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A Weak Solvability of the Navier–Stokes Equation with Navier's Boundary Condition around a Ball Striking the Wall

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Abstract We assume that B^t is a closed ball in $\mathbb{R}^3_+ := \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ x_3 > 0\}$, striking the wall (= the x_1,x_2–plane) at time $t_c \in (0,T)$. The speed of the ball at the instant of the collision need not be zero. Although a weak solution to the Navier–Stokes equation with Dirichlet's no–slip boundary condition in $(\mathbb{R}^3_+ \smallsetminus B^t) \times (0,T)$ does not exist if the speed of the stroke is non–zero, we prove that such a solution may exist if Dirichlet's boundary condition is replaced by Navier's slip boundary condition.

1 Motivation, introduction and notation

The existence of a weak solution to the Navier–Stokes equation in a fixed domain $\Omega \subset \mathbb{R}^3$ on a given time interval $(0,T)$ belongs to fundamental results of the qualitative theory of the Navier–Stokes equation. (See e.g. J. Leray 1934 [17], E. Hopf 1952 [15], O. A. Ladyzhenskaya 1969 [16], J. L. Lions 1969 [18], R. Temam 1977 [26] or G. P. Galdi 2000 [10].)

Of all results on the existence of the weak solution in domains with given moving boundaries, we cite the papers by H. Fujita and N. Sauer 1970 [7] (the boundary of a variable domain Ω^t consists of a finite number of moving simple closed surfaces of the class C^3, the distance of any two of these surfaces is never less than $d_0 > 0$) and J. Neustupa 2007 [19] (Ω^t has an arbitrary shape and smoothness, the assumptions on Ω^t involve simulation of collisions of bodies moving in a fluid).

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There exists a series of other works dealing with flows in time varying domains that concern the motion of one or more bodies in a fluid. The fluid and the bodies are studied as an interconnected system so that the position of the bodies in the fluid is not apriori known. The weak solvability of such a problem, provided the bodies do not touch each other or they do not strike the boundary, was proved by B. Desjardins and M. J. Esteban 1999 [3], 2000 [4], K. H. Hoffmann V. N. Starovoitov 1999 [13] (the 2D case), C. Conca, J. San Martín and M. Tucsnak 2000 [2] and M. D. Gunzburger, H. C. Lee, G. Seregin 2000 [12]. The analogous result, without the assumption on the lack of collisions, was proved by J. San Martín, V. N. Starovoitov and M. Tucsnak 2002 [20] (the 2D case), K. H. Hoffmann, V. N. Starovoitov 2000 [14] (the motion of a "small" ball in a fluid filling a "large ball") and E. Feireisl 2003 [6] (in a 3D bounded domain, the author uses the contact condition that once two bodies touch one another, they remain stuck together forever).

All the mentioned authors consider the homogeneous Dirichlet boundary condition for velocity on the boundary of Ω^t. The motion of the so called "self–propelled bodies" (which produce certain velocity profile on their surface), together with the motion of the fluid around them, was studied except others by G. P. Galdi, see the survey paper [11].

None of the mentioned papers provides the existence of a weak solution to the Navier–Stokes equation at the geometrical configuration when the fluid fills a domain Ω^t around a solid ball striking a wall with a finite non–zero speed. Moreover, it follows from results of V. N. Starovoitov 2003 [21] that the weak solution with the no–slip Dirichlet boundary condition in such a situation cannot exist. Paper [19], where the no–slip boundary condition is also considered, provides the weak solution only if the ball strikes the wall with the speed that tends to zero as time approaches the instant of the collision. (With a non–zero speed, the body must have another shape than the ball, see [19].)

This state motivated us to study the Navier–Stokes equation in the described domain Ω^t with boundary conditions that enable the fluid to slip on the boundary. We assume that the motion of the ball is given. We use Navier's boundary condition and we prove the global in time existence of a weak solution under the restriction that the speed of the ball is "sufficiently small" at times close to the instant of the collision – see Theorem 1. The considered case of a ball moving in a fluid and striking perpendicularly the wall represents a sample example. We actually prepare a generalization concerning flow around moving bodies of various shapes which may collide one with another. Nevertheless, the basic techniques is developed in the present paper. It is based on the construction of Rothe approximations.

A series of steps require a different approach than in the case of homogeneous Dirichlet's boundary condition. For instance, Sobolev's imbedding inequalities cannot be used in a standard fashion because the constants in these inequalities now depend on time. Other difficulties appear in the part where we treat the limit transition in the nonlinear term and we therefore need an information on a strong convergence of a sequence of approximations in an appropriate norm. (The argument based on the Lions–Aubin lemma cannot be used in a usual way – see Section 6.)

The time–variable domain Ω^t. We suppose that $(0,T)$ is a bounded time interval and $t_c \in (0,T)$. We denote by $\mathbb{R}^3_+$ the half–space $\{\mathbf{x} = (x_1,x_2,x_3) \in \mathbb{R}^3;\, x_3 > 0\}$, by $\overline{\mathbb{R}}^3_+$ the closure of $\mathbb{R}^3_+$ and by $\partial\mathbb{R}^3_+$ the boundary of $\mathbb{R}^3_+$ (= the x_1, x_2–plane).

Further, we denote by B^t the closed ball in $\overline{\mathbb{R}}^3_+$ with radius R and center $S^t = (0,\,0,\,\delta^t + R)$. We suppose that δ^t (the distance of the ball B^t from $\partial\mathbb{R}^3_+$) is a continuous function of t for $t \in [0,T]$ such that $\delta^{t_c} = 0$ and

(i) δ^t is decreasing on $[0,t_c]$ and increasing on $[t_c,T]$,

(ii) $\dot{\delta}^t$ (the derivative of δ^t) is bounded on the intervals $[0,t_c)$ and $(t_c,T]$,

(iii) $\ddot{\delta}^t$ (the second derivative of δ^t) is integrable on $(0,T)$.

We put $\Omega^t := \mathbb{R}^3_+ \smallsetminus B^t$. The boundary of Ω^t is denoted by Γ^t. Ω^t represents the space filled by the fluid and B^t represents a solid ball which moves in the fluid and strikes the fixed wall $\partial\mathbb{R}^3_+$ at time $t = t_c$. We assume, for simplicity, that ball B^t does not rotate and all its particles have only the translational velocity. Thus, the velocity of the "material points" on the boundary Γ^t of Ω^t is

$$\mathbf{V}^t(\mathbf{x}) := \begin{cases} (0,0,\dot{\delta}^t) & \text{for } t \neq t_c \text{ and } \mathbf{x} \in \partial B^t, \\ \mathbf{0} & \text{for } t \neq t_c \text{ and } \mathbf{x} \in \partial\mathbb{R}^3_+. \end{cases}$$

Notation of norms and function spaces.

- $(\,.\,,.\,)_{2;\Omega^t}$ is the scalar product and $\|\,.\,\|_{2;\Omega^t}$ is the norm in $L^2(\Omega^t)$ or in $L^2(\Omega^t)^3$ or in $L^2(\Omega^t)^9$, respectively. The meaning of $(\,.\,,.\,)_{2;\Gamma^t}$ and $\|\,.\,\|_{2;\Gamma^t}$ is analogous.
- $\|\,.\,\|_{q;\Omega^t}$ is the norm in $L^q(\Omega^t)$ or in $L^q(\Omega^t)^3$ or in $L^q(\Omega^t)^9$, respectively.
- $C^\infty_\sigma(\Omega^t)$ is the space of infinitely differentiable divergence–free vector–functions in $\overline{\Omega}^t$ with a compact support in $\overline{\Omega}^t$ and zero normal component on Γ^t.
- $W^{1,2}_\sigma(\Omega^t)$ is the closure of $C^\infty_\sigma(\Omega^t)$ in $W^{1,2}(\Omega^t)^3$.
- $C^\infty_{0,\sigma}(\Omega^t)$ is a subspace of $C^\infty_\sigma(\Omega^t)$, containing functions with a compact support in Ω^t.
- $L^q_\sigma(\Omega^t)$ is the closure of $C^\infty_{0,\sigma}(\Omega^t)$ in $L^q(\Omega^t)^3$ (for $1 \leq q < +\infty$).

If $t \in (0,T) \smallsetminus \{t_c\}$ then $W^{1,2}_\sigma(\Omega^t) \hookrightarrow L^q(\Omega^t)^3$ for $2 \leq q \leq 6$. Using the characterization of $L^q_\sigma(\Omega^t)$ (see [8, p. 111]), we can verify that $W^{1,2}_\sigma(\Omega^t) \hookrightarrow L^q_\sigma(\Omega^t)$.

The initial–boundary value problem. Put $Q_{(0,T)} := \big\{(\mathbf{x},t);\ 0 < t < T,\ \mathbf{x} \in \Omega^t\big\}$ and $\Gamma_{(0,T)} := \big\{(\mathbf{x},t);\ 0 < t < T,\ \mathbf{x} \in \Gamma^t\big\}$.

Our aim is to prove the existence of a weak solution of the problem

$$\partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = \nu\Delta\mathbf{v} + \mathbf{f} \qquad \text{in } Q_{(0,T)}, \tag{1}$$

$$\operatorname{div}\mathbf{v} = 0 \qquad \text{in } Q_{(0,T)}, \tag{2}$$

$$\mathbf{v}\cdot\mathbf{n} = \mathbf{V}^t\cdot\mathbf{n} \qquad \text{in } \Gamma_{(0,T)}, \tag{3}$$

$$[\mathbb{T}_\mathrm{d}(\mathbf{v})\cdot\mathbf{n}]_\tau + K(\mathbf{v} - \mathbf{V}^t) = \mathbf{0} \qquad \text{in } \Gamma_{(0,T)}, \tag{4}$$

$$\mathbf{v} = \mathbf{v}_0 \qquad \text{in } \Omega^0 \times \{0\}. \tag{5}$$

The equations (1), (2) describe the motion of a viscous incompressible fluid in domain Ω^t. The symbols $\mathbf{v}$, p, ν, $\mathbf{f}$, $\mathbf{n}$ and $\mathbb{T}_\mathrm{d}(\mathbf{v})$ successively denote the velocity of the fluid, the pressure, the kinematic coefficient of viscosity, the specific external body force, the outer normal vector on the boundary of Ω^t and the dynamic stress tensor associated with the flow $\mathbf{v}$. The density of the fluid is supposed to be one. The subscript τ denotes the tangential component to Γ^t. Since the considered fluid is Newtonian, the dynamic stress tensor has the form $\mathbb{T}_\mathrm{d}(\mathbf{v}) = 2\nu\,(\nabla\mathbf{v})_s$ where $(\nabla\mathbf{v})_s$ is the symmetrized gradient of $\mathbf{v}$. Condition (3) expresses the impermeability of Γ^t. Condition (4) is due to H. Navier, who proposed in 1824 that the tangential component of the stress acting on the boundary should be proportional to the velocity of the fluid (relative with respect to the material boundary). We suppose (in accordance with physical arguments) that $K \geq 0$.

Introduction of function $\mathbf{a}^t$. In order to transform the inhomogeneous boundary condition (3) to the homogeneous one, we look for the solution $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{a}^t + \mathbf{u}$ where $\mathbf{a}^t$ is considered to be a known function satisfying the condition

$$\mathbf{a}^t \cdot \mathbf{n} \;=\; \mathbf{V}^t \cdot \mathbf{n} \qquad \text{a.e. in } \Gamma_{(0,T)} \tag{6}$$

and $\mathbf{u}$ is a new unknown function. The construction of an appropriate function $\mathbf{a}^t$ is presented in Section 2. We shall see that function $\mathbf{a}^t$ can be defined a.e. in $\mathbb{R}^3_+ \times [0,T]$ so that it is divergence–free and, in addition to the condition (6), it also satisfies the series of estimates

$$\|\nabla\mathbf{a}^t\|^2_{2;\Omega^t} \;\leq\; c_1\,(\dot{\delta}^t)^2\,\ln\Big(1+\frac{R}{\delta^t}\Big), \tag{7}$$

$$\big|\big(\partial_t\mathbf{a}^t,\phi\big)_{2;\Omega^t}\big| \;\leq\; c_2\,(\dot{\delta}^t)^2\,\|\nabla\phi\|_{2;\Omega^t} + c_3\,|\ddot{\delta}^t|\,\|\phi\|_{2;\Omega^t}, \tag{8}$$

$$\big|\big(\mathbf{a}^t\cdot\nabla\mathbf{a}^t,\phi\big)_{2;\Omega^t}\big| \;\leq\; c_4\,(\dot{\delta}^t)^2\,\|\nabla\phi\|_{2;\Omega^t}, \tag{9}$$

$$\|\mathbf{a}^t\|_{5;\Omega^t} \;\leq\; c_5\,\frac{|\dot{\delta}^t|}{(\delta^t)^{1/10}}, \tag{10}$$

$$\big|\big(\phi\cdot\nabla\mathbf{a}^t,\phi\big)_{2;\Omega^t}\big| \;\leq\; c_6\,|\dot{\delta}^t|\,\|\nabla\phi\|^2_{2;\Omega^t}, \tag{11}$$

$$K\,\big|\big(\mathbf{a}^t-\mathbf{V}^t,\phi\big)_{2;\Gamma^t}\big| \;\leq\; \tfrac{1}{16}\nu\,\|\nabla\phi\|^2_{2;\Omega^t} + c_7 \tag{12}$$

for $t \neq t_c$, all $\phi \in W^{1,2}_\sigma(\Omega^t)$, with constants c_1–c_7 which are independent of ϕ and t. Obviously, the right hand side of (7) is integrable on $(0,T)$ with any power $\alpha \geq 0$ and the right hand side of (10) is integrable on $(0,T)$ with any power $\alpha \in [1,10)$.

Using the continuous imbedding $L^6(\Omega^t) \hookrightarrow W^{1,2}(\Omega^t)$, we can also derive the estimate

$$\begin{aligned}\big|\big(\phi\cdot\nabla\mathbf{a}^t,\phi\big)_{2;\Omega^t}\big| &\leq \|\nabla\mathbf{a}^t\|_{2;\Omega^t}\,\|\phi\|^{1/2}_{2;\Omega^t}\,\|\phi\|^{3/2}_{6;\Omega^t} \\ &\leq c_8\,a(t)\,\|\phi\|^2_{2;\Omega^t} + \tfrac{1}{16}\nu\,\|\nabla\phi\|^2_{2;\Omega^t}\end{aligned} \tag{13}$$

where $a(t) := \|\nabla \mathbf{a}^t\|_{2;\Omega^t} + \|\nabla \mathbf{a}^t\|_{2;\Omega^t}^4$. Inequality (13) holds at times $t \neq t_c$ when domain Ω^t has the cone property and $W^{1,2}(\Omega^t)^3$ is therefore continuously imbedded into $L^6(\Omega^t)^3$. Constant c_8 depends on ν and it also generally depends on t through the cone parameters appearing in the definition of the cone property of Ω^t, see e.g. [1, p. 103]. However, if we use (13) only at times t such that $|t-t_c| > \kappa_0$ then c_8, although dependent on κ_0, can be considered to be independent of t. The value of κ_0 will be fixed by condition (iv) in Theorem 1.

We shall also see in Section 2 that the initial–value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t;t_0,\mathbf{x}_0) = \mathbf{a}^t\big(\mathbf{X}(t;t_0,\mathbf{x}_0)\big), \qquad \mathbf{X}(t_0;t_0,\mathbf{x}_0) = \mathbf{x}_0 \tag{14}$$

has a unique solution $X(t;t_0,\mathbf{x}_0)$, defined for a.a. $t_0 \in (0,T)$, all $t \in [0,T]$ and all $\mathbf{x}_0 \in \mathbb{R}^3_+$. The mapping $\mathbf{x}_0 \longmapsto \mathbf{X}(t;t_0,\mathbf{x}_0)$ is a 1–1 transformation of $\Omega^{t_0} \smallsetminus \ell^{t_0}$ onto $\Omega^t \smallsetminus \ell^t$ (where ℓ^{t_0} and ℓ^t are certain sets of measure zero), whose Jacobian equals one due to the incompressibility of the flow $\mathbf{a}^t$. This mapping can be used in order to transform volume integrals on Ω^{t_0} to volume integrals on Ω^t.

2 A formal study of the initial–boundary value problem (1)–(5) and the main theorem

A formal derivation of the weak formulation. The weak formulation of the problem (1)–(5) can be formally derived from the classical formulation if we multiply equation (1) by an appropriate test function ϕ, integrate in $Q_{(0,T)}$ and use all the conditions (2)–(5). Thus, assume that ϕ is an infinitely differentiable divergence–free vector–function in $\overline{\mathbb{R}}^3_+ \times [0,T]$ that has a compact support in $\overline{\mathbb{R}}^3_+ \times [0,T)$ and satisfies the condition $\phi \cdot \mathbf{n} = 0$ on $\Gamma_{(0,T)}$. Assume that $\mathbf{v}$ is a "sufficiently smooth" solution of (1)–(5) of the form $\mathbf{v} = \mathbf{a}^t + \mathbf{u}$ where $\mathbf{a}^t$ has all the properties named in the last paragraph of Section 1 and $\mathbf{u} \in L^2_\sigma(\Omega^t)$ for a.a. $t \in (0,T)$. The product $\{\partial_t \mathbf{v} + (\mathbf{v}^t \cdot \nabla)\mathbf{v}\} \cdot \phi$ equals the sum of $\{\partial_t \mathbf{v} + (\mathbf{a}^t \cdot \nabla)\mathbf{v}\} \cdot \phi$ and $\mathbf{u} \cdot \nabla \mathbf{v} \cdot \phi$. The integral of the first term can be treated as follows:

$$\begin{aligned}
&\int_0^T \int_{\Omega^t} \big\{\partial_t \mathbf{v}(\mathbf{x},t) + \mathbf{a}^t(\mathbf{x}) \cdot \nabla \mathbf{v}(\mathbf{x},t)\big\} \cdot \phi(\mathbf{x},t)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t + \int_{\Omega^0} \mathbf{v}_0(\mathbf{x}_0) \cdot \phi(\mathbf{x}_0,0)\, \mathrm{d}\mathbf{x}_0 \\
&= \int_0^T \int_{\Omega^0} \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{v}\big(\mathbf{X}(t;0,\mathbf{x}_0),t\big) \cdot \phi\big(\mathbf{X}(t;0,\mathbf{x}_0),t\big)\, \mathrm{d}\mathbf{x}_0\, \mathrm{d}t + \int_{\Omega^0} \mathbf{v}_0(\mathbf{x}_0) \cdot \phi(\mathbf{x}_0,0)\, \mathrm{d}\mathbf{x}_0 \\
&= -\int_0^T \int_{\Omega^t} \big\{\partial_t \phi(\mathbf{x},t) + \mathbf{a}^t(\mathbf{x}) \cdot \nabla \phi(\mathbf{x},t)\big\} \cdot \mathbf{v}(\mathbf{x},t)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t.
\end{aligned} \tag{15}$$

The integral of $\mathbf{u} \cdot \nabla \mathbf{v} \cdot \phi$ in Ω^t can be transformed to the negative integral of $\mathbf{u} \cdot \nabla \phi \cdot \mathbf{v}$ by means of the integration by parts. Further, we have

$$\int_{\Omega^t} \nu \Delta \mathbf{v} \cdot \phi\, \mathrm{d}\mathbf{x} = \int_{\Omega^t} \nu \Delta \mathbf{a}^t \cdot \phi\, \mathrm{d}\mathbf{x} + \int_{\Gamma^t} \nu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} \cdot \phi\, \mathrm{d}S - \int_{\Omega^t} \nu \nabla \mathbf{u} : \nabla \phi\, \mathrm{d}\mathbf{x}$$

$$= \int_{\Gamma^t} \nu \left[2\mathbf{n}\cdot(\nabla\mathbf{u})_s - \mathbf{n}\cdot\nabla\mathbf{u}\right]\cdot\phi \, \mathrm{d}S + \int_{\Omega^t} \left[\nu\Delta\mathbf{a}^t\cdot\phi - \nu\nabla\mathbf{u} : \nabla\phi\right] \mathrm{d}\mathbf{x}$$
$$= \int_{\Gamma^t} \mathbf{n}\cdot[2\nu(\nabla\mathbf{v})_s - 2\nu(\nabla\mathbf{a}^t)_s]\cdot\phi \, \mathrm{d}S + \int_{\Omega^t} \left[\nu\Delta\mathbf{a}^t\cdot\phi - 2\nu(\nabla\mathbf{u})_s : \nabla\phi\right] \mathrm{d}\mathbf{x}$$
$$= -\int_{\Gamma^t} K(\mathbf{v}-\mathbf{V}^t)\cdot\phi \, \mathrm{d}S - \int_{\Omega^t} 2\nu(\nabla\mathbf{v})_s : \nabla\phi \, \mathrm{d}\mathbf{x}. \tag{16}$$

We have used the identities

$$\int_{\Gamma^t} \mathbf{n}\cdot 2\nu(\nabla\mathbf{v})_s\cdot\phi \, \mathrm{d}S = \int_{\Gamma^t} [\mathbf{n}\cdot\mathbb{T}_{\mathrm{d}}(\mathbf{v})]_\tau\cdot\phi \, \mathrm{d}S = -\int_{\Gamma^t} K(\mathbf{v}-\mathbf{V}^t)\cdot\phi \, \mathrm{d}S,$$
$$\int_{\Gamma^t} \mathbf{n}\cdot\nabla\mathbf{u}\cdot\phi \, \mathrm{d}S = \int_{\Omega^t} (\nabla\mathbf{u})^T : \nabla\phi \, \mathrm{d}\mathbf{x},$$
$$\int_{\Omega^t} \nu\Delta\mathbf{a}^t\cdot\phi \, \mathrm{d}\mathbf{x} = \int_{\Gamma^t} 2\nu\,\mathbf{n}\cdot(\nabla\mathbf{a}^t)_s\cdot\phi \, \mathrm{d}S - \int_{\Omega^t} 2\nu(\nabla\mathbf{a}^t)_s : \nabla\phi \, \mathrm{d}\mathbf{x},$$

the first of whose follows from (4). The integral of $\nabla p\cdot\phi$ on Ω^t equals zero because the subspace of gradients of scalar functions is orthogonal to $L^2_\sigma(\Omega^t)$ in $L^2(\Omega^t)^3$. Thus, using (15) and (16), we obtain the integral identity

$$\int_0^T\int_{\Omega^t} \left\{-\mathbf{v}\cdot\partial_t\phi - \mathbf{v}\cdot\nabla\phi\cdot\mathbf{v} + 2\nu(\nabla\mathbf{v})_s : \nabla\phi\right\} \mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T\int_{\Gamma^t} K(\mathbf{v}-\mathbf{V}^t)\cdot\phi \, \mathrm{d}S\,\mathrm{d}t$$
$$= \int_0^T\int_{\Omega^t} \mathbf{f}\cdot\phi \, \mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\Omega^0} \mathbf{v}_0\cdot\phi(.,0)\,\mathrm{d}\mathbf{x}.$$

Replacing $\mathbf{v}$ by the sum $\mathbf{a}^t+\mathbf{u}$, we arrive at the definition:

Definition (the weak solution of (1)–(5)). Suppose that $\mathbf{u}_0 \in L^2_\sigma(\Omega^0)$ and $\mathbf{f} \in L^2(0,T;L^2(\Omega^t)^3)$. The function $\mathbf{v} \equiv \mathbf{a}^t+\mathbf{u}$ is called a *weak solution* of the problem (1)–(5) if $\mathbf{u} \in L^2(0,T;W^{1,2}_\sigma(\Omega^t)) \cap L^\infty(0,T;L^2_\sigma(\Omega^t))$ satisfies

$$\int_0^T\int_{\Omega^t} \left\{-(\mathbf{a}^t+\mathbf{u})\cdot\partial_t\phi - (\mathbf{a}^t+\mathbf{u})\cdot\nabla\phi\cdot(\mathbf{a}^t+\mathbf{u}) + 2\nu\,[\nabla(\mathbf{a}^t+\mathbf{u})]_s : \nabla\phi\right\} \mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$+ \int_0^T\int_{\Gamma^t} K(\mathbf{a}^t+\mathbf{u}-\mathbf{V}^t)\cdot\phi \, \mathrm{d}S\,\mathrm{d}t = \int_0^T\int_{\Omega^t} \mathbf{f}\cdot\phi \, \mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\Omega^0} [\mathbf{a}^0+\mathbf{u}_0]\cdot\phi(.,0)\,\mathrm{d}\mathbf{x} \tag{17}$$

for all divergence–free vector–functions $\phi \in C_0^\infty\big(\overline{\mathbb{R}^3_+}\times[0,T)\big)$, that satisfy the condition $\phi\cdot\mathbf{n} = 0$ on $\Gamma_{(0,T)}$.

The readers can verify that this definition enables us the "backward calculation", i.e. to show that if the weak solution $\mathbf{v}$ is "sufficiently smooth" and all other input data are also "sufficiently smooth" then there exists a pressure p so that the pair $\mathbf{v}$, p is a classical solution of (1)–(5).

We shall refer to the problem defined above as to the weak problem (17).

A formal derivation of the energy inequality. The energy inequality is a fundamental apriori estimate of a solution of the problem (1)–(5). An analogous estimate can be rigorously derived for appropriate approximations of the solution. However, in order to abstract from technical details connected with the approximations and to explain how we use the boundary conditions and apply estimates (7)–(13), we include the formal derivation of the energy inequality already in this section.

Lemma 1. *Suppose that*

(iv) there exists $\kappa_0 > 0$ so that $c_6\,|\dot{\delta}^t| < \frac{1}{4}\nu$ for $t_c - \kappa_0 < t < t_c + \kappa_0$.

(Recall that c_6 is the constant from inequality (1.11).) Then there exist non–negative integrable functions ω and G on $(0,T)$ such that if $(\mathbf{v},p) \equiv (\mathbf{a}^t + \mathbf{u}, p)$ is a smooth solution of the initial–boundary value problem (1)–(5) and $t \in (0,T)$ then

$$\|\mathbf{u}(\,.\,,t)\|^2_{2;\Omega^t} + \nu \int_0^t \|\nabla\mathbf{u}(\,.\,,s)\|^2_{2;\Omega^s}\,\mathrm{d}s + 2K \int_0^t \|\mathbf{u}(\,.\,,s)\|^2_{2;\Gamma^s}\,\mathrm{d}s \\ \leq \|\mathbf{u}_0\|^2_{2;\Omega^0} + \int_0^t \omega(s)\,\|\mathbf{u}(\,.\,,s)\|^2_{2;\Omega^s}\,\mathrm{d}s + G(t). \tag{18}$$

Proof. Assume that $t \neq t_c$, multiply equation (1) (where $\mathbf{v} = \mathbf{a}^t + \mathbf{u}$) by $\mathbf{u}$ and integrate in Ω^t. We obtain

$$\int_{\Omega^t} \big\{ [\partial_t(\mathbf{a}^t + \mathbf{u}) + \mathbf{a}^t \cdot \nabla(\mathbf{a}^t + \mathbf{u})] \cdot \mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{a}^t \cdot \mathbf{u} - \nu\Delta\mathbf{v} \cdot \mathbf{u} \big\}\,\mathrm{d}\mathbf{x} \;=\; \int_{\Omega^t} \mathbf{f} \cdot \mathbf{u}\,\mathrm{d}\mathbf{x}. \tag{19}$$

Now we estimate or rewrite the terms in (19):

- Following (16), we have

$$-\nu \int_{\Omega^t} \Delta\mathbf{v} \cdot \mathbf{u}\,\mathrm{d}\mathbf{x} \;=\; K \int_{\Gamma^t} |\mathbf{u}|^2\,\mathrm{d}S + K \int_{\Gamma^t} (\mathbf{a}^t - \mathbf{V}^t) \cdot \mathbf{u}\,\mathrm{d}S + \nu \int_{\Gamma^t} \mathbf{n} \cdot \nabla\mathbf{u} \cdot \mathbf{u}\,\mathrm{d}S \\ +\; \nu \int_{\Omega^t} |\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} + 2\nu \int_{\Omega^t} (\nabla\mathbf{a}^t)_s : \nabla\mathbf{u}\,\mathrm{d}\mathbf{x}.$$

- Using the identity $\nabla(\mathbf{u} \cdot \mathbf{n}) \cdot \mathbf{u} = 0$ (valid a.e. on Γ^t) and the negative semi–definiteness of the tensor $\nabla\mathbf{n}$ a.e. on Γ^t (following from the special geometry of Ω^t), we observe that $\int_{\Gamma^t} \mathbf{n} \cdot \nabla\mathbf{u} \cdot \mathbf{u}\,\mathrm{d}S \geq 0$. Therefore, using (12), we get

$$\int_{\Gamma^t} \mathbf{n} \cdot \nabla\mathbf{u} \cdot \mathbf{u}\,\mathrm{d}S + K \int_{\Gamma^t} (\mathbf{a}^t - \mathbf{V}^t) \cdot \mathbf{u}\,\mathrm{d}S \;\geq\; -\tfrac{1}{16}\nu\,\|\nabla\mathbf{u}\|^2_{2;\Omega^t} - c_7, \\ -\nu \int_{\Omega^t} \Delta\mathbf{v} \cdot \mathbf{u}\,\mathrm{d}\mathbf{x} \;\geq\; K\,\|\mathbf{u}\|^2_{2;\Gamma^t} + \tfrac{14}{16}\nu \int_{\Omega^t} |\nabla\mathbf{u}|^2\,\mathrm{d}\mathbf{x} - 16\nu \int_{\Omega^t} |\nabla\mathbf{a}^t|^2\,\mathrm{d}\mathbf{x} - c_7. \tag{20}$$

- Due to (8) and (9), we obtain (with $c_9 = [8\,(c_2^2 + c_4^2)/\nu] \cdot \operatorname{ess\,sup}(\dot{\delta}^t)^4$)

$$\left| \int_{\Omega^t} [\partial_t\mathbf{a}^t + \mathbf{a}^t \cdot \nabla\mathbf{a}^t] \cdot \mathbf{u}\,\mathrm{d}\mathbf{x} \right| \;\leq\; c_3\,|\ddot{\delta}^t|\,\|\mathbf{u}\|^2_{2;\Omega^t} + \tfrac{1}{4}\,c_3\,|\ddot{\delta}^t| + \tfrac{1}{16}\nu\,\|\nabla\mathbf{u}\|^2_{2;\Omega^t} + c_9.$$

- Using the transformation $\mathbf{x} \mapsto \mathbf{y} = \mathbf{X}(t+h;t,\mathbf{x})$ of $\Omega^t \smallsetminus \ell^t$ onto $\Omega^{t+h} \smallsetminus \ell^{t+h}$, we can rewrite the next integral as follows:

$$\int_{\Omega^t} \big[\partial_t \mathbf{u} + \mathbf{a}^t \cdot \nabla \mathbf{u}\big] \cdot \mathbf{u}\, \mathrm{d}\mathbf{x} \; = \; \bigg[\int_{\Omega^t} \frac{\mathrm{d}}{\mathrm{d}\vartheta}\, \frac{1}{2}\, \big|\mathbf{u}\big(\mathbf{X}(\vartheta\,; t, \mathbf{x}), \vartheta\big)\big|^2\, \mathrm{d}\mathbf{x}\bigg]_{\vartheta = t}$$
$$= \lim_{h\to 0}\, \frac{1}{2h} \left[\int_{\Omega^t} \Big(\big|\mathbf{u}\big(\mathbf{X}(t+h; t, \mathbf{x}), t+h\big)\big|^2 - \big|\mathbf{u}\big(\mathbf{X}(t; t, \mathbf{x}), t\big)\big|^2\Big)\, \mathrm{d}\mathbf{x}\right]$$
$$= \lim_{h\to 0}\, \frac{1}{2h} \left[\int_{\Omega^{t+h}} |\mathbf{u}(\mathbf{y}, t+h)|^2\, \mathrm{d}\mathbf{y} - \int_{\Omega^t} |\mathbf{u}(\mathbf{x}, t)|^2\, \mathrm{d}\mathbf{x}\right] \; = \; \frac{\mathrm{d}}{\mathrm{d}t}\, \frac{1}{2} \int_{\Omega^t} |\mathbf{u}|^2\, \mathrm{d}\mathbf{x}.$$

- Now, due to the inequalities (11) and (13) and denoting by χ_0 the characteristic function of the interval $(t_c - \kappa_0, t_c + \kappa_0)$, we can estimate

$$\left|\int_{\Omega^t} \mathbf{u} \cdot \nabla \mathbf{a}^t \cdot \mathbf{u}\, \mathrm{d}\mathbf{x}\right| \; \leq \; c_6\, \chi_0(t)\, |\dot{\delta}^t|\; \|\nabla \mathbf{u}\|_{2;\Omega^t}^2 + \tfrac{1}{16} \nu\, \|\nabla \mathbf{u}\|_{2;\Omega^t}^2 + c_8\, a(t)\, \|\mathbf{u}\|_{2;\Omega^t}^2 .$$

- Finally, by means of condition (iv) of the smallness of $|\dot{\delta}^t|$ on the interval $(t_c - \kappa_0, t_c + \kappa_0)$, the term $c_6\, \chi_0(t)\, |\dot{\delta}^t|\; \|\nabla \mathbf{u}\|_{2;\Omega^t}^2$ can also be absorbed by $\frac{14}{16} \nu\, \|\nabla \mathbf{u}\|_{2;\Omega^t}^2$ (see (20)).
- Substituting now all previous estimates or identities to (19), using inequality (7) and denoting $\omega(t) = 2c_3\, |\ddot{\delta}^t| + 2c_8\, a(t) + 1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \|\mathbf{u}\|_{2;\Omega^t}^2 + \nu\, \|\nabla \mathbf{u}\|_{2;\Omega^t}^2 + 2K\, \|\mathbf{u}\|_{2;\Gamma^t}^2 \; \leq \; \|\mathbf{f}\|_{2;\Omega^t}^2 + \omega(t)\, \|\mathbf{u}\|_{2;\Omega^t}^2$$
$$+ \tfrac{1}{2} c_3\, |\ddot{\delta}^t| + 16 c_1\, (\dot{\delta}^t)^2\, \ln\Big(1 + \frac{R}{\delta^t}\Big) + c_7 + c_9.$$

To complete the proof, we integrate this inequality on the time interval $(0, t)$. □

Our main theorem, whose proof is given in Sections 4–6, reads:

Theorem 1. *Suppose that function δ^t satisfies conditions (i)–(iii) and also the condition of smallness (iv). Then the weak problem (17) has a solution.*

3 Construction of the auxiliary function $\mathbf{a}^t$ and its properties

The purpose of this section is to define a divergence–free function $\mathbf{a}^t$ in $\mathbb{R}^3_+ \times [0, T]$ which has the properties named and used in Section 1: identity (6), essentially $\mathbf{a}^t \cdot \mathbf{n} = (0, 0, \dot{\delta}^t) \cdot \mathbf{n}$ in ∂B^t for $t \neq t_c$, and inequalities (7)–(13).

Except for the Cartesian coordinates x_1, x_2, x_3, we shall also use the cylindrical coordinates r, φ and x_3. Thus, $r^2 = x_1^2 + x_2^2$. The lower half of the surface of B^t coincides with the graph of the function

$$x_3 \; = \; g^t(r) \; := \; \delta^t + R - \sqrt{R^2 - r^2}; \qquad 0 \leq r \leq R.$$

Domains $\Omega^t(r_0)$, Ω_{ext} and Ω_{int}. Let us fix $r_0 := \frac{3}{4} R$. The crucial sub–domain of Ω^t, where the collision occurs, is (see Fig. 1)

$$\Omega^t(r_0) \;:=\; \big\{\mathbf{x} = (r,\varphi,x_3) \in \Omega^t;\; r < r_0,\;\; x_3 < g^t(r)\big\}. \tag{21}$$

We also denote by $\Omega_{\rm ext}$ the set of points $\mathbf{x} = (r,\varphi,x_3) \in \mathbb{R}^3_+$ such that either $r > \frac{21}{20}R$ or $x_3 > \max\{\delta^0;\delta^T\}$. The complementary set $\Omega_{\rm int}$ is defined as $\mathbb{R}^3_+ \smallsetminus \overline{\Omega}_{\rm ext}$.

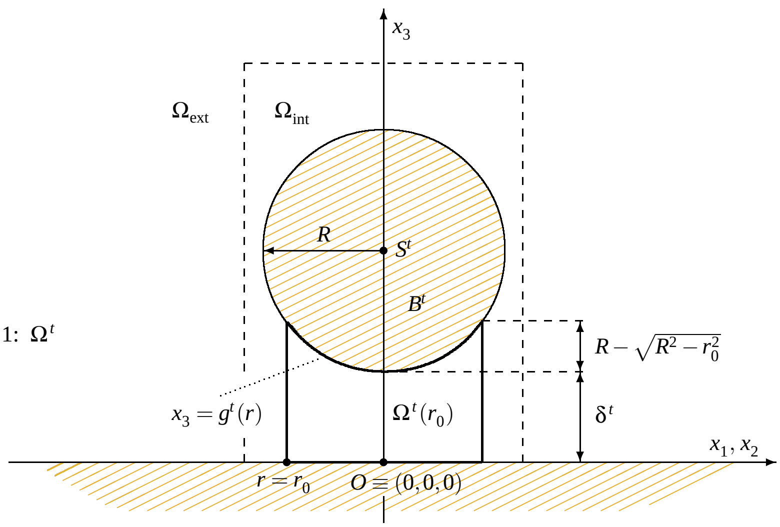

Fig. 1: Ω^t

An auxiliary function $\mathbf{b}^t$. Suppose that $t \in (0,T) \smallsetminus \{t_c\}$. We define

$$\beta^t \;=\; (\beta^t_r, \beta^t_\varphi, \beta^t_3) \;:=\; \Big(0,\; \frac{r x_3}{2g^t(r)},\; 0\Big)\,\dot\delta^t,$$

$$\mathbf{b}^t \;=\; (b^t_r, b^t_\varphi, b^t_3) \;:=\; \mathbf{curl}\,\beta^t \;=\; \Big(-\frac{r}{2g^t(r)},\; 0,\; -\frac{x_3 r\,\partial_r g^t(r)}{2g^t(r)^2} + \frac{x_3}{g^t(r)}\Big)\,\dot\delta^t \tag{22}$$

in the cylinder $r < r_0$, $0 < x_3 < \delta^t + R$. The derivative of $g^t(r)$ with respect to r is $\partial_r g^t(r) = r/\sqrt{R^2 - r^2}$. The function $\mathbf{b}^t$ is divergence–free and it satisfies the conditions of impermeability, $\mathbf{b}^t \cdot \mathbf{n} = -b^t_3 = 0$ for $x_3 = 0$, and for $x_3 = g^t(r)$,

$$\mathbf{b}^t \cdot \mathbf{n} \;=\; \Big(-\frac{r}{2g^t(r)},\; 0,\; -\frac{r\,\partial_r g^t(r)}{2g^t(r)} + 1\Big)\,\dot\delta^t \cdot \frac{\big(-\partial_r g^t(r),\, 0,\, 1\big)}{\sqrt{[g^t(r)]^2 + 1}} \;=\; (0,0,\dot\delta^t)\cdot\mathbf{n}$$

Thus, $\mathbf{b}^t \cdot \mathbf{n} = \mathbf{V}^t \cdot \mathbf{n}$ on the lower and upper parts of the boundary of $\Omega^t(r_0)$.

Two auxiliary cut–off functions. We shall use two cut–off functions: η_1 is an infinitely differentiable cut–off function of one variable such that

$$\eta_1(s) \;:=\; \begin{cases} 1 & \text{for} \quad s < R, \\ 0 & \text{for} \quad \frac{21}{20}R < s, \\ \in [0,1] & \text{for} \quad R \le s \le \frac{21}{20}R \end{cases}$$

and η_2^t is an infinitely differentiable cut–off function in $\overline{\mathbb{R}}_+^3$ whose support is a subset of $\{\mathbf{x} = (r, \varphi, x_3);\ r \leq r_0,\ x_3 \leq \delta^t + R\}$ and $\eta_2^t(\mathbf{x}) = 1$ for $r < \frac{1}{2}R$ and $0 \leq x_3 < g^t(r)$.

Let $\mathbf{e}_\varphi$ denote the unit vector in the direction of φ. Then $\mathbf{curl}\left[\frac{1}{2} r \dot{\delta}^t \mathbf{e}_\varphi\right] = (0, 0, \dot{\delta}^t)$. Furthermore, $\mathbf{curl}\left[\eta_1\left(|\mathbf{x} - S^t|\right) \frac{1}{2} r \dot{\delta}^t \mathbf{e}_\varphi\right]$ coincides with $(0, 0, \dot{\delta}^t)$ in B^t and it equals zero if $|\mathbf{x} - S^t| > \frac{21}{20}R$.

Definition of function $\mathbf{a}^t$. We put

$$\mathbf{a}^t(\mathbf{x}) := \mathbf{curl}\left[\eta_2^t(\mathbf{x})\, \beta^t(\mathbf{x}) + \left[1 - \eta_2^t(\mathbf{x})\right] \eta_1(|\mathbf{x} - S^t|)\, \tfrac{1}{2} r \dot{\delta}^t \mathbf{e}_\varphi\right]. \tag{23}$$

Then, in the important regions,

$$\mathbf{a}^t(\mathbf{x}) = \begin{cases} \mathbf{b}^t(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega^t(r_1) \text{ with } r_1 := \frac{1}{2}R = \frac{2}{3}r_0, \\ (0, 0, \dot{\delta}^t) & \text{for } \mathbf{x} \in B_+^t := \left\{\mathbf{x} \in \mathbb{R}_+^3;\ |\mathbf{x} - S^t| \leq R,\ x_3 > R + \delta^t\right\}, \\ \mathbf{0} & \text{for } \mathbf{x} \in \Omega_{\text{ext}}. \end{cases}$$

Obviously, $\mathbf{a}^t$ is divergence–free and satisfies the identity (6). It can be proved all the estimates (7)–(13) named in Section 1: Since $\mathbf{a}^t$ is smooth outside the critical region $\Omega^t(r_1)$, where the collision of the ball B^t with the x_1, x_2–plane occurs, and $\mathbf{a}^t = \mathbf{0}$ in Ω_{ext}, we can focus only on the behavior of $\mathbf{a}^t$ in $\Omega^t(r_1)$, where $\mathbf{a}^t = \mathbf{b}^t$.

Using the explicit form of $\mathbf{b}^t$, given by (22), one can show that $\mathbf{b}^t$ indeed satisfies the same estimates as (7)–(13). We only have to consider the norms or scalar products in $\Omega^t(r_1)$ instead of Ω^t on the left hand sides of (7)–(11). Similarly, $\mathbf{b}^t$ satisfies an estimate analogous to (12) with $\Gamma^t \cap \partial\Omega^t(r_1)$ instead of Γ^t.

We verify only two of the estimates in the rest of this section.

An estimate of $\left|\left(\phi \cdot \nabla \mathbf{b}^t, \phi\right)_{2;\Omega^t(r_1)}\right|$ with $\phi \in W_\sigma^{1,2}(\Omega^t)$. We consider this estimate to be crucial, because although domain $\Omega^t(r_1)$ is time–dependent, it provides an estimate with constant C independent of t.

Let us begin with the integral of $(\partial_r b_r^t)\, \phi_r^2$, where we can easily check that $|\partial_r b_r^t| = \left|\partial_r\left(r\dot{\delta}^t / 2g^t(r)\right)\right| \leq C|\dot{\delta}^t| / g^t(r)$. We put $\tilde{\phi}_r(r, x_3) := \int_0^{2\pi} \phi_r(r, \varphi, x_3)\, \mathrm{d}\varphi$. Since the flow ϕ is incompressible and it also satisfies the condition of impermeability $\phi \cdot \mathbf{n} = 0$ on Γ^t, we have

$$\int_0^{g^t(r)} \tilde{\phi}_r\, \mathrm{d}x_3 = \int_0^{g^t(r)} \int_0^{2\pi} \phi_r\, \mathrm{d}\varphi\, \mathrm{d}x_3 = \int_{\partial\Omega^t(r)} \phi \cdot \mathbf{n}\, \mathrm{d}S = \int_{\Omega^t(r)} \operatorname{div}\phi\, \mathrm{d}\mathbf{x} = 0 \tag{24}$$

(where $0 < r \leq r_1$). This implies that to each $r \in (0, r_1)$ there exists $x_3(r)$ between 0 and $g^t(r)$ such that $\tilde{\phi}_r(r, x_3(r)) = 0$. Using also the inequality $r^2 \leq 2Rg^t(r)$ and applying Poincaré's inequality (see e.g. [5, R. Dautray and J. L. Lions, p. 127]) to the integral $\int_0^{2\pi} \phi_r^2\, \mathrm{d}\varphi$, we obtain

$$\left|\int_{\Omega^t(r_1)} (\partial_r b_r^t)\, \phi_r^2\, \mathrm{d}\mathbf{x}\right| \leq C\, |\dot{\delta}^t| \int_0^{r_1} \frac{r\, \mathrm{d}r}{g^t(r)} \int_0^{g^t(r)} \mathrm{d}x_3 \left(\int_0^{2\pi} \phi_r^2\, \mathrm{d}\varphi\right)$$

$$
\begin{aligned}
&\leq C\,|\dot{\delta}^t| \int_0^{r_1} \frac{r\,\mathrm{d}r}{g^t(r)} \int_0^{g^t(r)} \mathrm{d}x_3 \left(4\pi \int_0^{2\pi} (\partial_\varphi \phi_r)^2\,\mathrm{d}\varphi + \frac{1}{2\pi} \left[\int_0^{2\pi} \phi_r\,\mathrm{d}\varphi \right]^2 \right) \\
&= C\,|\dot{\delta}^t| \int_0^{r_1} \frac{r\,\mathrm{d}r}{g^t(r)} \int_0^{g^t(r)} \mathrm{d}x_3 \int_0^{2\pi} (\partial_\varphi \phi_r)^2\,\mathrm{d}\varphi + C\,|\dot{\delta}^t| \int_0^{r_1} \frac{r\,\mathrm{d}r}{g^t(r)} \int_0^{g^t(r)} |\tilde{\phi}_r|^2\,\mathrm{d}x_3 \\
&\leq C\,|\dot{\delta}^t| \int_0^{r_1} \frac{r^3\,\mathrm{d}r}{g^t(r)} \int_0^{g^t(r)} \mathrm{d}x_3 \int_0^{2\pi} \frac{1}{r^2} \left[\partial_\varphi \phi_r(r,\varphi,x_3) \right]^2 \mathrm{d}\varphi \\
&\quad + C\,|\dot{\delta}^t| \int_0^{r_1} \frac{r\,\mathrm{d}r}{g^t(r)} \int_0^{g^t(r)} \left[\int_{x_3(r)}^{x_3} \partial_y \tilde{\phi}_r(r,y)\,\mathrm{d}y \right]^2 \mathrm{d}x_3 \\
&\leq C\,|\dot{\delta}^t| \int_0^{r_1} r\,\mathrm{d}r \int_0^{g^t(r)} \mathrm{d}x_3 \int_0^{2\pi} \frac{1}{r^2} \left[\partial_\varphi \phi_r(r,\varphi,x_3) \right]^2 \mathrm{d}\varphi \\
&\quad + C\,|\dot{\delta}^t| \int_0^{r_1} g^t(r)\, r\,\mathrm{d}r \int_0^{g^t(r)} \left| \partial_{x_3} \tilde{\phi}_r(r,x_3) \right|^2 \mathrm{d}x_3 \;\leq\; C\,|\dot{\delta}^t| \int_{\Omega^t(r_1)} |\nabla \phi_r|^2\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

The generic constant C is always independent of t. The integrals of $(\partial_3 b_3^t)\,\phi_3^2$ and $(\partial_r b_3^t)\,\phi_r\,\phi_3$ can be treated similarly. (Here we can use the identity $\phi_3(r,\varphi,0)=0$.) Thus, we finally estimate the modulus of $\left(\phi \cdot \nabla \mathbf{b}^t,\, \phi\right)_{2;\Omega^t(r_1)}$ by $C\,|\dot{\delta}^t|\;\|\nabla \phi\|^2_{2;\Omega^t(r_1)}$.

An estimate of the surface integral of $(\mathbf{b}^t - \mathbf{V}^t)\cdot\phi$**.** We estimate the product $(\mathbf{b}^t - \mathbf{V}^t)\cdot\phi$ on the "lower part" $\Gamma_0^t(r_1) := \{\mathbf{x} = (r,\varphi,x_3) \in \Gamma^t;\; r < r_1,\; x_3 = 0\}$ and on the "upper part" $\Gamma_1^t(r_1) := \{\mathbf{x} = (r,\varphi,x_3) \in \Gamma^t;\; r < r_1,\; x_3 = g^t(r)\}$ of $\Gamma^t \cap \partial\Omega^t(r_1)$. Using the explicit forms of $\mathbf{b}^t - \mathbf{V}^t$ on $\Gamma_0^t(r_1)$ and $\Gamma_1^t(r_1)$ and the identity $\phi_3 = \partial_r g^t(r)\,\phi_r$ on $\Gamma_1^t(r_1)$ (following from the condition $\phi\cdot\mathbf{n}=0$), we get

$$
\begin{aligned}
&\left| \int_{\Gamma_0^t(r_1)} (\mathbf{b}^t - \mathbf{V}^t)\cdot\phi\;\mathrm{d}S + \int_{\Gamma_1^t(r_1)} (\mathbf{b}^t - \mathbf{V}^t)\cdot\phi\;\mathrm{d}S \right| \\
&= \left| \frac{\dot{\delta}^t}{2} \int_0^{r_1} \int_0^{2\pi} \left[\frac{r^2}{g^t(r)}\,\phi_r(r,\varphi,0) + \frac{r^2}{g^t(r)} \left(1 + \left[\partial_r g^t(r)\right]^2 \right) \phi_r\big(r,\varphi,g^t(r)\big) \right] \mathrm{d}\varphi\,\mathrm{d}r \right| \\
&= \left| \frac{\dot{\delta}^t}{2} \int_0^{r_1} \left[\frac{r^2}{g^t(r)}\,\tilde{\phi}_r(r,0) + \frac{r^2}{g^t(r)} \left(1 + \left[\partial_r g^t(r)\right]^2 \right) \tilde{\phi}_r\big(r,g^t(r)\big) \right] \mathrm{d}r \right| \\
&\leq C \left| \int_0^{r_1} \frac{r^2}{g^t(r)} \left(\int_{x_3(r)}^{0} \partial_\sigma \tilde{\phi}_r(r,\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}r \right| \\
&\quad + C \left| \int_0^{r_1} \frac{r^2}{g^t(r)} \left(1 + \left[\partial_r g^t(r)\right]^2 \right) \left(\int_{x_3(r)}^{g^t(r)} \partial_\sigma \tilde{\phi}_r(r,\sigma)\,\mathrm{d}\sigma \right) \mathrm{d}r \right| \\
&\leq \left| \int_0^{r_1} r \int_0^{g^t(r)} \left(\varepsilon\, \left| \partial_{x_3} \tilde{\phi}_r(r,x_3) \right|^2 + C(\varepsilon) \right) \mathrm{d}x_3\,\mathrm{d}r \right|.
\end{aligned}
$$

The generic constant again C does not depend on t. Choosing sufficiently small $\varepsilon > 0$, we obtain an inequality that further enables us to arrive at (12).

The initial–value problem (14). Suppose that $t_0 \in [0,T]$ and $\mathbf{x}_0 \in \Omega^{t_0} \smallsetminus l^{t_0}$ (where l^{t_0} is the open line segment in Ω^{t_0} with the end points $(0,0,0)$ and $(0,0,\delta^{t_0})$). Then the initial–value problem (14) has a unique solution $\mathbf{X}(t;t_0,\mathbf{x}_0)$ defined for $t\in[0,T]$ by the Carathéodory theorem. The trajectory of the solution stays in $\Omega^t\smallsetminus\ell^t$ (where ℓ^t is defined by analogy with ℓ^{t_0}) due to the condition $\mathbf{a}^t\cdot\mathbf{n}=\mathbf{V}^t\cdot\mathbf{n}$ satisfied by function $\mathbf{a}^t$ on Γ^t. The mapping $\mathbf{x}_0\mapsto\mathbf{X}(t;t_0,\mathbf{x}_0)$ is a 1–1 regular mapping of $\Omega^{t_0}\smallsetminus\ell^{t_0}$ onto $\Omega^t\smallsetminus\ell^t$, whose Jacobian equals one. Note that if $\mathbf{x}_0\in\Omega_{\mathrm{ext}}$ then $\mathbf{X}(t;t_0,\mathbf{x}_0)=\mathbf{x}_0$ independently of t and t_0 because $\mathbf{a}^t(\mathbf{x}_0,t)=\mathbf{0}$ for all $0\le t\le T$.

4 The time discretized boundary value problems

The time–discretization. Let $n\in\mathbb{N}$ and $k\in\{0;1;\dots;n\}$. We put $h:=T/n$, $t_k:=kh$, $\Omega_k:=\Omega^{t_k}$ and $\Gamma_k:=\Gamma^{t_k}$. We can assume without loss of generality that the critical time t_c of the collision differs from all the time instants t_k.

The stationary boundary value problems. We put $\mathbf{U}_0:=\mathbf{u}_0$. We successively solve, for $k=1,\dots,n$, a sequence of these stationary boundary value problems: given $\mathbf{U}_{k-1}\in L^2_\sigma(\Omega_{k-1})$ and $\mathbf{f}_k\in L^2(\Omega_k)^3$, we look for $\mathbf{U}_k$, P_k such that

$$\begin{aligned}
&\mathbf{U}_k(\mathbf{x})-\mathbf{U}_{k-1}\big(\mathbf{X}(t_{k-1};t_k,\mathbf{x})\big)+h\,\mathbf{U}_k(\mathbf{x})\cdot\big\{[\nabla\mathbf{a}]_k(\mathbf{x})+\nabla\mathbf{U}_k(\mathbf{x})\big\}+h\nabla P_k(\mathbf{x})\\
&\qquad=\nu\,h\big\{\mathrm{Div}\,[\nabla\mathbf{a}]_k(\mathbf{x})+\Delta\mathbf{U}_k(\mathbf{x})\big\}+\mathbf{A}_k(\mathbf{x})+h\mathbf{f}_k(\mathbf{x}) \qquad \text{in } \Omega_k, && (25)\\
&\mathrm{div}\,\mathbf{U}_k(\mathbf{x})=0 \qquad \text{in } \Omega_k, && (26)\\
&\mathbf{U}_k\cdot\mathbf{n}=0 \qquad \text{in } \Gamma_k, && (27)\\
&\big[(\mathbb{T}_\mathrm{d})_k\cdot\mathbf{n}\big]_\tau+K\,(\mathbf{a}_k+\mathbf{U}_k-\mathbf{V}_k)=\mathbf{0} \qquad \text{in } \Gamma_k, && (28)
\end{aligned}$$

The meaning of the functions $\mathbf{A}_k$, $[\nabla\mathbf{a}]_k$, $\mathbf{f}_k$, $\mathbf{a}_k$, $\mathbf{V}_k$ and $(\mathbb{T}_\mathrm{d})_k$ is explained below:

$$\mathbf{A}_k(\mathbf{x}):=-\mathbf{a}^{t_k}(\mathbf{x})+\mathbf{a}^{t_{k-1}}\big(\mathbf{X}(t_{k-1};t_k,\mathbf{x})\big)=-\int_{t_{k-1}}^{t_k}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}^t\big(\mathbf{X}(t;t_k,\mathbf{x})\big)\,\mathrm{d}t,$$

$$[\nabla\mathbf{a}]_k(\mathbf{x}):=\frac{1}{h}\int_{t_{k-1}}^{t_k}\nabla\mathbf{a}^t(\mathbf{x})\,\mathrm{d}t,\qquad \mathbf{f}_k(\mathbf{x}):=\frac{1}{h}\int_{t_{k-1}}^{t_k}\mathbf{f}(\mathbf{x},t)\,\mathrm{d}t$$

for $\mathbf{x}\in\Omega_k$ and $(\mathbb{T}_\mathrm{d})_k:=2\nu\big\{[\nabla\mathbf{a}]_k+\nabla\mathbf{U}_k\big\}_s$ on Γ_k. Denoting by $\mathbf{e}_3$ the unit vector $(0,0,1)$, we define for $\mathbf{x}\in\Gamma^s$ and $t,s\in[0,T]$ (such that $s\le t$)

$$\mathbf{Y}(t;s,\mathbf{x}):=\begin{cases}\mathbf{x}+\big(\delta^t-\delta^s\big)\,\mathbf{e}_3 & \text{if } \mathbf{x}\in\partial B^s,\\ \mathbf{x} & \text{if } \mathbf{x}\in\text{the } x_1,x_2\text{–plane.}\end{cases}\tag{29}$$

The mapping $\mathbf{x}\mapsto\mathbf{Y}(t;s,\mathbf{x})$ represents the shift of the "material point" $\mathbf{x}$ on the boundary of the flow field in the time interval $[s,t]$. Now we denote for $\mathbf{x}\in\Gamma_k$

$$\mathbf{a}_k(\mathbf{x}):=\frac{1}{h}\int_{t_{k-1}}^{t_k}\mathbf{a}^t\big(\mathbf{Y}(t;t_k,\mathbf{x})\big)\,\mathrm{d}t,\qquad \mathbf{V}_k(\mathbf{x}):=\frac{1}{h}\int_{t_{k-1}}^{t_k}\mathbf{V}^t\big(\mathbf{Y}(t;t_k,\mathbf{x})\big)\,\mathrm{d}t.$$

Note that the term $\mathbf{a}^t \cdot \nabla \mathbf{u}$, which appears in equation (1) if we write $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{a}^t + \mathbf{u}$, is now related to the difference at the beginning of (25):

$$\mathbf{U}_k(\mathbf{x}) - \mathbf{U}_{k-1}\big(\mathbf{X}(t_{k-1}; t_k, \mathbf{x})\big) = \int_{t_{k-1}}^{t_k} \nabla \mathbf{U}_k\big(\mathbf{X}(t; t_k, \mathbf{x})\big) \cdot \mathbf{a}^t\big(\mathbf{X}(t; t_k, \mathbf{x})\big)\, \mathrm{d}t.$$

The weak formulation of the BV problem (25)–(28). We can get rid of pressure P_k in the classical formulation (25)–(28) if we formally multiply equation (25) by a test function Φ_k from $W^{1,2}_\sigma(\Omega_k)$. Furthermore, we integrate by parts in the "viscous term" and we use the boundary conditions (27) and (28) in the same way as the conditions (3) and (4) were used in (16). Thus, we arrive at the weak formulation: *we look for* $\mathbf{U}_k \in W^{1,2}_\sigma(\Omega_k)$ *such that*

$$\begin{aligned}
&\int_{\Omega_k} \big\{\mathbf{U}_k(\mathbf{x}) - \mathbf{U}_{k-1}\big(\mathbf{X}(t_{k-1}; t_k, \mathbf{x})\big) + h\,\mathbf{U}_k(\mathbf{x}) \cdot \big\{[\nabla \mathbf{a}]_k(\mathbf{x}) + \nabla \mathbf{U}_k(\mathbf{x})\big\}\big\} \cdot \Phi_k(\mathbf{x})\, \mathrm{d}\mathbf{x} \\
&\qquad + \int_{\Omega_k} 2\nu\, h \big\{[\nabla \mathbf{a}]_k(\mathbf{x}) + \nabla \mathbf{U}_k(\mathbf{x})\big\}_s : \nabla \Phi_k(\mathbf{x})\, \mathrm{d}\mathbf{x} \\
&\qquad + \int_{\Gamma_k} K h \big[\mathbf{a}_k(\mathbf{x}) + \mathbf{U}_k(\mathbf{x}) - \mathbf{V}_k(\mathbf{x})\big] \cdot \Phi_k(\mathbf{x})\, \mathrm{d}S \\
&= \int_{\Omega_k} h \mathbf{f}_k(\mathbf{x}) \cdot \Phi_k(\mathbf{x})\, \mathrm{d}\mathbf{x} + \int_{\Omega_k} \mathbf{A}_k(\mathbf{x}) \cdot \Phi_k(\mathbf{x})\, \mathrm{d}\mathbf{x} \qquad (30)
\end{aligned}$$

for all $\Phi_k \in W^{1,2}_\sigma(\Omega_k)$. The solvability of this nonlinear elliptic problem can be proved by standard methods, particularly of theory of the steady Navier–Stokes equation. We refer e.g. to the book [9] by G. P. Galdi for the corresponding techniques. The coerciveness of an associated quadratic form follows from the next estimates.

Apriori estimates of solutions of the BV problem (30). Using $\Phi_k = \mathbf{U}_k$ in (30), we obtain:

$$\begin{aligned}
&\frac{1}{2}\, \|\mathbf{U}_k\|^2_{2;\Omega_k} + \frac{1}{2} \int_{\Omega_k} \big|\mathbf{U}_k(\mathbf{x}) - \mathbf{U}_{k-1}\big(\mathbf{X}(t_{k-1}; t_k \mathbf{x})\big)\big|^2\, \mathrm{d}\mathbf{x} + \nu\, h \int_{\Omega_k} (\nabla \mathbf{U}_k)_s : \nabla \mathbf{U}_k\, \mathrm{d}\mathbf{x} \\
&\quad + \int_{\Gamma_k} K h |\mathbf{U}_k|^2\, \mathrm{d}S \;\le\; \frac{1}{2}\, \|\mathbf{U}_{k-1}\|^2_{2;\Omega_{k-1}} + \left| h \int_{\Omega_k} \mathbf{f}_k \cdot \mathbf{U}_k\, \mathrm{d}\mathbf{x} \right| + \left| \int_{\Omega_k} \mathbf{A}_k \cdot \mathbf{U}_k\, \mathrm{d}\mathbf{x} \right| \\
&+ \left| h \int_{\Omega_k} \mathbf{U}_k \cdot [\nabla \mathbf{a}]_k \cdot \mathbf{U}_k\, \mathrm{d}\mathbf{x} \right| + \left| \nu\, h \int_{\Omega_k} \big([\nabla \mathbf{a}]_k\big)_s : \nabla \mathbf{U}_k\, \mathrm{d}\mathbf{x} \right| + \left| \int_{\Gamma_k} K\, (\mathbf{a}_k - \mathbf{V}_k) \cdot \mathbf{U}_k\, \mathrm{d}S \right|.
\end{aligned}$$

The integral of $(\nabla \mathbf{U}_k)_s : \nabla \mathbf{U}_k$ can be estimated from below by $\frac{1}{2}\, \|\nabla \mathbf{U}_k\|^2_{2;\Omega_k}$ by means of the integration by parts, the identity $\nabla(\mathbf{n} \cdot \mathbf{U}_k) \cdot \mathbf{U}_k = 0$ (valid on Γ_k) and the negative semi–definiteness of $\nabla \mathbf{n}$ on Γ_k. The integrals on the right hand side can be treated by analogy with the procedure explained in Section 1, which now leads us to a discrete variant of the energy inequality (18):

$$\|\mathbf{U}_j\|^2_{2;\Omega_j} + \sum_{k=1}^{j} \int_{\Omega_k} \big|\mathbf{U}_k(\mathbf{x}) - \mathbf{U}_{k-1}\big(\mathbf{X}(t_{k-1};t_k,\mathbf{x})\big)\big|^2 \,\mathrm{d}\mathbf{x} + \nu\, h \sum_{k=1}^{j} \|\nabla\mathbf{U}_k\|^2_{2;\Omega_k}$$
$$+\, 2Kh \sum_{k=1}^{j} \|\mathbf{U}_k\|^2_{2;\Gamma_k} \;\leq\; \|\mathbf{U}_0\|^2_{2;\Omega_0} + \sum_{k=1}^{j} \omega_k \|\mathbf{U}_k\|^2_{2;\Omega_k} + \sum_{k=1}^{j} g_k \tag{31}$$

for $j = 1,\dots,n$, where ω_k, g_k are certain positive numbers, depending on the same quantities as functions ω and G in (18), and satisfying the estimates $\sum_{k=1}^{n} \omega_k \leq c_{11}$ and $\sum_{k=1}^{n} g_k \leq c_{12}$ (with appropriate constants c_{11} and c_{12} independent of n).

5 The non–stationary approximations, their estimates and weak convergence

We define for $t_{k-1} < t \leq t_k$ (where $k = 1,\dots,n$)

$$\mathbf{u}^n(\mathbf{x},t) := \begin{cases} \mathbf{U}_k(\mathbf{x}) & \text{if } \mathbf{x} \in \overline{\Omega}_k, \\ \mathbf{0} & \text{if } \mathbf{x} \in \mathbb{R}^3_+ \smallsetminus \overline{\Omega}_k, \end{cases} \qquad \mathbb{U}^n(\mathbf{x},t) := \begin{cases} \nabla\mathbf{U}_k(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_k, \\ \mathbb{O} & \text{if } \mathbf{x} \in \mathbb{R}^3_+ \smallsetminus \Omega_k, \end{cases}$$

$$\mathbf{u}^n_*(\mathbf{x},t) := \mathbf{u}^n\big(\mathbf{Y}(t_k;t,\mathbf{x}),t\big) \;=\; \mathbf{U}_k\big(\mathbf{Y}(t_k;t,\mathbf{x})\big) \quad \text{if } \mathbf{x} \in \Gamma^t.$$

Estimates of the sequences $\{\mathbf{u}^n\}$, $\{\mathbb{U}^n\}$ and $\{\mathbf{u}^n_*\}$. Inequality (31) implies that there exist $c_{13}(h) > 0$ and $c_{14}(h) > 0$ such that both $c_{13}(h)$ and $c_{14}(h)$ tend to zero as $h \to 0+$ and

$$\big[1 - c_{13}(h)\big]\, \|\mathbf{u}^n(\,.\,,t)\|^2_{2;\mathbb{R}^3_+} + \nu \int_0^t \|\mathbb{U}^n(\,.\,,s)\|^2_{2;\mathbb{R}^3_+}\,\mathrm{d}s + 2K \int_0^t \|\mathbf{u}^n_*(\,.\,,s)\|^2_{2;\Gamma^s}\,\mathrm{d}s$$
$$\leq\; \|\mathbf{u}_0\|^2_{2;\Omega^0} + \int_0^t \lambda_n(s)\, \|\mathbf{u}^n(\,.\,,s)\|^2_{2;\mathbb{R}^3_+}\,\mathrm{d}s + c_{12} + c_{14}(h) \tag{32}$$

where $\lambda_n(s) := \omega_k$ for $t_{k-1} < s \leq t_k$. Applying Gronwall's lemma, we deduce that there exists $c_{15} > 0$ (depending on c_{11}, c_{12} and $\|\mathbf{u}_0\|_{2;\Omega^0}$) such that for all $n \in \mathbb{N}$ so large that $c_{13}(h) \leq \frac{1}{2}$ and for all $t \in (0,T)$, we have

$$\|\mathbf{u}^n(\,.\,,t)\|_{2;\mathbb{R}^3_+} \;\leq\; c_{15}\,. \tag{33}$$

Using this estimate in (32), we observe that there exist c_{16} and c_{17} independent of n and such that

$$\int_0^T \|\mathbb{U}^n(\,.\,,s)\|^2_{2;\mathbb{R}^3_+}\,\mathrm{d}s \;\leq\; c_{16}\,, \qquad \int_0^T \|\mathbf{u}^n_*(\,.\,,s)\|^2_{2;\Gamma^s}\,\mathrm{d}s \;\leq\; c_{17}\,. \tag{34}$$

Inequalities (33) and (34) conversely yield:

$$\|\mathbf{U}_k\|_{2;\Omega_k} \leq c_{15} \quad (k=1,\dots,n) \qquad \text{and} \qquad h\sum_{k=1}^{n} \|\nabla \mathbf{U}_k\|^2_{2;\Omega_k} \leq c_{16}\,. \tag{35}$$

Weak convergence of selected subsequences. Estimates (33) and (34) imply that there exist subsequences of $\{\mathbf{u}^n\}$, $\{\mathbb{U}^n\}$ and $\{\mathbf{u}^n_*\}$ (we shall denote them again by $\{\mathbf{u}^n\}$, $\{\mathbb{U}^n\}$ and $\{\mathbf{u}^n_*\}$ in order not to complicate the notation) and functions $\mathbf{u} \in L^\infty(0,T;L^2(\mathbb{R}^3_+)^3)$, $\mathbb{U} \in L^2(0,T;L^2(\mathbb{R}^3_+)^9)$ and $\mathbf{u}_* \in L^2(\Gamma_{(0,T)})^3$ such that

$$\mathbf{u}^n \rightharpoonup \mathbf{u} \qquad \text{weakly–}* \text{ in } L^\infty(0,T;L^2(\mathbb{R}^3_+)^3) \qquad \text{for } n\to+\infty, \tag{36}$$

$$\mathbb{U}^n \rightharpoonup \mathbb{U} \qquad \text{weakly in } L^2(0,T;L^2(\mathbb{R}^3_+)^9) \qquad \text{for } n\to+\infty, \tag{37}$$

$$\mathbf{u}^n_* \rightharpoonup \mathbf{u}_* \qquad \text{weakly in } L^2(\Gamma_{(0,T)})^3 \qquad \text{for } n\to+\infty \tag{38}$$

with the following relations between $\mathbf{u}$, $\mathbb{U}$ and $\mathbf{u}_*$:

Lemma 2. *a)* $\mathbb{U} = \nabla\mathbf{u}$ *in the sense of distributions in* $Q_{(0,T)}$,
b) $\mathbf{u} \in L^2(0,T;W^{1,2}_\sigma(\Omega^t))$,
c) $\mathbf{u}_* = \mathbf{u}$ *on* $\Gamma_{(0,T)}$ *(here* $\mathbf{u}$ *denotes the trace of function* $\mathbf{u}\big|_{Q_{(0,T)}}$ *on* $\Gamma_{(0,T)}$*).*

The proof can be made by standard techniques.

6 The limit function u: a solution of the weak problem (17)

Suppose that ϕ is a fixed infinitely differentiable divergence–free vector–function in $\overline{\mathbb{R}}^3_+ \times [0,T]$ with a compact support in $\overline{\mathbb{R}}^3_+ \times [0,T)$, such that $\phi\cdot\mathbf{n} = 0$ on $\Gamma_{[0,T]}$.

Using the relation between $\mathbf{u}^n$ and the solutions of the steady weak problem (30), one can verify that $\mathbf{u}^n$ (with $\mathbb{U}^n$ standing for $\nabla\mathbf{u}^n$ and $\mathbf{u}_*$ standing for the trace on $\Gamma_{(0,T)}$) satisfies the non–steady weak problem (17), up to a correction which tends to zero as $n\to+\infty$. (The intermediate step is to use (30) with $\Phi_k = \phi(\,.\,,t_k)$.)

Applying (36)–(38), we can pass to the limit as $n\to+\infty$ in all the linear terms. Thus, the limit of the nonlinear term (the integral of $\mathbf{u}^n\cdot\mathbb{U}^n\cdot\phi$) also exists. So we obtain:

$$\begin{aligned}
&\int_0^T \int_{\Omega^t} \big\{-[\partial_t\phi + \mathbf{a}^t\cdot\nabla\phi]\cdot(\mathbf{a}^t+\mathbf{u}) - \mathbf{u}\cdot\nabla\phi\cdot\mathbf{a}^t + 2\nu\,[\nabla(\mathbf{a}^t+\mathbf{u})]_s : \nabla\phi\,\big\}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\qquad + \lim_{n\to+\infty} \int_0^T \int_{\Omega^t} \mathbf{u}^n\cdot\mathbb{U}^n\cdot\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_0^T \int_{\Gamma^t} K\,[\mathbf{a}^t+\mathbf{u}-\mathbf{V}^t]\cdot\phi\,\mathrm{d}S\,\mathrm{d}t \\
&\qquad\qquad = \int_0^T \int_{\Omega^t} \mathbf{f}\cdot\phi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\Omega^0} (\mathbf{a}^0+\mathbf{u}_0)\cdot\phi(\,.\,,0)\,\mathrm{d}\mathbf{x}.
\end{aligned} \tag{39}$$

Comparing (39) with (17), we observe that in order to verify that $\mathbf{u}$ is a solution of the weak problem (17), **it is sufficient to show that there exists a subsequence of**

$\{\mathbf{u}^n\}$ (we shall denote it again by $\{\mathbf{u}^n\}$) such that

$$\lim_{n\to+\infty} \int_0^T \int_{\Omega^t} \mathbf{u}^n \cdot \mathbb{U}^n \cdot \phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}t \;=\; \int_0^T \int_{\Omega^t} \mathbf{u} \cdot \nabla\mathbf{u} \cdot \phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}t. \tag{40}$$

This limit procedure is not standard because of the variability of domain Ω^t and the choice of the test function ϕ, which generally has only the normal component equal to zero on $\Gamma_{(0,T)}$. We explain it in greater detail in the next six paragraphs.

Cutting–off function ϕ. Let $\varepsilon_1 > 0$ be given. Then, due to (33) and (34), there exists $\kappa_1 > 0$ so small that

$$\begin{aligned} \left| \int_{t_c-\kappa_1}^{t_c+\kappa_1} \int_{\Omega^t} \mathbf{u}^n \cdot \mathbb{U}^n \cdot \phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}t \right| &\leq c_{18} \left[\operatorname*{ess\,sup}_{0<t<T} \|\mathbf{u}^n(\,.\,,t)\|_{2;\Omega^t} \right] \int_{t_c-\kappa_1}^{t_c+\kappa_1} \|\mathbb{U}^n(\,.\,,t)\|_{2;\Omega^t} \, \mathrm{d}\mathbf{x} \\ &\leq c_{18}\, c_{15}\, \sqrt{2\kappa_1 c_{16}} \;<\; \varepsilon_1 \end{aligned} \tag{41}$$

for all $n \in \mathbb{N}$ sufficiently large. (Here c_{18} is the maximum of $|\phi|$ on $\mathbb{R}^3_+ \times [0,T]$.) Let η_3 be an infinitely differentiable cut–off function of variable t defined on the interval $[0,T]$, with values in $[0,1]$, such that

$$\eta_3(t) \;:=\; \begin{cases} 1 \text{ for } & t \in [0, t_c - \kappa_1] \cup [t_c + \kappa_1, T], \\ 0 \text{ for } & t \in \left[t_c - \frac{1}{2}\kappa_1, t_c + \frac{1}{2}\kappa_1\right], \end{cases}$$

The function $\phi^*(\mathbf{x},t) := \eta_3(t)\,\phi(\mathbf{x},t)$ equals zero for $t_c - \frac{1}{2}\kappa_1 \leq t \leq t_c + \frac{1}{2}\kappa_1$ and

$$\left| \int_{t_c-\kappa_1}^{t_c+\kappa_1} \int_{\Omega^t} \mathbf{u}^n \cdot \mathbb{U}^n \cdot (\phi - \phi^*) \, \mathrm{d}\mathbf{x}\, \mathrm{d}t \right| \;<\; \varepsilon_1$$

due to (41). Since ε_1 can be chosen arbitrarily small, it is sufficient to prove (40) with function ϕ^* instead of ϕ.

Approximation of function ϕ^*. Since each of the domains Ω_k (for $k = 1, \dots, n$) has the cone property (because all the time instants t_k differ from t_c), inequalities (35) and the Sobolev imbedding theorem imply that $\mathbf{U}_k \in L^6(\Omega_k)^3$. This means that $\mathbf{u}^n(\,.\,,t) \in L^6(\mathbb{R}^3_+)^3$ for all $t \in (0,T)$. Moreover, if we restrict ourselves to times $t \in I(\kappa_1)$, where

$$I(\kappa_1) \;:=\; \left[0, t_c - \tfrac{1}{2}\kappa_1\right) \cup \left(t_c + \tfrac{1}{2}\kappa_1, T\right],$$

then the cone parameters in the definition of the cone property of domain Ω^t can be chosen to be independent of t. Hence the constants in the imbedding inequalities also become independent of t and we obtain the uniform estimate $\|\mathbf{u}^n(\,.\,,t)\|_{6;\mathbb{R}^3_+} \leq C\left(\|\mathbf{u}^n(\,.\,,t)\|_{2;\mathbb{R}^3_+} + \|\mathbb{U}^n(\,.\,,t)\|_{2;\mathbb{R}^3_+}\right)$ for all $t \in I(\kappa_1)$. From this information and from (34), we can deduce that the product $\mathbf{u}^n \cdot \mathbb{U}^n$ belongs to $L^2\big(I(\kappa_1); L^1(\mathbb{R}^3_+)^3\big) \cap L^1\big(I(\kappa_1); L^{3/2}(\mathbb{R}^3_+)^3\big)$. By interpolation, we obtain the inclusion $\mathbf{u}^n \cdot \mathbb{U}^n \in L^r\big(I(\kappa_1); L^s(\mathbb{R}^3_+)^3\big)$ for $r \geq 1$, $s \geq 1$ such that $2/r + 3/s = 4$. Particularly, $\mathbf{u}^n \cdot \mathbb{U}^n \in L^{5/4}\big(I(\kappa_1); L^{5/4}(\mathbb{R}^3_+)^3\big)$.

Function ϕ^* can be approximated by infinitely differentiable divergence–free vector–functions that have a compact support in $Q_{[0,T)}$ with an arbitrary accuracy in the norm of the space $L^5\big(I(\kappa_1);\, L^5(\Omega^t)^3\big)$. Hence, given $\varepsilon_2>0$, there exists such a vector–function ϕ^{**} which satisfies

$$\left| \int_0^T \int_{\Omega^t} \mathbf{u}^n \cdot \mathbb{U}^n \cdot \phi^* \, \mathrm{d}\mathbf{x} - \int_0^T \int_{\Omega^t} \mathbf{u}^n \cdot \mathbb{U}^n \cdot \phi^{**} \, \mathrm{d}\mathbf{x} \right| \; < \; \varepsilon_2$$

for all $n\in\mathbb{N}$ sufficiently large. Since ε_2 can be chosen to be arbitrarily small, we can prove (40) only with the function ϕ^{**} instead of ϕ (respectively instead of ϕ^*).

Partition of function ϕ^{**}. Let $m\in\mathbb{N}$. We denote $\tau_j = jT/m$ (for $j=0,\dots,m$). There exist $m+1$ infinitely differentiable functions $\theta_0,\,\dots,\,\theta_m$ on $[0,T]$ with their values in the interval $[0,1]$ such that $\operatorname{supp}\theta_0 \subset I_0 := [\tau_0,\tau_1)$, $\operatorname{supp}\theta_j \subset I_j := (\tau_{j-1},\tau_{j+1})$ (for $j=1,\dots,m-1$), $\operatorname{supp}\theta_m \subset I_m := (\tau_{m-1},\tau_m]$ and $\sum_{j=0}^m \theta_j(t) = 1$ for $0\le t\le T$. Now we put $\phi_j^{**} := \theta_j\,\phi^{**}$ (for $j=0,1,\dots,m$). The functions ϕ_j^{**} are divergence–free, they have compact supports in Q_{I_j} (where $Q_{I_j} = \big\{(\mathbf{x},t)\in \mathbb{R}^3\times[0,T];\ t\in I_j,\ \mathbf{x}\in\Omega^t\big\}$) and

$$\sum_{j=0}^m \phi_j^{**} \; = \; \phi^{**} \qquad \text{in } \; Q_{[0,T]}.$$

Denote by K_j be the orthogonal projection of $\operatorname{supp}\phi_j^{**}$ into $\mathbb{R}^3$. If m is large enough then the distance between K_j and Γ^t is greater than one half of the distance between $\operatorname{supp}\phi^{**}$ and $\Gamma_{[0,T]}$ for all $t\in I_j$. Thus, there exists a bounded open set Ω_j^* in $\mathbb{R}^3$ with the boundary of the class $C^{1,1}$ such that $K_j\subset\Omega_j^*\subset\overline{\Omega_j^*}\subset\Omega^t$ for all $t\in I_j$. So, we conclude that in order to prove (40), it is sufficient to treat (40) separately with $\phi=\phi_j^{**}$ (for $j=0,1,\dots,m$) and to show that

$$\lim_{n\to+\infty} \int_{I_j}\int_{\Omega_j^*} \mathbf{u}^n\cdot\nabla\mathbf{u}^n\cdot\phi_j^{**}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;=\; \int_{I_j}\int_{\Omega_j^*} \mathbf{u}\cdot\nabla\mathbf{u}\cdot\phi_j^{**}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t. \tag{42}$$

The local Helmholtz decomposition of function $\mathbf{u}^n$. We denote by P_σ^j the Helmholtz projection in $L^2(\Omega_j^*)^3$. Put $\mathbf{w}_n^j := P_\sigma^j\mathbf{u}^n$. The function $(I-P_\sigma^j)\mathbf{u}^n$ has the form $\nabla\varphi_n^j$ for an appropriate scalar function φ_n^j. (42) can now be written as

$$\begin{aligned}\lim_{n\to+\infty} \int_{I_j}\int_{\Omega_j^*} \Big[& \mathbf{w}_n^j\cdot\nabla\mathbf{w}_n^j\cdot\phi_j^{**} + \mathbf{w}_n^j\cdot\nabla^2\varphi_n^j\cdot\phi_j^{**} + \nabla\varphi_n^j\cdot\nabla\mathbf{w}_n^j\cdot\phi_j^{**} \\ & + \nabla\varphi_n^j\cdot\nabla^2\varphi_n^j\cdot\phi_j^{**}\Big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \;=\; \int_{I_j}\int_{\Omega_j^*} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\phi_j^{**}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.\end{aligned} \tag{43}$$

Since $\nabla\varphi_n^j\cdot\nabla^2\varphi_n^j = \nabla\big(\frac{1}{2}|\nabla\varphi_n^j|^2\big)$ and $\phi_j^{**}(\,.\,,t)\in L^2_\sigma(\Omega_j^*)$, the integral of $\nabla\varphi_n^j\cdot\nabla^2\varphi_n^j\cdot\phi_j^{**}$ on Ω_j^* equals zero.

The convergence (36) and (37), the coincidence of $\mathbb{U}^n$ with $\nabla\mathbf{u}^n$ on $\Omega_j^*\times I_j$ and the boundedness of operator P_σ^j in $L^2(\Omega_j^*)^3$ and in $W^{1,2}(\Omega_j^*)^3$ imply that

$$\mathbf{w}_n^j \rightharpoonup \mathbf{w}^j = P_\sigma^j \mathbf{u}, \quad \text{and} \quad \nabla \varphi_n^j \rightharpoonup \nabla \varphi^j = (I - P_\sigma^j)\mathbf{u} \quad \text{for } n \to +\infty \tag{44}$$

weakly in $L^2(I_j; W^{1,2}(\Omega_j^*)^3)$ and weakly–$*$ in $L^\infty(I_j; L^2_\sigma(\Omega_j^*))$.

Strong convergence of a subsequence of $\{\mathbf{w}_n^j\}$. We are going to show that there exists a subsequence of $\{\mathbf{w}_n^j\}$ that tends to $\mathbf{w}^j$ strongly in $L^2(I_j; L^2_\sigma(\Omega_j^*))$ as $n \to +\infty$. We shall therefore use the next lemma, see J. L. Lions [18, Theorem 5.2].

Lemma 3. *Let $0 < \gamma < \frac{1}{2}$ and let H_0, H and H_1 be Hilbert spaces such that $H_0 \hookrightarrow\hookrightarrow H \hookrightarrow H_1$. Let $\mathcal{H}^\gamma(\mathbb{R}; H_0, H_1)$ denote the Banach space $\big\{ w \in L^2(\mathbb{R}; H_0);\ |\vartheta|^\gamma\, \hat{w}(\vartheta) \in L^2(\mathbb{R}; H_1) \big\}$ with the norm*

$$|\!|\!| w |\!|\!|_{\gamma;\mathbb{R}} \ := \ \big(\|w\|^2_{L^2(\mathbb{R};H_0)} + \|\, |\vartheta|^\gamma\, \hat{w}(\vartheta)\|^2_{L^2(\mathbb{R};H_1)} \big)^{1/2}.$$

(Here $\hat{w}(\vartheta)$ is the Fourier transform of $w(t)$.) Let $\mathcal{H}^\gamma(a,b; H_0, H_1)$ further denote the Banach space of restrictions of functions from $\mathcal{H}^\gamma(\mathbb{R}; H_0, H_1)$ onto the interval (a,b), with the norm

$$|\!|\!| w |\!|\!|_{\gamma;(a,b)} \ := \ \inf\ |\!|\!| z |\!|\!|_{\gamma;\mathbb{R}}$$

where the infimum is taken over all $z \in \mathcal{H}^\gamma(\mathbb{R}; H_0, H_1)$ such that $z = w$ a.e. in (a,b). Then $\mathcal{H}^\gamma(0,T; H_0, H_1) \hookrightarrow\hookrightarrow L^2(a,b; H)$.

Consider $j \in \{1; \dots; m\}$ fixed. We shall use Lemma 3 with $(a,b) = I_j$, $H_0 = W^{1,2}_\sigma(\Omega_j^*)$, $H = L^2_\sigma(\Omega_j^*)$ and $H_1 = W^{-1,2}_{0,\sigma}(\Omega_j^*)$. (Here $W^{-1,2}_{0,\sigma}(\Omega_j^*)$ denotes the dual to $W^{1,2}_{0,\sigma}(\Omega_j^*)$ where $W^{1,2}_{0,\sigma}(\Omega_j^*)$ is the closure of $C^\infty_{0,\sigma}(\Omega_j^*)$ in $W^{1,2}(\Omega_j^*)^3$. The space $W^{1,2}_{0,\sigma}(\Omega_j^*)$ can be characterized as the space of functions from $W^{1,2}_\sigma(\Omega_j^*)$ that have the trace on $\partial\Omega_j^*$ equal to zero.) We claim that $\{\mathbf{w}_n^j\}$ is bounded in the space $\mathcal{H}^\gamma(I_j; H_0, H_1)$. The boundedness of $\{\mathbf{w}_n^j\}$ in $L^2(I_j; H_0)$ follows from (33), (34), from the coincidence of $\mathbb{U}^n$ with $\nabla \mathbf{u}^n$ on $\Omega_j^* \times I_j$ and from the boundedness of operator P_σ^j in $L^2(\Omega_j^*)^3$ and in $W^{1,2}(\Omega_j^*)^3$. Thus, we only need to verify that $\{|\vartheta|^\gamma\, \hat{\mathbf{w}}_n^j\}$ is bounded in the space $L^2(I_j; H_1)$, i.e. in $L^2(I_j; W^{-1,2}_{0,\sigma}(\Omega_j^*))$. Let $\mathbf{z}_n^j$ be an extension by zero of $\mathbf{w}_n^j$ from the time interval I_j onto $\mathbb{R}$. Then

$$\hat{\mathbf{z}}_n^j(\vartheta) \ = \ \int_{-\infty}^{+\infty} \mathrm{e}^{-2\pi\,\mathrm{i}\,t\vartheta}\, \mathbf{w}_n^j(t)\, \mathrm{d}t \ = \ \sum_{k \in \Lambda_j^n} \int_{t_{k-1}}^{t_k} \mathrm{e}^{-2\pi\,\mathrm{i}\,t\vartheta}\, P_\sigma^j \mathbf{U}_k\, \mathrm{d}t \tag{45}$$

where Λ_j^n is the set of such indices $k \in \{1; \dots; n\}$ that $[\mathbb{R}^3 \times (t_{k-1}, t_k)] \cap \operatorname{supp} \phi_j^{**} \neq \emptyset$. Λ_j^n has the form $\Lambda_j^n = \{l; l+1; \dots; q\}$ where $1 \le l \le q \le n$. Calculating the integrals in (45), we obtain

$$\hat{\mathbf{z}}_n^j(\vartheta) \ = \ \sum_{k=l}^{q} \frac{1}{2\pi\,\mathrm{i}\,\vartheta} \left[\mathrm{e}^{-2\pi\,\mathrm{i}\,t_{k-1}\vartheta} - \mathrm{e}^{-2\pi\,\mathrm{i}\,t_k\vartheta} \right] P_\sigma^j \mathbf{U}_k$$

$$= \ \frac{1}{2\pi\,\mathrm{i}\,\vartheta} \left[\mathrm{e}^{-2\pi\,\mathrm{i}\,t_{l-1}}\, P_\sigma^j \mathbf{U}_l - \mathrm{e}^{-2\pi\,\mathrm{i}\,t_q}\, P_\sigma^j \mathbf{U}_q \right] + \frac{1}{2\pi\,\mathrm{i}\,\vartheta} \sum_{k=l+1}^{q} \mathrm{e}^{-2\pi\,\mathrm{i}\,t_{k-1}\vartheta}\, [P_\sigma^j \mathbf{U}_k - P_\sigma^j \mathbf{U}_{k-1}].$$

Since $\boldsymbol{\Omega}_j^* \subset \boldsymbol{\Omega}^s$ for all $s \in I_j$, we also have $\boldsymbol{\Omega}_j^* \subset \boldsymbol{\Omega}_k$ for all $k \in \Lambda_j^n$ (if n is large enough). If $|\vartheta| \le 1$ then, using (45) and (35), we can estimate the norm of $|\vartheta|^\gamma\, \hat{\mathbf{z}}_n^j(\vartheta)$ in $W_{0,\sigma}^{-1,2}(\boldsymbol{\Omega}_j^*)$ as follows:

$$\big\|\, |\vartheta|^\gamma\, \hat{\mathbf{z}}_n^j(\vartheta) \big\|_{-1,2;\boldsymbol{\Omega}_j^*} \le C(\boldsymbol{\Omega}_j^*)\, |\vartheta|^\gamma \sum_{k=l}^{q} h\, \|\mathbf{U}_k\|_{2:\boldsymbol{\Omega}_j^*} \;\le\; C(\boldsymbol{\Omega}_j^*)\; |\vartheta|^\gamma. \tag{46}$$

If $|\vartheta| > 1$ then we must proceed more subtly:

$$\begin{aligned}
\big\|\, |\vartheta|^\gamma\, \hat{\mathbf{z}}_n^j(\vartheta) \big\|_{-1,2;\boldsymbol{\Omega}_j^*} &\le \frac{|\vartheta|^{\gamma-1}}{2\pi} \left(\|P_\sigma^j \mathbf{U}_l\|_{-1,2;\boldsymbol{\Omega}_j^*} + \|P_\sigma^j \mathbf{U}_q\|_{-1,2;\boldsymbol{\Omega}_j^*} \right) \\
&\quad + \frac{|\vartheta|^{\gamma-1}}{2\pi} \sum_{k=l+1}^{q} \|P_\sigma^j \mathbf{U}_k - P_\sigma^j \mathbf{U}_{k-1}\|_{-1,2;\boldsymbol{\Omega}_j^*} \\
&\le C(\boldsymbol{\Omega}_j^*)\, |\vartheta|^{\gamma-1} \left(\|\mathbf{U}_l\|_{2:\boldsymbol{\Omega}_j^*} + \|\mathbf{U}_q\|_{2:\boldsymbol{\Omega}_j^*} \right) \\
&\quad + \frac{|\vartheta|^{\gamma-1}}{2\pi} \sum_{k=l+1}^{q} \sup_{\psi_k} \frac{1}{\|\psi_k\|_{1,2;\boldsymbol{\Omega}_j^*}} \left| \int_{\boldsymbol{\Omega}_j^*} (\mathbf{U}_k - \mathbf{U}_{k-1}) \cdot \psi_k \, \mathrm{d}\mathbf{x} \right|
\end{aligned} \tag{47}$$

where the supremum is taken over all $\psi_k \in W_{0,\sigma}^{1,2}(\boldsymbol{\Omega}_j^*)$ such that $\|\psi_k\|_{1,2;\boldsymbol{\Omega}_j^*} > 0$. The sum in (47) can be estimated by $\mathcal{S}_1 + \mathcal{S}_2$ where

$$\begin{aligned}
\mathcal{S}_1 &= \sum_{k=l+1}^{q} \sup_{\psi_k} \frac{1}{\|\psi_k\|_{1,2;\boldsymbol{\Omega}_j^*}} \left| \int_{\boldsymbol{\Omega}_j^*} \left[\mathbf{U}_k(\mathbf{x}) - \mathbf{U}_{k-1}\big(\mathbf{X}(t_{k-1}; t_k, \mathbf{x})\big) \right] \cdot \psi_k(\mathbf{x})\, \mathrm{d}\mathbf{x} \right|, \\
\mathcal{S}_2 &= \sum_{k=l+1}^{q} \sup_{\psi_k} \frac{1}{\|\psi_k\|_{1,2;\boldsymbol{\Omega}_j^*}} \left| \int_{\boldsymbol{\Omega}_j^*} \left[\mathbf{U}_{k-1}(\mathbf{x}) - \mathbf{U}_{k-1}\big(\mathbf{X}(t_{k-1}; t_k, \mathbf{x})\big) \right] \cdot \psi_k(\mathbf{x})\, \mathrm{d}\mathbf{x} \right|.
\end{aligned}$$

The function ψ_k, extended by zero to $\mathbb{R}^3_+ \smallsetminus \boldsymbol{\Omega}_j^*$, belongs to $W_\sigma^{1,2}(\boldsymbol{\Omega}_k)$. Hence the integral of $\left[\mathbf{U}_k(\mathbf{x}) - \mathbf{U}_{k-1}\big(\mathbf{X}(t_{k-1}; t_k, \mathbf{x})\big)\right] \cdot \psi_k(\mathbf{x})$ on $\boldsymbol{\Omega}_j^*$ equals the integral of the same function in $\boldsymbol{\Omega}_k$ and it can be therefore expressed by means of (30). Thus, $\mathcal{S}_1$ can be estimated:

$$\begin{aligned}
\mathcal{S}_1 \le \sum_{k=l+1}^{q} \sup_{\psi_k} \frac{1}{\|\psi_k\|_{1,2;\boldsymbol{\Omega}_j^*}} \bigg| &-h \int_{\boldsymbol{\Omega}_k} \mathbf{U}_k(\mathbf{x}) \cdot [\nabla \mathbf{a}]_k(\mathbf{x}) \cdot \psi_k(\mathbf{x})\, \mathrm{d}\mathbf{x} \\
&- h \int_{\boldsymbol{\Omega}_k} \mathbf{U}_k(\mathbf{x}) \cdot \nabla \mathbf{U}_k(\mathbf{x}) \cdot \psi_k(\mathbf{x})\, \mathrm{d}\mathbf{x} - h \int_{\boldsymbol{\Omega}_k} \nu \left\{ [\nabla \mathbf{a}]_k(\mathbf{x}) + \nabla \mathbf{U}_k(\mathbf{x}) \right\}_s : \nabla \psi_k(\mathbf{x})\, \mathrm{d}\mathbf{x} \\
&- \int_{\Gamma_k} K \left[\mathbf{a}_k(\mathbf{x}) + \mathbf{U}_k(\mathbf{x}) - \mathbf{V}_k(\mathbf{x}) \right] \cdot \psi_k(\mathbf{x})\, \mathrm{d}S \\
&+ \int_{\boldsymbol{\Omega}_k} h \mathbf{f}_k(\mathbf{x}) \cdot \psi_k(\mathbf{x})\, \mathrm{d}\mathbf{x} + \int_{\boldsymbol{\Omega}_k} \mathbf{A}_k(\mathbf{x}) \cdot \psi_k(\mathbf{x})\, \mathrm{d}\mathbf{x} \bigg|.
\end{aligned}$$

The surface integral on Γ_k equals zero because the function ψ_k is zero on Γ_k. The right hand side can be estimated by $C(\boldsymbol{\Omega}_j^*)$ by means of (7), (35), standard inequalities based on the Sobolev imbedding theorem (applied in $\boldsymbol{\Omega}_j^*$) and the Hölder inequality. Let us show the procedure in greater detail, for example, in the case of the terms containing the product $\mathbf{U}_k\cdot\nabla\mathbf{U}_k\cdot\psi_k$:

$$\begin{aligned}
&\sum_{k=l+1}^{q}\sup_{\psi_k}\frac{1}{\|\psi_k\|_{1,2;\Omega_j^*}}\left|h\int_{\Omega_k}\mathbf{U}_k\cdot\nabla\mathbf{U}_k\cdot\psi_k\,\mathrm{d}\mathbf{x}\right|\\
&\le C(\Omega_j^*)\left(\sum_{k=l+1}^{q}h\int_{\Omega_k}|\nabla\mathbf{U}_k|^2\,\mathrm{d}\mathbf{x}\right)^{1/2}\left(\sum_{k=l+1}^{q}h\int_{\Omega_k}|\mathbf{U}_k|^3\,\mathrm{d}\mathbf{x}\right)^{1/3}\\
&\le C(\Omega_j^*)\left(h\sum_{k=l+1}^{q}\|\nabla\mathbf{U}_k\|_{2;\Omega_k}^{3/2}\,\|\mathbf{U}_k\|_{6;\Omega_k}^{3/2}\right)^{1/3}\\
&\le C(\Omega_j^*)\left[h\sum_{k=l+1}^{q}\|\nabla\mathbf{U}_k\|_{2;\Omega_k}^{3/2}\left(\|\mathbf{U}_k\|_{2;\Omega_k}^{3/2}+\|\nabla\mathbf{U}_k\|_{2;\Omega_k}^{3/2}\right)\right]^{1/3}\\
&\le C(\Omega_j^*)\left[1+\left(h\sum_{k=1}^{n}\|\nabla\mathbf{U}_k\|_{2;\Omega_k}^{2}\right)^{3/4}\right]^{1/3}\le\ C(\Omega_j^*).
\end{aligned}$$

Here the constant $C(\Omega_j^*)$ also depends on the right hand sides of (7) and (35). In order to estimate $\mathcal{S}_2$, we use the identities

$$\begin{aligned}
\mathbf{U}_{k-1}(\mathbf{x})-\mathbf{U}_{k-1}\big(\mathbf{X}(t_{k-1};t_k,\mathbf{x})\big)\ &=\ \int_{t_{k-1}}^{t_k}\frac{\mathrm{d}}{\mathrm{d}\xi}\,\mathbf{U}_{k-1}\big(\mathbf{X}(\xi;t_k,\mathbf{x})\big)\,\mathrm{d}\xi\\
&=\int_{t_{k-1}}^{t_k}\mathbf{a}^{\xi}\big(\mathbf{X}(\xi;t_k,\mathbf{x})\big)\cdot\nabla\mathbf{U}_{k-1}\big(\mathbf{X}(\xi;t_k,\mathbf{x})\big)\,\mathrm{d}\xi\,.
\end{aligned}$$

Then the sum $\mathcal{S}_2$ can be estimated by means of (10) and (35) as follows:

$$\begin{aligned}
\mathcal{S}_2&\le C(\Omega_j^*)\,\sup_{\psi_k}\frac{\|\psi_k\|_{6;\Omega_j^*}}{\|\psi_k\|_{1,2;\Omega_j^*}}\sum_{k=l+1}^{q}\left[\int_{t_{k-1}}^{t_k}\int_{\Omega_j^*}\big|\nabla\mathbf{U}_{k-1}\big(\mathbf{X}(\xi;t_k,\mathbf{x})\big)\big|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\xi\right]^{1/2}\\
&\qquad\cdot\left[\int_{t_{k-1}}^{t_k}\left(\int_{\Omega_j^*}\big|\mathbf{a}^{\xi}\big(\mathbf{X}(\xi;t_k,\mathbf{x})\big)\big|^3\,\mathrm{d}\mathbf{x}\right)^{2/3}\mathrm{d}\xi\right]^{1/2}\\
&\le C(\Omega_j^*)\left[\sum_{k=l+1}^{q}\int_{t_{k-1}}^{t_k}\int_{\Omega_{k-1}}\big|\nabla\mathbf{U}_{k-1}(\mathbf{x})\big|^2\,\mathrm{d}\mathbf{x}\,\mathrm{d}\xi\right]^{1/2}\left[\int_0^T\int_{\Omega^{\xi}}\big|\mathbf{a}^{\xi}(\mathbf{x})\big|^5\,\mathrm{d}\mathbf{x}\,\mathrm{d}\xi\right]^{1/5}\\
&\le C(\Omega_j^*).
\end{aligned}$$

Substituting the estimates of $\mathcal{S}_1$ and $\mathcal{S}_2$ to (47), we finally obtain

$$\big\||\vartheta|^{\gamma}\,\hat{\mathbf{z}}_n^j(\vartheta)\big\|_{-1,2;\Omega_j^*}\ \le\ C(\Omega_j^*)\,|\vartheta|^{\gamma-1}. \tag{48}$$

The constant $C(\Omega_j^*)$ is independent of n. Recall that inequality (48) holds for $|\vartheta|>1$. Since the exponent γ satisfies $0<\gamma<\frac{1}{2}$, the right hand side of (48) is integrable on $(-\infty,-1)\cup(1,+\infty)$ with power 2. This, together with (46), implies that the sequence $\{|\vartheta|^\gamma\,\hat{\mathbf{z}}_n^j(\vartheta)\}$ is bounded in $L^2(\mathbb{R}; W_{0,\sigma}^{-1,2}(\Omega_j^*))$. Consequently, the sequence $\{\mathbf{w}_n^j\}$ is bounded in $\mathcal{H}^\gamma\big(I_j; W_\sigma^{1,2}(\Omega_j^*), W_{0,\sigma}^{-1,2}(\Omega_j^*)\big)$. This space is reflexive, hence there exists a subsequence (we denote it again by $\{\mathbf{w}_n^j\}$) which converges weakly in $\mathcal{H}^\gamma\big(I_j; W_\sigma^{1,2}(\Omega_j^*), W_{0,\sigma}^{-1,2}(\Omega_j^*)\big)$. Due to (44), the limit must be $\mathbf{w}^j$. Applying now Lemma 3, we have: $\mathbf{w}_n^j \longrightarrow \mathbf{w}^j = P_\sigma^j\mathbf{u}$ strongly in $L^2\big(I_j; L^2(\Omega_j^*)^3\big)$. This strong convergence, together with the weak convergence (44), enables us to pass to the limit in the first three terms on the left hand side of (43). The procedure is standard (see e.g. J. L. Lions [18] or R. Temam [26]), therefore we omit the details. Using also the equation

$$\int_{\Omega_j^*} (\nabla\varphi\cdot\nabla)\nabla\varphi\cdot\phi_j^{**}\,\mathrm{d}\mathbf{x} = 0,$$

following from the inclusion $\phi_j^{**}\in L_\sigma^2(\Omega_j^*)$ and from the identity $(\nabla\varphi\cdot\nabla)\nabla\varphi = \nabla\big(\frac{1}{2}|\nabla\varphi|^2\big)$, we can verify the validity of (43), and consequently also the validity of (40). This confirms that $\mathbf{u}$ is a weak solution of the weak problem (17). The proof of Theorem 1 is thus completed.

7 Concluding remarks

Energy inequality for the weak solution. The limit processes (36)–(38) and Lemma 2 imply that the limit function $\mathbf{u}$, which is a solution of (17), satisfies the same estimates (33) and (34) as the approximations. Inequality (33) thus provides an estimate of the kinetic energy associated with the flow $\mathbf{u}$ at a.a. times $t\in(0,T)$ and the first inequality in (34) estimates the dissipation of this energy in the time interval $(0,T)$. The question whether $\mathbf{u}$ also satisfies the energy inequality (18), formally derived in Section 2 (see Lemma 1), is open. To obtain (18), it would be necessary to make the limit transition in inequality (32) (which is a discrete equivalent of (18)). Here we need a piece of information on the strong convergence of a subsequence $\{\mathbf{u}^n\}$ in $L^2(0,T; L_\sigma^2(\Omega^t))$ in order to control the second term on the right hand side of (32). This is, however, a problem because we have only obtained the strong convergence of appropriate local interior Helmholtz projections of $\mathbf{u}^n$ in Section 6. It was sufficient for the limit transition (40), but it does not enable us to treat the integral on the right hand side of (32) in a similar way.

The condition of smallness (iv). Condition (iv) (see Lemma 1) requires a sufficient smallness of the speed of the ball B^t at times close to the critical instant t_c of the collision of the ball with the wall. We need this condition because estimate (13), based on the continuous imbedding $W^{1,2}(\Omega^t))\hookrightarrow L^6(\Omega^t)$, cannot be used in order to estimate the approximations at times close to t_c. (The constant in the imbedding inequality increases “too rapidly” to infinity as $t\longrightarrow t_c$.) Thus, we use estimate (11) instead of (13) at times close to t_c and since we need the right hand side to be absorbed by the “viscous term”, it must be “sufficiently small”.

Flow around a body of a general shape striking the wall. We have mainly used the information on the shape of B^t (i.e. that it is a ball) in the region close to the point of the collision of B^t with the wall. (Particularly, the shape of B^t influences the form of function g^t in Section 2. With another function g^t, we would obtain other inequalities than (7)–(13) for function $\mathbf{a}^t$.) Thus, Theorem 1 could be generalized in such a way that instead of the ball B^t we would speak on a compact body of another (however sufficiently smooth) shape, which coincides with a ball in the neighborhood of the point of the collision.

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