NON-OPERATOR REFLEXIVE SUBSPACE LATTICE

KAMILA KLIŚ-GARLICKA AND VLADIMIR MÜLLER

ABSTRACT. In [1] various types of closedness of subspace lattice was studied. In particular, the authors defined operator reflexivity which can be regarded as a one-point closedness of the lattice. They asked if all subspace lattices are operator reflexive. In this work we give an example that the answer is negative.

Let H be a Hilbert space. By $\mathcal{B}(H)$ we denote the algebra of all bounded linear operators on H and by $\mathcal{P}(H)$ the lattice of all orthogonal projections on H. A SOT-closed sublattice of $\mathcal{P}(H)$, containing the trivial projections 0 and I is called a *subspace lattice*.

Recall that for any set $\mathcal{S}$ of operators the *operator-reflexive hull* of $\mathcal{S}$ is defined as

$$\operatorname{ref}\mathcal{S} = \{A \in \mathcal{B}(H) : Ax \in \overline{\mathcal{S}x} \text{ for all } x \in H\}.$$

It was proved in [1] that if $\mathcal{L}$ is a subspace lattice then

$$\operatorname{ref}\mathcal{L} = \{P \in \mathcal{P}(H) : Px \in \overline{\mathcal{L}x} \text{ for all } x \in H\}.$$

Recall after [1] that a projection lattice $\mathcal{L}$ is called *operator reflexive* (or *1-closed*) if $\mathcal{L} = \operatorname{ref}\mathcal{L}$. In [1] authors proved that operator reflexive lattices are always SOT-closed, but they asked if all subspace lattices are operator reflexive. Here we intend to proof that it is not so.

Let M be a subspace of a Hilbert space H. We denote by P_M the orthogonal projection onto M. Let $M, L \subset H$ be subspaces. Write

$$\delta(M, L) = \sup\{\operatorname{dist}\{x, L\} : x \in M, \|x\| \le 1\}.$$

Denote by $\hat{\delta}(M, L) = \max\{\delta(M, L), \delta(L, M)\}$ the *gap* between M and L. It is well-known, see [2], p. 197, that $\hat{\delta}(M, L) = \|P_M - P_L\|$. Moreover, if $\hat{\delta}(M, L) < 1$ then $\dim M = \dim L$.

Lemma 1. *Let H be a finite-dimensional Hilbert space, $M, L \subset H$ subspaces, $\dim M = \dim L$, $\dim H = 2\dim M$. Let $\varepsilon > 0$. Then there exists a subspace $M' \subset H$ such that $\hat{\delta}(M', M) \le \varepsilon$ and $M' \cap L = \{0\}$.*

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Proof. We may assume that $\varepsilon < 1$. We have $\dim H = \dim M + \dim L = \dim(M \cap L) + \dim(M + L)$. Hence $\dim(M \cap L) = \dim(M + L)^\perp$. Let $V : M \cap L \to (M + L)^\perp$ be a surjective isometry.

Let $M' = (I + \varepsilon V)(M \cap L) \oplus (M \ominus (M \cap L))$. Clearly M' is a subspace and $\dim M' = \dim M$.

Suppose that $u \in M' \cap L$. Then $u = (I + \varepsilon V)x + y$ for some $x \in M \cap L$ and $y \in M \ominus (M \cap L)$. We have $u - x - y \in M + L$ and $\varepsilon V x \perp (M + L)$, so $\varepsilon V x = 0 = u - x - y$. Thus $x = 0$ and $u = y$. Hence $y \in M \cap L$, and so $y = 0$ and $u = 0$. Consequently, $M' \cap L = \{0\}$.

Suppose that $u \in M$, $\|u\| = 1$. Then $u = x + y$ for some $x \in M \cap L$ and $y \in M \ominus (M \cap L)$ with $\|x\|^2 + \|y\|^2 = \|u\|^2 = 1$. Then $\operatorname{dist}\{u, M'\} \le \|u - (I + \varepsilon V)x - y\| = \|\varepsilon V x\| \le \varepsilon$. Hence $\delta(M, M') \le \varepsilon$.

Conversely, let $v \in M'$, $\|v\| = 1$. Then $v = (I + \varepsilon V)x + y$ for some $x \in M \cap L$ and $y \in M \ominus (M \cap L)$. Since $\varepsilon V x \perp y$, we have $\|(I + \varepsilon V)x\| \le 1$. Since $\varepsilon V x \perp x$, we have $\|x\| \le 1$. Thus

$$\operatorname{dist}\{v, M\} \le \|v - (x + y)\| = \|\varepsilon V x\| \le \varepsilon$$

and so $\hat{\delta}(M', M) \le \varepsilon$. □

Lemma 2. *Let H be a finite dimensional Hilbert space,* $\dim H = 2n$, *let $M_1, \dots, M_k, L \subset H$ be n-dimensional subspaces, let $\varepsilon > 0$. Then there exists a subspace $L' \subset H$ such that $\hat{\delta}(L', L) \le \varepsilon$ and $L' \cap M_i = \{0\} \quad (i = 1, \dots, k)$.*

Proof. We prove the statement by induction on k. For $k = 1$ the statement was proved in Lemma 1. Suppose that the statement is true for some $k - 1$ and let $M_1, \dots, M_k, L, \varepsilon$ be given.

By the induction assumption, there exists a subspace $L'' \subset H$ such that $\hat{\delta}(L, L'') \le \varepsilon/2$ and $L'' \cap M_i = \{0\} \quad (i = 1, \dots, k - 1)$.

By a compactness argument, there exists $\delta > 0$ such that $\operatorname{dist}\{x, L''\} \ge \delta$ whenever $1 \le i \le k - 1$, $x \in M_i$, $\|x\| = 1$. By Lemma 1, there exists $L' \subset H$ such that $\hat{\delta}(L', L'') \le \min\{\varepsilon/2, \delta/2, \}$ and $L' \cap M_k = \{0\}$.

We have $\hat{\delta}\{L', L\} = \|P_{L'} - P_L\| \le \|P_{L'} - P_{L''}\| + \|P_{L''} - P_L\| \le \varepsilon$.

We show that $L' \cap M_i = \{0\} \quad (i = 1, \dots, k-1)$. Fix $i \in \{1, \dots, k-1\}$ and suppose that there exists $x \in L' \cap M_i$, $\|x\| = 1$. Then there exists $x' \in L''$ with $\|x' - x\| \le \hat{\delta}(L', L'') \le \delta/2$, a contradiction with the definition of δ. Hence $L' \cap M_i = \{0\} \quad (i = 1, \dots, k)$. □

Let H be the Hilbert space with an orthonormal basis $e_1, e_2, \dots$ For $k \in \mathbb{N}$ let $H_k = \bigvee\{e_1, \dots, e_k\}$. Denote by S_H the unit sphere in H. Fix a sequence $(x_n, y_n)_{n=1}^\infty$ dense in $S_H \times S_H$ such that for each $n \in \mathbb{N}$ the vectors x_n, y_n are linearly independent and $\langle x_n, y_n \rangle \ne 0$. Moreover, we may assume that all the vectors x_n, y_n have finite support, i.e., $x_n, y_n \in \bigcup_{k\in\mathbb{N}} H_k$ for each $n \in \mathbb{N}$.

Fix a sequence $(t_n)_{n=1}^{\infty} \subset (0,1)$ consisting of mutually distinct numbers.

Lemma 3. *There exist subspaces $M_n \subset H \quad (n \in \mathbb{N})$ such that:*

(i) $M_n \cap M_m = \{0\} \quad (m, n \in \mathbb{N}, m \neq n)$;
(ii) $M_n \vee M_m = H \quad (m, n \in \mathbb{N}, m \neq n)$;
(iii) $\|P_{M_n} x_n - \langle x_n, y_n\rangle y_n\| \leq 1/n$;
(iv) *there is a constant $c > 0$ such that for all $m, n \in \mathbb{N}$, $m \neq n$,*

$$\max_{j=1,2,3} \|P_{M_n} e_j - P_{M_m} e_j\| \geq c;$$

(v) *there is an increasing sequence of positive integers $(k_n)_{n=1}^{\infty}$ such that each M_n can be written as*

$$M_n = F_n \oplus \bigvee\{e_{2j+1} + t_n e_{2j+2} : j \geq k_n\},$$

where $F_n \subset H_{2k_n}$ is a k_n-dimensional subspace.

Proof. We construct the subspaces M_n by induction on n. Let $n \in \mathbb{N}$ and suppose that the subspaces $M_1, \dots, M_{n-1}$ satisfying (i)–(v) have already been constructed.

Let $L_n = \bigvee\{x_n, y_n\}$. By assumption, $\dim L_n = 2$. Fix $j_n \in \{1, 2, 3\}$ such that

$$\operatorname{dist}\{e_{j_n}, L_n\} = \max_{i=1,2,3} \operatorname{dist}\{e_i, L_n\}.$$

Clearly there is a constant $c > 0$ such that $\max_{i=1,2,3} \operatorname{dist}\{e_i, L\} \geq 4c$ for each 2-dimensional subspace $L \subset H$. Hence $\operatorname{dist}\{e_{j_n}, L_n\} \geq 4c$. Let $u_n = \frac{P_{L_n^\perp} e_{j_n}}{\|P_{L_n^\perp} e_{j_n}\|}$. Fix $k_n > \max\{k_{n-1}, 2\}$ such that $x_n, y_n \in H_{2k_n - 1}$. Since $u_n = \frac{e_{j_n} - P_{L_n} e_{j_n}}{\|P_{L_n^\perp} e_{j_n}\|}$ and $e_{j_n} \in L_n + L_n'$, thus $u_n \in H_{2k_n-1}$.

Let $L_n' = \bigvee\{u_n, e_{2k_n}\}$. Then $\dim L_n' = 2$ and $L_n' \perp L_n$. Let F_n' be any k_n-dimensional subspace of H_{2k_n} such that $y_n \in F_n'$, $u_n + e_{2k_n} \in F_n'$ and $\dim(H_{2k_n} \ominus (L_n + L_n')) \cap F_n' = k_n - 2$.

For $s = 1, \dots, n-1$ let $E_s \subset H_{2k_n}$ be defined by

$$E_s = F_s \oplus \bigvee\{e_{2j+1} + t_s e_{2j+2} : k_s \leq j < k_n\}.$$

By Lemma 2 for the subspaces $E_1, \dots, E_{n-1}, F_n'$ there exists a subspace $F_n \subset H_{2k_n}$ such that $F_n \cap E_s = \{0\} \quad (s = 1, \dots, n-1)$ and $\hat{\delta}\{F_n, F_n'\} < \min\{\frac{1}{n}, c\}$. Note that this implies that $\dim F_n = k_n$ and $F_n \vee E_s = H_{2k_n} \quad (s = 1, \dots, n-1)$.

Let $M_n = F_n \oplus \bigvee\{e_{2j+1} + t_n e_{2j+2} : j \geq k_n\}$. We show that M_n satisfies (i)–(v). Condition (v) follows from the definition. Since $t_m \neq t_n$ for $m < n$, we have $M_m \cap M_n = \{0\}$ and $M_m \vee M_n = H$.

We have $P_{F_n'} x_n = \langle x_n, y_n \rangle y_n$ and $\|P_{F_n} - P_{F_n'}\| = \hat{\delta}\{F_n, F_n'\} \leq \frac{1}{n}$. Hence

$$\|P_{M_n} x_n - \langle x_n, y_n \rangle y_n\| = \|P_{F_n} x_n - P_{F_n'} x_n\| \leq \frac{1}{n}.$$

Let Q be the orthogonal projection onto the 1-dimensional subspace generated by e_{2k_n}. Let $m < n$. We have $P_{M_m} e_{j_n} \in H_{2k_m}$, and so $QP_{M_m} e_{j_n} = 0$. Furthermore

$$\begin{aligned} \|QP_{M_n} e_{j_n}\| = \|QP_{F_n} e_{j_n}\| &\geq \|QP_{F_n'} e_{j_n}\| - \|Q(P_{F_n'} - P_{F_n}) e_{j_n}\| \\ &\geq \|QP_{F_n'} e_{j_n}\| - \hat{\delta}\{F_n', F_n\} \geq \|QP_{F_n'} e_{j_n}\| - c \end{aligned}$$

and

$$\begin{aligned} \|QP_{F_n'} e_{j_n}\| &= \|QP_{L_n \cap F_n'} e_{j_n} + QP_{L_n' \cap F_n'} e_{j_n}\| = \|QP_{L_n' \cap F_n'} e_{j_n}\| \\ &= \|QP_{L_n' \cap F_n'}(u_n \cdot \|P_{L_n^\perp} e_{j_n}\|)\| \geq 4c \cdot \|QP_{L_n' \cap F_n'} u_n\| = 4c \cdot \|Q\tfrac{u_n + e_{2k_n}}{2}\| = 2c. \end{aligned}$$

Hence

$$\|P_{M_n} e_{j_n} - P_{M_m} e_{j_n}\| \geq \|QP_{M_n} e_{j_n} - QP_{M_m} e_{j_n}\| \geq 2c - c = c.$$

□

Theorem 4. *There exists a strongly closed lattice $\mathcal{L} \subset \mathcal{P}(H)$ which is not operator reflexive.*

Proof. Let M_n be the subspaces constructed in the previous lemma. Let $\mathcal{L} = \{0, I, P_{M_n} : n \in \mathbb{N}\}$. Clearly $\mathcal{L}$ is a lattice and $\mathcal{L} \neq \mathcal{P}(H)$. We show that $\mathcal{L}$ is strongly closed. It is sufficient to show that the set $\{P_{M_n} : n \in \mathbb{N}\}$ is strongly closed. Let $P \in \mathcal{P}(H)$, $P \in \{P_{M_n} : n \in \mathbb{N}\}^{-SOT}$. Let $c > 0$ be the number from the previous lemma.

Let $x \in H$. Then there exists $n(x) \in \mathbb{N}$ such that

$$\|P_{M_{n(x)}} x - Px\| < \frac{c}{2}$$

and

$$\|P_{M_{n(x)}} e_j - Pe_j\| < \frac{c}{2} \qquad (j = 1, 2, 3).$$

Moreover, $n(x)$ is determined uniquely and is independent of the choice of $x \in H$. Indeed, let $y \in H$ and let $n(y) \in \mathbb{N}$ satisfies

$$\|P_{M_{n(y)}} x - Px\| < \frac{c}{2}$$

and

$$\|P_{M_{n(y)}} e_j - Pe_j\| < \frac{c}{2} \qquad (j = 1, 2, 3).$$

For $j = 1, 2, 3$ we have

$$\|P_{M_{n(x)}} e_j - P_{M_{n(y)}} e_j\| \le \|P_{M_{n(x)}} e_j - P e_j\| + \|P e_j - P_{M_{n(y)}} e_j\| < c.$$

Hence $n(x) = n(y)$. Furthemore, $P_{M_{n(x)}} x = Px$. Indeed, for each $\delta \in (0, \frac{c}{2})$ there exists $r \in \mathbb{N}$ such that

$$\|P_{M_r} x - Px\| < \delta$$

and

$$\|P_{M_r} e_j - P e_j\| < \frac{c}{2} \qquad (j = 1, 2, 3).$$

Hence $r = n(x)$ and $\|P_{M_{n(x)}} x - Px\| < \delta$. Since $\delta > 0$ was arbitrary, we have $P_{M_{n(x)}} x = Px$ and $P = P_{M_{n(x)}}$.

Hence $\mathcal{L}$ is closed in the strong operator topology.

On the other hand, the operator-reflexive hull of $\mathcal{L}$ is the whole lattice $\mathcal{P}(H)$. To see this, let $P \in \mathcal{P}(H)$ and $x \in H$, $\|x\| = 1$. If $Px = 0$ then obviously $Px \in \{Qx : Q \in \mathcal{L}\}^-$. Let $Px \neq 0$ and $y = \frac{Px}{\|Px\|}$. Then there is a sequence (n_k) such that $n_k \to \infty$, $x_{n_k} \to x$ and $y_{n_k} \to y$. Thus

$$Px = \langle x, y \rangle y = \lim_{k \to \infty} \langle x_{n_k}, y_{n_k} \rangle y_{n_k} = \lim_{k \to \infty} P_{M_{n_k}} x_{n_k} =$$
$$= \lim_{k \to \infty} P_{M_{n_k}} x \in \{Qx : Q \in \mathcal{L}\}^-,$$

and so P is in the operator-reflexive hull of $\mathcal{L}$. □

References

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Kamila Kliś-Garlicka, Institute of Mathematics, University of Agriculture, Al.Mickiewicza 24/28, 30-059 Kraków, Poland

E-mail address: `rmklis@cyf-kr.edu.pl`

Vladimir Müller, Institute of Mathematics, Academy of Sciences of the Czech Republic, Žitna 25, 115 67, Praha 1, Czech Republic

E-mail address: `muller@math.cas.cz`