \pm w(t) \geq 0\}$.

*Proof.* a)For any $\alpha \in R_+$, we put $G_1 = ([a, a+\alpha] \cup [b-\alpha, b]) \cap G$. In view of the condition (3.19) we can choose $\alpha \in ]0, (b-a)/2[$ such that if $G_2 = G \setminus G_1$, $t_a = \inf\{G_2\}$ and $t_b = \sup\{G_2\}$, then

$$a < t_a,\ t_b < b, \tag{3.22}$$

and $\int_{G_1} |f(s, (-1)^i r) w(s)| ds \neq 0$, $\int_{G_2} |f(s, (-1)^i r)| ds \neq 0$. From these inequalities, by the conditions (3.18) and $f_1 \in E(N_w)$ where $f_1$ is non-decreasing in the second argument, there follows the existence of $\delta_0 > 0$ and $\varepsilon^* > 0$ such that

$$\int_{G_2 \setminus U_{\varepsilon^*}} |f_1(s,x)| ds - \int_{U_{\varepsilon^*}} |f_1(s,x)| ds \geq 0 \quad \text{for} \quad (-1)^i x \geq r, \tag{3.23}$$

$$\int_{G_1 \setminus U_{\varepsilon^*}} |f_1(s,x) w(s)| ds > \delta_0 \qquad \text{for} \quad (-1)^i x \geq r. \tag{3.24}$$

Now we put $I^* = [t_a^*,\ t_b^*]$, where $t_a^* = \frac{a + \min(t_a, t_1)}{2}$ and $t_b^* = \frac{\max(t_k, t_b) + b}{2}$. In view of (3.22), we obtain

$$G_2 \subset I^*,\ N_w \subset I^*,\ w(t_a^*) \neq 0,\ w(t_b^*) \neq 0. \tag{3.25}$$

Then it is clear that there exists $\gamma_1 > 0$ such that for any $\gamma \in ]0, \gamma_1[$ the equation $|w(t)| = \gamma$, on the set $I^*$, has only $t_{\gamma,i}, t_{\gamma,i}^*$ $(i = 1, ..., k)$ solutions and

$$t_{\gamma,i} < t_i < t_{\gamma,i}^* \ (i = 1, ..., k), \tag{3.26}$$

$$|w(t)| \leq \gamma \quad \text{for} \quad t \in H_\gamma, \qquad |w(t)| > \gamma \quad \text{for} \quad t \in I^* \setminus H_\gamma, \tag{3.27}$$

where $H_\gamma = \cup_{i=1}^k [t_{\gamma,i}, t_{\gamma,i}^*]$, and

$$\lim_{\gamma \to +0} t_{\gamma,i} = \lim_{\gamma \to +0} t_{\gamma,i}^* = t_i \ \ (i = 1, ..., k). \tag{3.28}$$

The relations (3.26) and (3.28) imply that there exist $\gamma \in ]0, \gamma_1]$ and $\varepsilon_1 \in ]0, \varepsilon^*]$ such that

$$U_{\varepsilon_1} \subset H_\gamma \subset U_{\varepsilon^*}. \tag{3.29}$$

Moreover, from the inclusion $G_1 \subset G$ it is clear that

$$G \setminus U_{\varepsilon_1} = \big[\big(G \setminus G_1\big) \setminus U_{\varepsilon_1}\big] \cup \big(G_1 \setminus U_{\varepsilon_1}\big),\ \big[\big(G \setminus G_1\big) \setminus U_{\varepsilon_1}\big] \cap \big(G_1 \setminus U_{\varepsilon_1}\big) = \emptyset,$$

and thus

$$\mathbb{I}(G, U_{\varepsilon_1}, x) = \int_{G_1 \setminus U_{\varepsilon_1}} |f_1(s,x) w(s)| ds + \mathbb{I}(G_2, U_{\varepsilon_1}, x) \quad \text{for} \quad (-1)^i x \geq r.$$

By virtue of (3.23), (3.25), (3.27), and (3.29), we get

$$\mathbb{I}(G_2, U_{\varepsilon_1}, x) \geq \gamma \Big( \int_{G_2 \setminus H_\gamma} |f_1(s,x)| ds - \int_{H_\gamma} |f_1(s,x)| ds \Big) \geq$$

$$\geq \gamma \Big( \int_{G_2 \setminus U_{\varepsilon^*}} |f_1(s,x)| ds - \int_{U_{\varepsilon^*}} |f_1(s,x)| ds \Big) \geq 0$$

for $(-1)^i x \geq r$. In view of the last two relations, (3.24), and the fact that $U_{\varepsilon_1} \subset U_{\varepsilon^*}$, we conclude that the inequality (3.20) holds.

b) First consider the case when

$$\int_{D_n^+} |f_1(s,x)w(s)|ds = 0 \ \text{ for } \ x \geq r, \ n \in N. \qquad (3.30)$$

By (3.3) and the definitions of the sets $D_n^\pm$ and $U_\varepsilon^\pm$ we get

$$\lim_{n\to+\infty} \text{mes}(U_\varepsilon^\pm \setminus D_n^\pm) = 0. \qquad (3.31)$$

Then in view of (3.30) and the fact that for any $\varepsilon > 0$ and $n \in N$

$$U_\varepsilon^\pm = (U_\varepsilon^\pm \cap D_n^\pm) \cup (U_\varepsilon^\pm \setminus D_n^\pm), \quad (U_\varepsilon^\pm \cap D_n^\pm) \cap (U_\varepsilon^\pm \setminus D_n^\pm) = \emptyset, \ (3.32)$$

we have $\int_{U_\varepsilon^+} |f_1(s,x)w(s)|ds = \int_{U_\varepsilon^+ \setminus D_n^+} |f_1(s,x)w(s)|ds$ for $x \geq r$, $n \in N$, and $\varepsilon > 0$. Thus in view of (3.31) we get $\int_{U_\varepsilon^+} |f_1(s,x)w(s)|ds = 0$. From the last equality and (3.30) we conclude that

$$I(D_n^+, U_\varepsilon^+, x) = 0 \quad \text{for} \quad x \geq r, \ n \in N, \ \varepsilon > 0. \qquad (3.33)$$

Therefore in this case $(3.21_1)$ is true.

Now, consider the case when for some $r_1 \geq r$ there exists $n_0 \in N$ such that

$$\int_{D_n^+} |f_1(s,x)w(s)|ds \neq 0 \ \text{ for } \ x \geq r_1, \ n \geq n_0. \qquad (3.34)$$

It is clear that there exist $\eta > 0$ and $\varepsilon_2 > 0$ such that

$$\int_{U_\varepsilon^+} |f_1(s,x)w(s)|ds \leq \delta_1 \ \text{ for } \ r \leq x \leq r_1 + \eta, \ \varepsilon \leq \varepsilon_2,$$

and then

$$I(D_n^+, U_\varepsilon^+, x) \geq -\delta_1 \quad \text{for} \quad r \leq x \leq r_1 + \eta, \ n \geq n_0, \ \varepsilon \leq \varepsilon_2. \quad (3.35)$$

On the other hand, from the fact that $f_1$ is non-decreasing in the second argument (3.18) and (3.34) it is clear that $\int_{D_{n_0}^+} |f_1(s, r_1 + \eta)w(s)|ds \neq 0$. Therefore from item a) of our lemma with $G = D_n^+$, and the inclusions $D_{n_0}^+ \subset D_n^+$, $U_\varepsilon^+ \subset U_\varepsilon$ for $n \geq n_0$, $\varepsilon > 0$ we get $I(D_n^+, U_\varepsilon^+, x) \geq \delta_0$ for $x \geq r_1 + \eta$, $n \geq n_0$, $\varepsilon \leq \varepsilon_2$. From this inequality and (3.35) we obtain $(3.21_1)$ in second case too. Analogously one can prove $(3.21_2)$. □

**Lemma 3.3.** *Let all the conditions of Lemma3.1 be fulfilled and there exist $r > 0$ such that the condition* (3.18) *holds where $f_1 \in K(I; R)$. Then*

$$\lim_{n\to+\infty} \inf \int_s^t f_1(\xi, u_n(\xi)) \text{sgn} u_n(\xi) d\xi \geq 0 \quad \textit{for} \quad a \leq s < t \leq b. \quad (3.36)$$

*Proof.* Let

$$\gamma_r^*(t) \overset{def}{=} \sup\{|f_1(t,x)| : |x| \le r\} \quad \text{for} \quad t \in I. \tag{3.37}$$

Then, according to (3.1), (3.2), and (3.18), we obtain the estimate

$$\int_s^t f_1(\xi, u_n(\xi))\mathrm{sgn} u_n(\xi) d\xi \ge$$

$$\ge - \int_{[s,t]\cap A_{n,1}} \gamma_r^*(\xi) d\xi + \int_{[s,t]\cap A_{n,2}} |f_1(\xi, u_n(\xi))| d\xi$$

for $a \le s < t \le b$, $n \in N$. This estimate and (3.7) imply (3.36). □

**Lemma 3.4.** *Let* $r > 0$, $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$, $w$ *be a nonzero solution of the problem* (0.3), (0.4), *the condition* (3.3) *hold and*

$$|v_n^{(i)}(t) - w^{(i)}(t)| \le 1/2n \quad for \quad t \in I, \; n \in N, \; (i = 0, 1) \tag{3.38}$$

*where* $v_n(t) = u_n(t)||u_n||_C^{-1}$ *for* $t \in I$,

$$N_w = \emptyset, \tag{3.39}$$

*and*

$$u_n(a) = 0, \; u_n(b) = 0. \tag{3.40}$$

*Let, moreover,* $f_1 \in K(I;R)$, $h_1 \in L(I;R)$, *there exist the numbers* $\varepsilon > 0$, $n_0 \in N$ *and the functions* $f^+, f^- \in L(I;R_+)$ *such that*

$$\begin{aligned} f_1(t,x) &\le -f^-(t) \quad for \quad x \le -r, \\ f^+(t) &\le f_1(t,x) \quad for \quad x \ge r, \end{aligned} \tag{3.41}$$

*on* $I$, *and*

$$- \int_a^b f^-(s)|w(s)|ds + \varepsilon||\gamma_r^*||_L \le - \int_a^b h_1(s)|w(s)|ds \le$$

$$\le \int_a^b f^+(s)|w(s)|ds - \varepsilon||\gamma_r^*||_L, \tag{3.42}$$

*when* $\gamma^*$ *is defined by* (3.37) *Then there exists* $n_1 \in N$ *such that*

$$\mathbb{M}_n \overset{def}{=} \int_a^b (h_1(s) + f_1(s, u_n(s)))w(s)ds \ge 0 \quad for \quad n \ge n_1. \tag{3.43}$$

*Proof.* It is not difficult to verify that all the assumptions of Lemma3.1 are satisfied. From the unique solvability of Caushy's problem for the equation (0.3) and the conditions (0.4) we conclude that $w'(a) \ne 0$ and $w'(b) \ne 0$. Therefore in view of (3.38)-(3.40) there exists $n_2 \in N$ such that

$$u_n(t)\mathrm{sgn} w(t) > 0 \qquad \text{for} \quad n \ge n_2, \; a < t < b. \tag{3.44}$$

Also, by (3.1) and (3.2) we gat the estimate

$$\mathbb{M}_n \geq -\int_{A_{n,1}} \gamma_r^*(s)|w(s)|ds + \int_a^b h_1(s)w(s)ds + \\ + \int_{A_{n,2}} f_1(s, u_n(s))w(s)ds, \tag{3.45}$$

where $\gamma_r^*$ is given by (3.37). Now, note that $f^- \equiv 0$, $f^+ \equiv 0$ if $f_1(t,x) \equiv 0$. Then by virtue of (3.7), we see that there exist $\varepsilon > 0$ and $n_1 \in N$ $(n_1 \geq n_2)$, such that $\int_a^b f^\pm(s)|w(s)|ds - \frac{\varepsilon}{2}||\gamma_r^*||_L \leq \int_{A_{n,2}} f^\pm(s)|w(s)|ds$ and $\frac{\varepsilon}{2}||\gamma_r^*||_L \geq \int_{A_{n,1}} \gamma_r^*(s)|w(s)|ds$ for $n \geq n_1$. By these inequalities, (3.3), (3.41), and (3.44), from (3.45) we obtain

$$\mathbb{M}_n \geq -\varepsilon||\gamma_r^*||_L + \int_a^b h_1(s)|w(s)|ds + \int_a^b f^+(s)|w(s)|ds$$

if $n \geq n_1$ and $w(t) \geq 0$. Analogously we obtain

$$\mathbb{M}_n \geq -\varepsilon||\gamma_r^*||_L - \int_a^b h_1(s)|w(s)|ds + \int_a^b f^-(s)|w(s)|ds,$$

for $n \geq n_1$ and $w(t) \leq 0$. From the last two estimates in view of (3.42) it follows that (3.43) is valid. □

**Lemma 3.5.** *Let* $r > 0$, $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$, $w$ *be an arbitrary nonzero solution of the problem* (0.3),(0.4), *and the conditions* (3.3), (3.18), (3.38), (3.40) *hold. Let, moreover the function* $f_1 \in E(N_w)$ *be non-decreasing in the second argument for* $|x| \geq r$, *and*

$$\int_{\Omega_w^+} |f_1(s,r)|ds + \int_{\Omega_w^-} |f_1(s,-r)|ds \neq 0. \tag{3.46}$$

*Then there exist* $\delta > 0$ *and* $n_1 \in N$ *such that if*

$$\left| \int_a^b h_1(s)w(s)ds \right| < \delta, \tag{3.47}$$

*the inequality* (3.43) *holds.*

*Proof.* It is not difficult to verify that all the assumption of Lemma 3.1 are satisfied. Then by the definition of the sets $B_{n,1}$, $B_{n,2}$, (3.1), (3.2), and (3.18) we obtain the estimate

$$\int_a^b f_1(s, u_n(s))w(s)ds \geq -\int_{A_{n,1}} \gamma_r^*(s)|w(s)|ds + \widehat{\mathbb{M}}_n \tag{3.48}$$

where

$$\widehat{\mathbb{M}}_n = -\int_{B_{n,2}} |f_1(s, u_n(s))w(s)|ds + \int_{B_{n,1}} |f_1(s, u_n(s))w(s)|ds.$$

On the other hand from unique solvability of Cauchy's problem for the equation (0.3) it is clear that

$$w'(a) \neq 0,\ w'(b) \neq 0,\ w'(t_i) \neq 0 \qquad \text{for } i = 1, ..., k. \tag{3.49}$$

In view of (3.14) and (3.46), there exists $n_2 \geq n_0$ such that

$$\int_{C^+_{n_2,1}} |f_1(s,r)w(s)|ds \neq 0 \tag{$3.50_1$}$$

and/or

$$\int_{C^-_{n_2,1}} |f_1(s,-r)w(s)|ds \neq 0. \tag{$3.50_2$}$$

From $(3.50_1)$ and $(3.50_2)$ in view of (3.6) it follows that

$$\int_{D^+_{n,1}} |f_1(s,r)w(s)|ds \neq 0 \text{ and /or } \int_{D^-_{n,1}} |f_1(s,-r)w(s)|ds \neq 0 \tag{3.51}$$

for $n > n_2$ respectively, i.e., all the assumptions of Lemma 3.2 are satisfied with $G = D^+_n$ and/or $G = D^-_n$. Then there exist $0 < \varepsilon_0 \leq \min\{\varepsilon_1, \varepsilon_2\}$, $n_3 \geq n_2$, and $\delta_0 > 0$ such that

$$\mathbb{I}(D^+_n, U^+_{\varepsilon_0}, x) \geq \delta_0 \quad \text{for} \quad x \geq r,\ n \geq n_3 \tag{$3.52_1$}$$

if $(3.50_1)$ holds,

$$\mathbb{I}(D^-_n, U^-_{\varepsilon_0}, x) \geq \delta_0 \quad \text{for} \quad x \leq -r,\ n \geq n_3, \tag{$3.52_2$}$$

if $(3.50_2)$ holds, and

$$\begin{aligned} &\mathbb{I}(D^+_n, U^+_{\varepsilon_0}, x) \geq -\delta_0/2, \quad \text{for} \quad x \geq r,\ n \geq n_3, \\ &\mathbb{I}(D^-_n, U^-_{\varepsilon_0}, x) \geq -\delta_0/2, \quad \text{for} \quad x \leq -r,\ n \geq n_3. \end{aligned} \tag{3.53}$$

On the other hand the definition of the set $U_\varepsilon$ and (3.17), imply that there exists $n_4 > n_3$, such that

$$C^+_{n,2} \subset U^+_{\varepsilon_0},\ C^-_{n,2} \subset U^-_{\varepsilon_0} \quad \text{for} \quad n \geq n_4. \tag{3.54}$$

By this inclusion, (3.2), and (3.5) we obtain that for $n \geq n_4$

$$C^+_{n,1} = A^+_{n,2} \setminus C^+_{n,2} \supset D^+_{n_0} \setminus U^+_{\varepsilon_0}, C^-_{n,1} = A^-_{n,2} \setminus C^-_{n,2} \supset D^-_{n_0} \setminus U^+_{\varepsilon_0}. \tag{3.55}$$

Now, suppose $N_w \neq \emptyset$, and let there exists $n \geq n_4$ such that

$$B_{n,2} \neq \emptyset. \tag{3.56}$$

Then, by taking into account that $f_1$ is non-decreasing in the second argument for $|x| \geq r$, (3.3), (3.12), (3.18) and the definitions of the sets $B^+_{n,2}, B^-_{n,2}$, we obtain

$$\begin{aligned} &|f_1(t,u_n)| \leq f_1(t, \frac{||u_n||_C}{2n}) = |f_1(t, \frac{||u_n||_C}{2n})| \quad \text{for } t \in B^+_{n,2}, \\ &|f_1(t,u_n)| \leq -f_1(t, -\frac{||u_n||_C}{2n}) = |f_1(t, -\frac{||u_n||_C}{2n})| \text{ for } t \in B^-_{n,2}. \end{aligned} \tag{3.57}$$

Analogously from (3.3), (3.13), (3.18), and the definitions of the sets $C^+_{n,1}, C^-_{n,1}$, we obtain the estimates

$$|f_1(t,u_n(s))| \geq |f_1(t,\frac{||u_n||_C}{2n})| \quad \text{for} \quad t\in C^+_{n,1},$$
$$|f_1(t,u_n(s))| \geq |f_1(t,-\frac{||u_n||_C}{2n})| \quad \text{for} \quad t\in C^-_{n,1}. \tag{3.58}$$

Then from (3.1), (3.2), (3.9), (3.57) and respectively from (3.1), (3.2), (3.10), and (3.58) we have

$$\int_{B_{n,2}} |f_1(s,u_n(s))w(s)|ds \leq$$
$$\leq \int_{C^+_{n,2}} |f_1(s,\frac{||u_n||_C}{2n})w(s)|ds + \int_{C^-_{n,2}} |f_1(s,-\frac{||u_n||_C}{2n})w(s)|ds \tag{3.59}$$

and respectively

$$\int_{B_{n,1}} |f_1(s,u_n)w(s)|ds \geq$$
$$\geq \int_{C^+_{n,1}} |f_1(s,\frac{||u_n||_C}{2n})w(s)|ds + \int_{C^-_{n,1}} |f_1(s,-\frac{||u_n||_C}{2n})w(s)|ds. \tag{3.60}$$

Then if the condition (3.56) holds, from (3.59), (3.60), ($3.52_1$), ($3.52_2$), (3.53), (3.54), and (3.55) we get

$$\widehat{\mathbb{M}}_n \geq \mathbb{I}(D^+_n, U^+_{\varepsilon_0}, \frac{||u_n||_C}{2n}) + \mathbb{I}(D^-_n, U^-_{\varepsilon_0}, -\frac{||u_n||_C}{2n}) \geq \frac{\delta_0}{2}. \tag{3.61}$$

On the other hand, in view of (3.10), (3.18), the definition of the sets $A_{n,2}, B_{n,1}$ and the fact that $f_1$ is non-decreasing in the second argument, we obtain the estimate

$$\int_{B_{n,1}} |f_1(s,u_n(s))w(s)|ds \geq$$
$$\geq \int_{C^+_{n,1}} |f_1(s,r)w(s)|ds + \int_{C^-_{n,1}} |f_1(s,-r)w(s)|ds. \tag{3.62}$$

Now, suppose that there exists $n \geq n_4$ such that

$$B_{n,2} = \emptyset. \tag{3.63}$$

Thus from ($3.50_1$), ($3.50_2$) and (3.62),(3.63) there follows the existence of $\delta^* > 0$ such that $\widehat{\mathbb{M}}_n \geq \delta^*$. From this inequality and (3.61) it follows that in both cases when (3.56) or (3.63) are fulfilled the inequality

$$\widehat{\mathbb{M}}_n \geq \delta \quad \text{for} \quad n \geq n_4 \tag{3.64}$$

holds with $\delta = \min\{\delta_0/2, \delta^*\}$. Then from (3.48) by (3.7) and (3.64), we see that for any $\varepsilon > 0$ there exists $n_1 > n_4$ such that

$$\int_a^b f_1(s,u_n(s))w(s)ds \geq \delta - \varepsilon \quad \text{for} \quad n \geq n_1,$$

and then

$$\mathbb{M}_n \geq \delta + \int_a^b h_1(s)w(s)ds - \varepsilon \quad \text{for} \quad n \geq n_1. \tag{3.65}$$

If $N_w = \emptyset$, then in view of (3.3), (3.38), (3.40) and (3.49), the condition (3.63) holds, i.e., the inequality (3.65) holds too.

Consequently because $\varepsilon > 0$ is arbitrary, from (3.65) and (3.47) the inequality (3.43) follows. □

**Lemma 3.6.** *Let, all the conditions of Lemma 3.5, except* (3.47)*, be satisfied with* $f_1(t,x) = f_0(t)g_1(x)$ *where* $f_0 \in L(I;R_+)$*,* $g_1 \in C(R;R)$

$$\lim_{|x|\to+\infty} |g_1(x)| = +\infty. \tag{3.66}$$

*Then for any function* $h_1 \in L(I;R)$ *the inequality* (3.43) *holds.*

*Proof.* From the conditions of our Lemma it is clear that the relations (3.48)–(3.55),(3.57)– (3.60) and (3.62) with $f_1(t,x) = f_0(t)g_1(x)$ are fulfilled and the function $g_1$ is non-decreasing. Note now that, by the same way as the equality (3.33) in the Lemma 3.2, from the relations (3.31) and (3.32) there follows the existence of $\varepsilon_0$ and $n_0 \in N$ such that

$$\beta^{\pm} \equiv \int_{D_n^{\pm}\setminus U_{\varepsilon_0}^{\pm}} f_0(s)|w(s)|ds - \int_{U_{\varepsilon_0}^{\pm}} f_0(s)|w(s)|ds \geq 0, \tag{3.67}$$

for $n \geq n_0$. Now suppose that the condition $(3.50_1)$ i.e., $(3.52_1)$ holds and first consider the case when $n \geq n_4$ is such that (3.56) is fulfilled. From $(3.52_1)$ it follows that $|g_1(r)| > 0$ and $\beta^+ > 0$. Consequently in view of the fact that $g_1$ is non-decreasing we get $\mathbb{I}(D_n^+, U_{\varepsilon_0}^+, x) \geq |g_1(r)|\beta^+ > 0$ for $x \geq r$. By virtue of this last inequality and (3.67) we see that the inequality (3.61) is true with $\delta = |g_1(r)|\beta^+$, i.e., $\widehat{\mathbb{M}}_n \geq |g_1(r)|\beta^+ > 0$. Consider, now the case when $n \geq n_4$ is such that the condition (3.63) holds. Then by virtue of (3.14) and (3.46) from (3.62) we see that for arbitrary $\varepsilon_1 > 0$ there exists $n_5 \geq n_4$ such that $\widehat{\mathbb{M}}_n \geq |g_1(r)| \int_{\Omega_w^+} f_0(s)ds - \varepsilon_1 > 0$, if $n \geq n_5$. From the last two relation and (3.48) in view of (3.7) it follows that in any case (when (3.56) or (3.63) hold) there exist $\varepsilon_2 > 0$ and $n_1 \geq n_4$ such that $\int_a^b f_1(s,u_n(s))w(s)ds \geq \beta|g_1(r)| - \varepsilon_2 > 0$ for $n \geq n_1$ when $\beta = \min(\beta^+, \int_{\Omega_w^+} f_0(s)ds)$. From (3.66) and the last inequality it is clear that for any function $h_1$ we can choose $r > 0$ such that the inequality (3.43) will be true. Analogously one can proof (3.43) in the case when the inequality $(3.50_2)$ holds. □

**Lemma 3.7.** *Let* $r > 0$*,* $u_n \in \widetilde{C}'(I;R)$ $(n \in N)$*,* $w$ *be an arbitrary nonzero solution of the problem* (0.3),(0.4)*, and the conditions* (3.3)*,* (3.38)*, and* (3.40) *hold. Moreover let there exists* $n_0 \in N$ *and the* *functions* $\alpha, f^-, f^+ \in L(I, R_+)$ *such that the condition* (3.41) *is satisfied,*

$$\sup\{|f_1(t,x)| : x \in R\} = \alpha(t) \quad for \quad t \in I, \tag{3.68}$$

*and*

$$-\int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds + \varepsilon||\alpha||_L \le$$

$$\le -\int_a^b h_1(s)w(s)ds \le$$

$$\le \int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds - \varepsilon||\alpha||_L. \tag{3.69}$$

*Than there exists* $n_1 \in N$ *such that the inequality* (3.43) *holds.*

*Proof.* It is not difficult to verify that all the assumption of Lemma3.1 are satisfied. From (3.1), (3.2), and (3.68) we gat

$$\mathbb{M}_n \ge -\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + \int_{B_{n,1}} f_1(s,u_n)w(s)ds+ \\ +\int_a^b h_1(s)w(s)ds. \tag{3.70}$$

Also, by the definition of the set $B_{n,1}$ we obtain

$$\text{sgn}u_n(t) = \text{sgn}w(t) \quad \text{for} \quad t \in B^+_{n,1} \cup B^-_{n,1}. \tag{3.71}$$

Then, by (3.1), (3.2), (3.10), (3.41), and (3.71) from (3.70) we readily obtain the estimate

$$\mathbb{M}_n \ge -\int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds + \int_{C^+_{n,1}} f^+(s)|w(s)|ds+$$

$$+\int_{C^-_{n,1}} f^-(s)|w(s)|ds + \int_a^b h_1(s)w(s)ds. \tag{3.72}$$

Now, note that $f^- \equiv 0$, $f^+ \equiv 0$ if $f_1(t,x) \equiv 0$. Then by (3.7), (3.11), (3.14), and the inclusions $C^+_{n,1} \subset \Omega^+_w$, $C^-_{n,1} \subset \Omega^-_w$ we see that there exist $\varepsilon > 0$ and $n_1 \in N$ such that

$$\frac{1}{3}\varepsilon||\alpha||_L \ge \int_{A_{n,1}\cup B_{n,2}} \alpha(s)|w(s)|ds \\ \int_{\Omega^\pm_w} f^\pm(s)|w(s)|ds - \frac{1}{3}\varepsilon||\alpha||_L \le \int_{C^\pm_{n,1}} f^\pm(s)|w(s)|ds \tag{3.73}$$

if $n \ge n_1$. Let $w_1$ be an arbitrary solution of the problem (0.3),(0.4). First suppose that $w(t) \equiv w_1(t)$. By virtue of (3.72) and (3.73) we obtain

$$\mathbb{M}_n \ge -\varepsilon||\alpha||_L + \int_{\Omega^+_{w_1}} f^+(s)|w_1(s)|ds+$$

$$+\int_{\Omega^-_{w_1}} f^-(s)|w_1(s)|ds + \int_a^b h_1(s)w_1(s)ds.$$

Analogously, if $w(t) \equiv -w_1(t)$, we obtain

$$\mathbb{M}_n \geq -\varepsilon||\alpha||_L + \int_{\Omega^+_{-w_1}} f^+(s)|w_1(s)|ds +$$

$$+ \int_{\Omega^-_{-w_1}} f^-(s)|w_1(s)|ds - \int_a^b h_1(s)w_1(s)ds.$$

Now, by taking into account the fact that the problem (0.3),(0.4) has only two solutions (different only by sign) and the fact that

$$\int_{\Omega^\pm_{w_1}} l(s)|w_1(s)|ds = \int_a^b l(s)[w_1(s)]_\pm ds,$$

$$\int_{\Omega^\pm_{-w_1}} l(s)|w_1(s)|ds = \int_a^b l(s)[w_1(s)]_\mp ds,$$

for the arbitrary $l \in L(I, R)$, from the last two inequalities and (3.69) we immediately obtain (3.43). □

Now we consider the definitions of the sets $V_{10}((a, b))$ introduced and described in [12] (see [Definition 1.3, p. 2350])

**Definition 3.1.** We shall say that the function $p \in L([a, b])$ belongs to the set $V_{10}((a, b))$, if the initial value problem

$$u''(t) = p^*(t)u(t) \quad \text{for} \quad t \in I, \quad u(a) = 0, \ u'(a) = 1, \qquad (3.74)$$

for any function satisfying the inequality $p(t) \leq p^*(t)$ for $t \in I$ has no zeros in the set $]a, b]$.

**Lemma 3.8.** *Let $i \in \{1, 2\}$, $p \in L(I; R)$, $p_n(t) = p(t) + (-1)^i/n$ and $w_n \in \widetilde{C}'(I; R)$ $(n \in N)$ be a solution of the problem*

$$w''_n(t) = p_n(t)w_n(t) \quad \textit{for} \quad t \in I, \quad w_n(a) = 0, \ w_n(b) = 0. \qquad (3.75_n)$$

*Then:*

*a. There exists $n_0 \in N$ such that the problem $(3.75_n)$ has only a zero solution if $n \geq n_0$.*

*b. If $i = 2$ and $N_w = \emptyset$ where $w$ is the solution of the problem (0.3), (0.4), the inclusion $p_n \in V_{10}((a, b))$ for $n \in N$ holds.*

*Proof.* a. Let $N^*_{w_n}$ be the number of zeroes of the function $w_n$ on $I$. Now, assume to the contrary that there exists the sequence $\{w_n\}^{+\infty}_{n \geq n_0}$ of the nonzero solutions of the problem $(3.75_n)$.

Then if $i = 1$, from the fact that $p_n(t) < p_{n+1}(t)$ by Sturm's comparison theorem we obtain $N^*_{w_{n+1}} < N^*_{w_n}$ for $w_n \not\equiv 0$. From this inequality there follows the existence of $n_1 \in N$ such that $N^*_{w_{n_1}} = 2$, i.e., $N_{w_{n_1}} = \{a, b\}$. Then by Sturm's comparison theorem we see that $w_n \equiv 0$ for $n > n_1$, and this contradicts our assumption.

If $i = 2$, from the fact that $p_{n-1}(t) > p_n(t) > p(t)$ by Sturm's comparison theorem we obtain

$$N^*_{w_{n-1}} < N^*_{w_n} \quad n \in N, \tag{3.76}$$

and if $w$ is the solution of the problem (0.3), (0.4)

$$N^*_{w_n} < N^*_w \quad n \in N. \tag{3.77}$$

On the other hand from (3.76) it follows that there exists $n_1 \in N$ such that $N^*_{w_n} > N^*_w$ for $n > n_1$ and this contradicts (3.77).

b. Let $p_n(t) \leq p^*(t)$ and $u$ be the solution of the problem (3.74). Now, assume to the contrary that there exists $n \in N$ such that $p_n \notin V_{10}([a, b])$. Then there exists $t_0 \in ]a, b]$ such that $u(t_0) = 0$. Then in view of the fact that $p(t) < p^*(t)$ by Sturm's comparison theorem we obtain that $w$, the solution of the problem (0.3), (0.4) has zero in the interval $]a, t_0[$. Which contradicts our assumption that $N_w = \emptyset$. □

## 4. Proof of The Main Results

*Proof of Theorem 1.1.* Let $p_n(t) = p(t) + 1/n$ and for any $n \in N$ consider the problem

$$u''_n(t) = p_n(t)u_n(t) + f(t, u_n(t)) + h(t) \quad \text{for} \quad t \in I, \tag{4.1}$$

$$u_n(a) = 0, \quad u_n(b) = 0. \tag{4.2}$$

In view of the condition (1.1) and Lemma 3.8 the problem ($3.75_n$) has only zero solution for $n \geq n_0$ and the inclusion $p_n \in V_{10}((a, b))$ holds. Also from the conditions (1.3) it follows that $0 \leq f(t, x)\text{sgn}x$ for $t \in I$, $|x| \geq r$. From the last inequality and the inclusion $p_n \in V_{10}((a, b))$, as is well-known (see [12, Theorem 2.2, p.2367]), it follows that the problem (4.1),(4.2) has at last one solution, suppose $u_n$. In view of the condition (1.2) without loss of generality we can assume that there exists $\varepsilon^* > 0$ such that $h_0(t) \geq \varepsilon^*$ on $I$. Then $g(t)|x| + h_0(t) \geq \varepsilon^*$ for $x \in R$, $t \in I$. Consequently it is not difficult to verify that $u_n$ also is the solution of the equation

$$u''_n(t) = (p_n(t) + p_0(t, u_n(t))\text{sgn}u_n(t))u_n(t) + p_1(t, u_n(t)) \tag{4.3}$$

on the set $I$ where $p_0(t, x) = \frac{f(t,x)g(t)}{g(t)|x|+h_0(t)}$, $\quad p_1(t, x) = h(t) + \frac{f(t,x)h_0(t)}{g(t)|x|+h_0(t)}$. Now, assume that

$$\lim_{n\to+\infty} ||u_n||_C = +\infty \tag{4.4}$$

and $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then on $I$, for any $n \in N$

$$v''_n(t) = (p_n(t) + p_0(t, u_n(t))\text{sgn}u_n(t))v_n(t) + \frac{1}{||u_n||_C} p_1(t, u_n(t)), \tag{4.5}$$

$$v_n(a) = 0 \qquad v_n(b) = 0, \tag{4.6}$$

and

$$||v_n||_C = 1. \tag{4.7}$$

In view of the condition (1.2) the functions $p_0, p_1 \in L(I;R)$ are bounded respectively by the functions $g(t)$ and $h(t)+h_0(t)$. Then from (4.5) by virtue of (4.4), (4.6) and (4.7) we see that there exists $r_0>0$ such that $||v_n'||_C \leq r_0$. Consequently in view of (4.7), by Arzela-Ascoli lemma, without loss of generality we can assume that there exists $w_0 \in \widetilde{C}'(I,R)$ such that $\lim_{n\to+\infty} v_n^{(i)}(t) = w_0^{(i)}(t)$ $(i=0,1)$ uniformly on $I$. From the last equality and (4.4) there follows the existence of the increasing sequence $\alpha_k \in N,\ k\in N$ such that $||u_{\alpha_k}||_C \geq 2rk$ and $||v_{\alpha_k}^{(i)} - w_0^{(i)}||_C \leq 1/2k$ for $k \in N$. Without loss of generality we can suppose that $u_n \equiv u_{\alpha_n}$ and $v_n \equiv v_{\alpha_n}$. In this case we see that $u_n$ and $v_n$ are the solutions of the problems (4.1), (4.2) and (4.5), (4.6) respectively with $p_n(t) = p(t) + 1/\alpha_n$ for $t\in I,\ n\in N$, and that the inequalities

$$||u_n||_C \geq 2rn, \quad ||v_n^{(i)} - w_0^{(i)}||_C \leq 1/2n \quad \text{for} \quad n\in N, \tag{4.8}$$

are fullfild. Analogously, because the functions $p_0, p_1 \in L(I;R)$ are bounded in view of (4.4), without loss of generality we can assume that there exists the function $\widetilde{p} \in L(I;R)$ such that

$$\lim_{n\to+\infty} \frac{1}{||u_n||_C^j} \int_a^t p_j(s,u_n(s))\text{sgn}u_n(s)ds = (1-j)\int_a^t \widetilde{p}(s)ds \tag{4.9$_j$}$$

uniformly on $I$ for $(j=0,1)$. By virtue of (4.7)–(4.9$_j$) $(j=0,1)$ from (4.5) we obtain

$$w_0''(t) = (p(t)+\widetilde{p}(t))w_0(t), \tag{4.10}$$

$$w_0(a)=0,\ w_0(b)=0, \tag{4.11}$$

$$||w_0||_C = 1. \tag{4.12}$$

From the conditions (1.3), and (4.8) it is clear that all the assumptions of Lemma 3.3 with $f_1(t,x)=f(t,x)$ are satisfied, and then from (4.9$_j$) $(j=0)$ we obtain $\int_s^t \widetilde{p}(\xi)d\xi \geq 0$ for $a\leq s<t\leq b$, i.e.,

$$\widetilde{p}(t)\geq 0 \quad \text{for} \quad t\in I. \tag{4.13}$$

Now, assume that $\widetilde{p}\not\equiv 0$ and $w$ is a solution of the problem (0.3), (0.4). Then using Sturm's comparison theorem, for the equations (0.3) and (4.10), from (4.13) we see that there exists the point $t_0 \in ]a,b[$ such that $w(t_0)=0$ which contradicts (1.1), i.e., our assumption is invalid and $\widetilde{p}\equiv 0$. Consequently, $w_0$ is a solution of the problem (0.3), (0.4), i.e.,

$$w(t) = w_0(t) \quad \text{for} \quad t\in I. \tag{4.14}$$

Consequently, multiplying the equations (4.1) and (0.3) respectively on $w$ and $-u_n$,and by integrating their sum from $a$ to $b$, in view of the conditions (4.2) and (0.4) we obtain

$$-\frac{1}{\alpha_n}\int_a^b w(s)u_n(s)ds = \int_a^b (h(s)+f(s,u_n(s)))w(s)ds \tag{4.15}$$

for $n \geq n_0$. Then by (4.8) and (4.14) we get

$$\int_a^b (h(s) + f(s, u_n(s)))w(s)ds < 0 \quad \text{for} \quad n \geq n_0. \tag{4.16}$$

On the other hand, in view the conditions (1.1)– ($1.4_1$), (4.2), and (4.8) it is clear that all the assumption of Lemma 3.4 with $f_1(t, x) = f(t, x)$, $h_1(t) = h(t)$ are fulfilled. Then the inequality (3.43) is true, which contradicts (4.16). I.e., assumption (4.4) is invalid and there exists $r_1 > 0$ such that $||u_n||_C \leq r_1$ for $n \in N$. Consequently from (4.1) and (4.2) it is clear that there exists $r_1' > 0$ such that $||u_n'||_C \leq r_1'$ and $|u_n''(t)| \leq \sigma(t)$ for $t \in I$, $n \in N$, where $\sigma(t) = (1 + |p(t)|)r_1 + |h(t)| + \gamma_{r_1}(t)$. Hence, by the Arzela-Ascoli lemma, without loss of generality we can assume that there exists the function $u_0 \in \widetilde{C}'(I; R)$ such that $\lim_{n\to+\infty} u_n^{(i)}(t) = u_0^{(i)}(t)$ $(i = 0, 1)$ uniformly on $I$ and that $u_0$ is the solution of the problem (0.1), (0.2). □

*Proof of Theorem 1.2.* Let $p_n(t) = p(t) - 1/n$ and for any $n \in N$ consider the problems (4.1), (4.2) and ($3.75_n$). In view of the Lemma 3.8 the problem ($3.75_n$) has only zero solution if $n \geq n_0$. Then, as is well-known (see [9, Theorem 1.1, p.345]), from the conditions (1.7), (1.9) it follows that the problem (4.1), (4.2) has at least one solution, suppose $u_n$. Now suppose that (4.4) is fulfilled and $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then the conditions (4.6) and (4.7) are fulfilled,

$$v_n''(t) = p_n(t)v_n(t) + \frac{1}{||u_n||_C}(f(s, u_n(s))) + h(s)). \tag{4.17}$$

Then in view the conditions (1.7) and (1.9), from (4.17) there follows the existence of $r_0 > 0$ such that $||v_n'||_C \leq r_0$. Consequently in view (4.7) by the Arzela-Ascoli lemma, without loss of generality we can assume that there exists the function $w_0 \in \widetilde{C}'(I, R)$ such that $\lim_{n\to+\infty} v_n^{(i)}(t) = w_0^{(i)}(t)$ $(i = 0, 1)$ uniformly on $I$. Now, analogously as in the proof of Theorem 1.1, we can choose the sequence $\{\alpha_k\}_{n=1}^{+\infty}$ from $N$ such that, if we suppose $u_n = u_{\alpha_n}$ then the coditions (4.8) will by true when the functions $u_n$ and $v_n$ are the solutions of the broblems (4.1), (4.2) and (4.17), (4.6) respectively with $p_n(t) = p(t) - 1/\alpha_n$ for $t \in I$, $n \in N$. From (4.17) by virtue of (4.6), (4.8) and (1.9) we obtain (4.14). Consequently, analogously as (4.15) in the proof of the Theorem 1.1 we obtain

$$\frac{1}{\alpha_n}\int_a^b w(s)u_n(s)ds = \int_a^b (h(s) + f(s, u_n(s)))w(s)ds \tag{4.18}$$

for $n \geq n_0$. Now note that in view of the conditions (1.1), (1.8), ($1.4_2$), (4.2), and (4.8), all the assumptions of the Lemma 3.4 with $f_1(t, x) = -f(t, x)$, $h_1(t) = -h(t)$ are satisfied. Hence, analogously as in the proof of the Theorem 1.1, from (4.18) by Lemma 3.4 we see that the problem (0.1), (0.2) has at least one solution. □

*Proof of Theorem 1.3.* Let $p_n(t) = p(t) + (-1)^i/n$ and for any $n \in N$ consider the problems (4.1), (4.2), and ($3.75_n$). In view of the condition (1.13) and the fact that $(-1)^i f(t;x)$ is non-decreasing in the second argument for $|x| \geq r$, we obtain

$$\lim_{n\to+\infty} \frac{1}{||z_n||_C} \int_a^b |f(s, z_n(s))|ds = 0 \tag{4.19}$$

for an arbitrary sequence $z_n \in C(I;R)$ with $\lim_{n\to+\infty} ||z_n||_C = +\infty$. Also, in view of Lemma 3.8 the problem ($3.75_n$) in the case $i = 0$ as in the case $i = 2$ has only a zero solution for $n \geq n_0$. Then as it is well-known (see [9, Theorem 1.1, p. 345]) from the inequality (4.19) it follows that the problem (4.1), (4.2) has at least one solution, suppose $u_n$. Now suppose that (4.4) is fulfilled and $v_n(t) = u_n(t)||u_n||_C^{-1}$. Then (4.6), (4.7) and (4.17) are fulfilled too. Thus, by the conditions (4.7) and (4.19), from (4.17) we get the existence of $r_0 > 0$ such that $||v_n'||_C \leq r_0$. Consequently in view of (4.7) by the Arzela-Ascoli lemma, without loss of generality we can assume that there exists the function $w_0 \in \widetilde{C}'(I, R)$ such that $\lim_{n\to+\infty} v_n^{(i)}(t) = w_0^{(i)}(t)$ $(i = 0, 1)$ uniformly on $I$. Now, analogously as in the proof of Theorem 1.1, we can choose the sequence $\{\alpha_k\}_{n=1}^{+\infty}$ from $N$ such that, if we suppose $u_n = u_{\alpha_n}$, the conditions (4.8) will by true when the functions $u_n$ and $v_n$ are the solutions of the problems (4.1), (4.2) and (4.17), (4.6) respectively with $p_n(t) = p(t) + (-1)^i/\alpha_n$ for $t \in I,\ n \in N$. From (4.17) by virtue of (4.6), (4.8) and (1.13) we obtain (4.14). Consequently, analogously as (4.15) in the proof of the Theorem 1.1 we obtain

$$-\frac{1}{\alpha_n} \int_a^b w(s)u_n(s)ds = (-1)^i \int_a^b (h(s) + f(s, u_n(s)))w(s)ds \tag{4.20}$$

for $n \in N$. Now note that in view the conditions (1.11), (1.12), (1.14), (4.2) and (4.8), all the assumptions of Lemma 3.5 with $f_1(t,x) = (-1)^i f(t,x),\ h_1(t) = (-1)^i h(t)$ are satisfied. Hence, analogously as in the proof of the Theorem 1.1, from (4.20) by Lemma 3.5 we obtain that the problem (0.1), (0.2) has at least one solution. □

*Proof of Corollary 1.1.* From the condition (1.15) we immediately obtain (1.14). Then all the conditions of Theorem 1.3 are fulfilled. □

*Proof of Theorem 1.4.* The proof is the same as the proof of theorem 1.3. The only difference is that instead of Lemma 3.5 we will use Lemma 3.6. □

*Proof of Theorem 1.5.* From (1.21) it is clear that for an arbitrary sequence $z_n \in C(I;R)$ such that $\lim_{n\to+\infty} ||z_n||_C = +\infty$, the equality (4.19) is valid. From (4.19) and Lemma 3.7, analogously as in the proof of Theorem 1.3 from (4.19) and Lemma 3.5, we see that the problem (0.1), (0.2) has at least one solution. □

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## References

[1] R. P. Agarwal and I. Kiguradze, *Two-point boundary value problems for higher-order linear differential equations with strong singularities.* Boundary Value Problems 2006, 1-32; Article ID 83910.

[2] S. Ahmad, *A resonance problem in which the nonlinearity may grow linearly* . Proc.Amer.Math.Soc.**92** (1984),381–384.

[3] M. Arias, *Existence results on the one-dimensional Dirichlet problem suggested by the piecewise linear case.*Proc. Amer. Math. Soc.**97** (1986)No.1,121–127.

[4] C. De Coster, P. Habets *Upper and Lower Solutions in the theory of ODE boundary value problems.*Nonlinear Analysis And Boundary Value Problems For Ordinary Differential Equations, SprringerWienNewYork(1996) No.371,1–119.

[5] R. Conti , *Equazioni differenziali ordinarie quasilineari con condizioni lineari.* Ann. Mat. Pura ed Appl., (1962) No.57, 49–61.

[6] E. Landesman, A. Lazer, *Nonlinear Perturbations of linear elliptic boundary value problems at resonance,* J. Math. Mech. **19** (1970),609–623.

[7] P. Drabek, *On The Resonance Problem With Nonlinearity which has arbitrary linear growth.* J. Math. Anal. Appl.**127** (1987),435–442.

[8] P. Drabek, *Solvability and Bifurkations of Nonlinear Equations.* University of West Bohemia Pilsen (1991),1–231.

[9] R. Iannacci, M.N. Nkashama *Nonlinear Two Point Boundary Value Problems At resonance Without Landesman-Lazer Condition.* Proc. Amer. Math. Soc.**106**(1989)No.4,943–952.

[10] R. Iannacci, M.N. Nkashama *Nonlinear Boundary Value Problems At resonance.* Nonlinear Anal.**6**(1987),919–933.

[11] R. Kannan, J.J. Nieto, M.B. Ray *A Class of Nonlinear Boundary Value Problems Without Landesman-Lazer Condition.* J. Math. Anal. Appl.**105**(1985),1–11.

[12] I. Kiguradze, B. Shekhter, *Singular boundary value problems for second order ordinary differential equations.* (Russian) Itogi Nauki Tekh., ser. Sovrem. Probl. Mat., Noveish. Dostizheniya
**30** (1987), 105–201; English transl.: J. Sov. Math. **43** (1988), No. 2, 2340–2417.

[13] I. Kiguradze, *Nekotorie Singularnie Kraevie Zadachi dlja Obiknovennih Differencialnih Uravneni.* Tbilisi University(1975),1–351.

[14] I. Kiguradze, *On a singular two-point boundary value problem.* (Russian) Differentsial' nye Uravneniya **5** (1969), No. 11, 2002-2016; English transl.: Differ. Equations 5 (1969), 1493-1504.

[15] I. Kiguradze, *On some singular boundary value problems for nonlinear second order ordinary differential equations.*(Russian) Differentsial' nye Uravneniya **4** (1968), No. 10, 1753-1773; English transl.: Differ. Equations **4** (1968), 901-910.

[16] I. Kiguradze, *On a singular boundary value problem.* J. Math. Anal. Appl. **30** (1970), No. 3, 475-489.

[17] J. Kurzveil, *Generalized ordinary Differential equations.* Czechosl. Mat. J., **8** (1958), No. 3, 360–388.

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