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Shape sensitivity analysis of the Neumann problem of the Laplace equation in the Half-Space

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1 Introduction, formulation of the problem and main theorems

Shape optimization for the Neumann problem of the Laplace equation is important for application and also from the numerical point of view. Mathematical analysis of such problem in the half space is not available. In this paper we prove the shape differentiability of solutions in appropriate weighted Sobolev spaces which describe the behavior of solutions at infinity. We will consider two different perturbations of domain to get the existence of weak Gateaux material derivative and in the second case the existence of Fréchet material derivatives.

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Firstly we give the description of the problem and introduce the appropriate functions spaces.

We consider the shape sensitivity analysis of the following model problem

$$-\Delta u = f \ \text{ in } \ \Omega, \ \frac{\partial u}{\partial n} = g \ \text{ on } \Gamma, \tag{1.1}$$

where $\Omega = \mathbb{R}^N_+$ and $\Gamma = \mathbb{R}^{N-1}$.
The same analysis can be performed in an unbounded domain Ω.

We consider the mapping $T_t : \mathbb{R}^N \to \mathbb{R}^N$ associated with the velocity field $V(t, x)$ which is compactly supported with respect to the spatial variable x. The mapping is given by the system of differential equations

$$\frac{d}{dt}x(t) = V(t, x(t)), \ x(0) = X, \tag{1.2}$$

with the solution denoted by $x(t) = x(t, X), t \in (-\delta, \delta), X \in \mathbb{R}^N$.
The variable domain $\Omega_t = T_t(\Omega)$ is defined in the usual way,

$$\Omega_t = \{x \in \mathbb{R}^N | x = x(t, X), X \in \Omega\}.$$

In order to define the Fréchet derivatives, we also consider transformations of the following type

$$\mathcal{H}_\xi = \mathcal{I} + \xi\theta, \tag{1.3}$$

where θ is a smooth vector field defined on $\mathbb{R}^N$ such that

$$\theta \in W^{k,\infty}(\mathbb{R}^N, \mathbb{R}^N), \tag{1.4}$$

with $-\delta < \xi < \delta$. This type of parametrization of domains is studied, e.g. by Murat - Simon [12] and Pironneau [11].

By the first approach the so - called Gateaux shape derivatives are obtained. The second approach leads directly to the Fréchet derivatives of shape functionals. The both approaches are equivalent, see Delfour - Zolesio [10].

1.1 Gateaux derivatives of solutions

We use the transformations T_t in order to define the perturbed domains Ω_t. Therefore, we consider the Neumann problem in Ω_t, which is called *(perturbed problem)*

$$-\Delta u_t = f_t \ \text{ in } \Omega_t, \ \frac{\partial u_t}{\partial n_t} = g_t \text{ on } \Gamma_t. \tag{1.5}$$

We would like to introduce some compatibility conditions but we cannot require that f_t, g_t satisfy compatibility condition

$$\int_{\Omega_t} f_t \, dx = \int_{\partial\Omega_t} g_t \, d\sigma, \tag{1.6}$$

since this condition doesn't have meaning for arbitrary data. To avoid this difficulty we suppose that for given elements f and g there are the extensions which is denoted by f and g such that the extended functions are defined on the sets Ω_t, Γ_t, $\forall t \in [0, \epsilon_0)$, $\epsilon_0 > 0$, respectively. Then we define

$$f_t := f\Big|_{\Omega_t} - \frac{1}{|\Omega_t|}\int_{\Omega_t} f dx,$$

$$g_t := g\Big|_{\Gamma_t} - \frac{1}{|\Gamma_t|}\int_{\Gamma_t} g d\sigma.$$

Remark 1.1 :
Let us point out that by such definitions of f_t and g_t we have the nontrivial shape derivatives $f' \neq 0$ and $g' \neq 0$ in general and also that by our definition

$$\int_{\Omega_t} f_t \, dx = 0$$

$$\int_{\Gamma_t} g_t \, d\sigma = 0,$$

hence (1.6) holds for such f_t and g_t.
The transported solution to the fixed domain is denoted by $u^t = u_t \circ T_t$, $f^t = f_t \circ T_t$, $g^t = g_t \circ T_t$ and the transported solution satisfies the following equation along with the boundary conditions

$$\begin{array}{rcll} -\frac{1}{\gamma}\ \mathrm{div}\ (A(t)\nabla u^t) & = & f^t & \text{in } \Omega \\ \nabla u^t \cdot \eta^t & = & g^t & \text{on } \Gamma, \end{array} \tag{1.7}$$

where

$$A(t) = det(DT_t)^* DT_t^{-1} DT_t^{-1}, \tag{1.8}$$

$$\eta^t = DT_t^{-1} n^t,$$

$$\gamma = det(DT_t).$$

Remark 1.2 : By n_t we denote the external normal on Γ_t and $\frac{\partial u_t}{\partial n_t} = \nabla u_t \cdot n_t = g_t$. The transport of the gradient is $\nabla u_t \circ T_t = {}^*DT_t^{-1} \cdot \nabla(u_t \circ T_t) = {}^*DT_t^{-1} \cdot \nabla u^t$, moreover

$$(\nabla u_t \cdot n_t) \circ T_t = g_t \circ T_t,$$

$$(\nabla u_t \circ T_t) \cdot (n_t \circ T_t) = g^t,$$
$$({}^*DT_t^{-1} \cdot \nabla u^t) \cdot n_t \circ T_t = g^t,$$
$$(\nabla u^t)^T \cdot DT_t^{-1} \cdot n^t = g^t.$$

Now, the derivative of the latter equality leads to the relation

$$\nabla \dot{u} \cdot n + \nabla u \cdot (-DV) \cdot n + \nabla u \cdot I \cdot \dot{n} = \dot{g},$$

where $\dot{u}$, $\dot{n}$, $\dot{g}$ denote material derivatives.
It implies that

$$\frac{\partial \dot{u}}{\partial n} = \nabla u \cdot DV \cdot n - \nabla u \cdot \dot{n} + \dot{g}. \tag{1.9}$$

The material derivative $\dot{u}$ of the solutions to (1.7) satisfies the following boundary value problem

$$\begin{array}{rcl} \Delta \dot{u} & = & \dot{f} + \text{ div } Vf + \text{ div } (A'(0)\nabla u) \text{ in } \Omega \\ \frac{\partial \dot{u}}{\partial n} & = & \dot{g} - \nabla u \cdot \dot{\eta} \text{ on } \Gamma \end{array} \tag{1.10}$$

Our first aim is to prove the existence of material derivative of weak type for the general transformation T_t:

Main Theorem 1.

If $f \in W_1^{0,2}(\mathbb{R}^3_+)$ and $g \in W_1^{1/2,2}(\mathbb{R}^2)$ then the material derivative $\dot{u} \in W_1^{1,2}(\mathbb{R}^N_+)$ is given by a unique solution to problem (1.10).

1.2 Fréchet derivatives of solutions

We are also interested in Fréchet differentiability of solutions to (1.1) with respect to perturbed domain. To this end we investigate the transformation $\mathcal{H}_\xi$ in order to define the perturbed domains Ω_ξ. We consider the problem (1.1), its *variational formulation* is the following:

Find $u \in W_0^{1,2}(\Omega)$ *such that :*

$$\forall w \in W_0^{1,2}(\Omega) \ \int_\Omega \nabla u \cdot \nabla w dx = \int_\Omega f \ w \ dx - \langle g, w\rangle_{W_0^{-\frac{1}{2},2}(\Gamma)\times W_0^{\frac{1}{2},2}(\Gamma)}, \quad (1.11)$$

see Section 2 for the definition of spaces $W_)^{1,2}(\Omega)$.
We describe the properties of transformation $\mathcal{H}_\xi$ defined by (1.3) for the vector field θ .

Let Θ_k be the space of vector fields from $C^k(\mathbb{R}^N, \mathbb{R}^N)$ and we denote $\|.\|_k$ the usual norm for $k \geq 1$ and $N = 2, 3$. We denote

$$\mathcal{D}_k := \{\theta \in \Theta_k, \|\theta\|_k < 1\}.$$

For $\theta \in \mathcal{D}_k$ the mapping $\mathcal{I} + \theta$ is a C^k - diffeomorphism, where $\mathcal{I}$ is the identity mapping.

Let $\theta\xi$ be a vector field in Θ_k. For simplicity we denote its norm in Θ_k as $|\xi\theta| = |\xi|\|\theta\|_{C^k(\mathbb{R}^N,\mathbb{R}^N)}$. For the transformation $\mathcal{H}_\xi = I + \xi\theta$ we denote $\Omega_\xi = \mathcal{H}_\xi(\Omega)$. For $|\xi|$ small enough, $\mathcal{H}_\xi$ is an diffeomorphism. As a consequence, there exists a solution $u_\xi \in W_0^{1,2}(\Omega_\xi)$ of variational equation

$$\forall v \in W_(^{1,2}\Omega_\xi) \ \int_{\Omega_\xi} \nabla u_\xi \cdot \nabla v dx = \int_{\Omega_\xi} f_\xi \ v \ dx. \quad (1.12)$$

After the transformation to the fixed domain, where $u^\xi = u_\xi \circ \mathcal{H}_\xi \in W_0^{1,2}(\Omega)$ we get the following variational formulation satisfying

$$\forall w \in W_0^{1,2}(\Omega), \ \int_\Omega (D\mathcal{H}_\xi^T)^{-1}\nabla u^\xi \cdot (D\mathcal{H}_\xi^T)^{-1}\nabla w \ q_\xi \ dx = \int_\Omega f^\xi \ w \ q_\xi dx \ , \quad (1.13)$$

where $f^\xi = f_\xi \circ \mathcal{H}_\xi$, q_ξ is the Jacobian of the transformation $\mathcal{H}_\xi$, $D\mathcal{H}_\xi$ is the Jacobian matrix:

$$D\mathcal{H}_\xi = I + \xi D\theta \quad (1.14)$$

$$q_\xi = \det \mathrm{D}\mathcal{H}_\xi = 1 + \xi \ \mathrm{div} \ \theta + \xi^N \ \det D\theta. \quad (1.15)$$

As in [14] the Taylor expansion for u^ξ leads to

$$u^\xi = u + \xi u^1(\theta) + \tilde{u}(\xi\theta), \quad (1.16)$$

where

$$\begin{aligned}
&\|u^\xi - u\|_{W_0^{1,2}(\Omega)} \le c|\xi|\|\theta\|, \\
&\|\tilde{u}(\xi\theta)\|_{W_0^{1,2}(\Omega)} = \|u^\xi - u - u^1(\theta)\|_{W_0^1(\Omega)} \le c|\xi|^2\|\theta\|^2.
\end{aligned} \tag{1.17}$$

Let J and E are functionals associated to the equations (1.5) we can define

$$J(\Omega_\xi) = E(\xi) = -\frac{1}{2}\int_{\Omega_\xi} \|\nabla u_\xi\|^2 dy. \tag{1.18}$$

We can prove that $E(\xi)$ has the following expansion

$$E(\xi) = E(0) + \xi E'(0)(\theta) + \tilde{E}(\xi\theta), \tag{1.19}$$

with the estimate

$$|\tilde{E}(\xi)| \le c|\xi|^2\|\theta\|^2. \tag{1.20}$$

Formula (1.17) shows the Fréchet differentiability of the first order for solutions and (1.19) for the energy functional.

Main Theorem 2.

If $f \in W_1^{0,2}(\mathbb{R}_+^N)$, $g \in W_1^{1/2,2}(\mathbb{R}^{N-1})$ then the material derivative $\dot{u} \in W_1^{1,2}(\mathbb{R}_+^N)$ is given by a unique solution to problem (1.10), which is same as before, but the strong convergence in the energy space.

Remark 1.3 : *Comparison of notations from 1.1 and 1.2.*

$$\begin{aligned}
A(t) &= \det\,(DT_t)^* DT_t^{-1} \cdot DT_t^{-1}, \\
A(\xi) &= q_\xi^{-1}(D\mathcal{H}_\xi)^{-1}(D\mathcal{H}_\xi^T)^{-1}, \\
D\mathcal{H}_\xi &= I + \xi D\theta, \\
DF_\xi^T &= (I + \xi D\theta)^T = I + \xi D\theta^T, \\
q_\xi &= \det\,(D\mathcal{H}_\xi^T) = \det\,(D\mathcal{H}_\xi), \\
q_\xi^{-1} &= \det\,[(D\mathcal{H}_\xi^T)^{-1}].
\end{aligned}$$

$$A'(0) = \lim_{t\to 0}\frac{A(t) - A(0)}{t} = \operatorname{div} V(0)I - {}^*DV(0) - DV(0)$$

We can also write

$$\lim_{|\xi|\to 0}\frac{A(\xi) - A(0)}{|\xi|} = \operatorname{div}\,\theta I - {}^*D\theta - D\theta.$$

So it is clear that the both approaches result in the same formula for the first order shape sensitivity analysis.

2 Notation and Mathematical Preliminaries

We introduce a class of weighted spaces for the Neumann boundary value problem and give some preliminary lemmas.

Let $\mathbb{R}^N_+ = \{(x', x_N) \in \mathbb{R}^N;\ x_N > 0\}$ be the upper half-space of $\mathbb{R}^N (N \geq 2)$ and denote by $\Gamma = \{(x', 0); x' \in \mathbb{R}^{N-1}\}$ its boundary.

We denote by $L^p(\mathbb{R}^N_+)$ the Lebesgue space, by $W^{p,k}(\mathbb{R}^N_+)$ the Sobolev space. The Sobolev spaces with radial weight have been introduced and studied by many authors : Hanouzet [6], Kudrjacev [21], Kufner [19], Kufner and Opic [20]. The Sobolev spaces with logarithmic weight were studied by Lizorkin [15], Leroux [16], Giroire [17], Girault [18], Amrouche and his collaborators [1], [2], [8], Boulmezaoud [4],[5], etc.

Let Ω be an open set of $\mathbb{R}^N$ and let us consider the basic weight

$$\rho(r) = (\sqrt{1+r^2}, \quad \lg \rho = \ \ln \ (2 + r^2).$$

with $r = \left(\sum_{i=1}^N x_i^2\right)^{1/2}$ being the distance to the origin. Given an integer $m \in \mathcal{N}$ and a real number $\alpha \in \mathbb{R}$, we define the weighted space

As usual, $\mathcal{D}(\mathbb{R}^N)$ denotes the space of indefinitely differentiable functions with compact supports and $\mathcal{D}^{'}(\mathbb{R}^N)$ denotes its dual space, called the space of distributions. For any nonnegative integers N and m , real numbers $p > 1$, α and β and setting

$$\begin{aligned} k = k(m,N,p,\alpha) &= -1 && \text{if } \ \tfrac{N}{p} + \alpha \notin \{1,\ldots,m\}, \\ k = k(m,N,p,\alpha) &= m - \tfrac{N}{p} - \alpha && \text{if } \ \tfrac{N}{p} + \alpha \in \{1,\ldots,m\}, \end{aligned}$$

we define the following space:

$$\begin{aligned} &W^{m,p}_{\alpha,\beta}(\Omega) = \{u \in \mathcal{D}'(\Omega);\ 0 \leq |\lambda| \leq k,\ \rho^{\alpha-m+|\lambda|}(\lg\rho)^{\beta-1} D^\lambda u \in L^p(\Omega); \\ &k+1 \leq |\lambda| \leq m,\ \rho^{\alpha-m+|\lambda|}(\lg\rho)^{\beta} D^\lambda u \in L^p(\Omega)\}. \end{aligned} \tag{2.1}$$

In the case $\beta = 0$, we simply denote the above space by $W^{m,p}_\alpha(\Omega)$. Note that $W^{m,p}_{\alpha,\beta}(\Omega)$ is a reflexive Banach space equipped with its natural norm:

$$\begin{aligned} \|u\|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big[&\textstyle\sum_{0\leq|\lambda|\leq k} \|\rho^{\alpha-m+|\lambda|}(\lg\rho)^{\beta-1} D^\lambda u\|^p_{L^p(\Omega)} \\ &+ \textstyle\sum_{k+1\leq|\lambda|\leq m} \|\rho^{\alpha-m+|\lambda|}(\lg\rho)^{\beta} D^\lambda u\|^p_{L^p(\Omega)}\Big]^{1/p}. \end{aligned} \tag{2.2}$$

We also define the semi-norm:

$$|u|_{W^{m,p}_{\alpha,\beta}(\Omega)} = \Big(\sum_{|\lambda|=m} \|\rho^{\alpha}(\lg\rho)^{\beta} D^{\lambda} u\|^{p}_{L^{p}(\Omega)} \Big)^{1/p}, \tag{2.3}$$

and for any integer q, we denote by P_q the space of polynomials in N variables of the degree smaller than or equal to q, with the convention that P_q is reduced to $\{0\}$ for negative q. The weights in definition (2.1) are chosen so that the corresponding space satisfies two properties:

$$\mathcal{D}(\overline{\mathbb{R}^N_+}) \quad \text{is dense in} \quad W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+), \tag{2.4}$$

and the Poincaré-type inequality holds in $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$.

Theorem 2.1. *Let α and β be two real numbers and $m \geq 1$ an integer not satisfying simultaneously*

$$\frac{N}{p} + \alpha \in \{1, \ldots, m\} \quad \textit{and} \quad (\beta - 1)p = -1. \tag{2.5}$$

Then the semi-norm $|\cdot|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}$ defines on $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)/P_{q'}$ a norm which is equivalent to the quotient norm, with $q' = inf\,(q, m-1)$, where q is the highest degree of the polynomials contained in $W^{m,p}_{\alpha}(\mathbb{R}^N_+)$.

Proof. see [1].

Now, we define the space

$$\mathring{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+) = \overline{\mathcal{D}(\mathbb{R}^N_+)}^{\|\cdot\|_{W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)}}$$

and the dual space of $\mathring{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ is denoted by $W^{-m,p'}_{-\alpha,-\beta}(\mathbb{R}^N_+)$, where p' is the conjugate of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 2.2. *Under the assumptions of Theorem 1.1, the semi-norm (2.3) is a norm on $\mathring{W}^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$ which is equivalent to the full norm (2.2).*

Proof. see [1].

In the sequel, for any integer $q \geq 0$, we shall use the following polynomial spaces:

$\mathcal{P}_q$ (respectively $\mathcal{P}^{\Delta}_q$) is the space of polynomials (respectively harmonic polynomials) of degree $\geq q$,

$\mathcal{P}'_g$ is the subspace of the polynomials in $\mathcal{P}_q$ depending only on the $N-1$ first variables $x' = (x_1, ..., x_{N_1})$,

2.1 The spaces of traces

In order to define the traces of functions of $W^{m,p}_{\alpha,\beta}(\mathbb{R}^N_+)$, we introduce for any $\sigma \in]0,1[$ the space:

$$W^{\sigma,p}_0(\mathbb{R}^N) = \{u \in \mathcal{D}'(\mathbb{R}^N);\ w^{-\sigma}u \in L^p(\mathbb{R}^N), \\ \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+\sigma p}} dxdy\Big)^{1/p} < \infty\}, \tag{2.7}$$

where

$$w = \rho \qquad \text{if} \quad \tfrac{N}{p} \neq \sigma, \\ w = \rho(\ \lg\rho)^{1/\sigma} \quad \text{if} \quad \tfrac{N}{p} = \sigma,$$

and $e_1, \dots, e_N$ is a canonical basic of $\mathbb{R}^N$. $W^{\sigma,p}_0(\mathbb{R}^N)$ is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W^{\sigma,p}_0(\mathbb{R}^N)} = \Big(\Big\|\frac{u}{w^\sigma}\Big\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+\sigma p}} dxdy\Big)^{1/p}. \tag{2.8}$$

If u is a function defined on $\mathbb{R}^N_+$, we denote its traces on $\Gamma = \mathbb{R}^{N-1}$ by:

$x' \in \mathbb{R}^{N-1}$, $\gamma_0 u(x') = u(x',0), \dots, \gamma_j u(x') = \frac{\partial^j u}{\partial x^j_N}(x',0)$. In the same way as in [3], we can prove the following trace lemma:

Lemma 2.3. *For any integer $m \geq 1$ and real number α, the mapping*

$$\gamma : \mathcal{D}(\overline{\mathbb{R}^N_+}) \to \prod_{j=0}^{m-1} \mathcal{D}(\mathbb{R}^{N-1}) \\ u \mapsto (\gamma_0 u, \dots, \gamma_{m-1} u)$$

can be extended by continuity to a linear and continuous mapping, still denoted by γ, from $W^{m,p}_\alpha(\mathbb{R}^N_+)$ onto $\prod_{j=0}^{m-1} W^{m-j-\frac{1}{p},p}_\alpha(\mathbb{R}^{N-1})$. Moreover

$$Ker\ \gamma = \mathring{W}^{m,p}_\alpha(\mathbb{R}^N_+).$$

Proof: see [3].

3 Neumann problem in the half space

In the section we recall the known results for the problem Neumann problem in the half space see [2].

$$-\Delta u = f \ \text{in}\ \ \mathbb{R}^N_+,\ \frac{\partial u}{\partial x_N} = g \ \text{on}\ \mathbb{R}^{N-1}. \tag{3.1}$$

Theorem 3.1. *Let*

$$\frac{N}{p'} \neq 1, \tag{3.2}$$

let $f \in W_1^{0,p}(\mathbb{R}_+^N)$ *satisfying the compatibility condition*

$$\int_\Omega f \ dx = \ 0, \ \textit{if}\ p' > N$$

then problem (3.1) with $g = 0$ *has a unique solution* $u \in W_1^{2,p}(\mathbb{R}_+^N)/\mathcal{P}^\Delta_{[1-N/p]}$.

Remark 3.1 Let us note that $W_1^{0,p}(\mathbb{R}_+^N) \subset W_0^{-1,p}(\mathbb{R}_+^N)$ iff $\frac{N}{p'} \neq 1$. In the case $\frac{N}{p'} = 1$ the previous result holds provided $f \in W_0^{-1,p}(\mathbb{R}_+^N) \cap W_1^{0,p}(\mathbb{R}_+^N)$ without compatibility conditions and problem (3.1) has a unique solution in $W_1^{2,p}(\mathbb{R}_+^N)/\mathcal{P}_{[1-N/p]}$.

Theorem 3.2. *Let* $\frac{N}{p'} \neq 1$, $f \in W_1^{0,p}(\mathbb{R}_+^N)$, $g \in W_0^{-1/p,p}(\Gamma)$. *We suppose the following condition holds:*

$$\int_\Omega f = \langle g, 1\rangle_{W_0^{-1/p,p}(\Gamma)\times W_0^{1/p.p'}(\Gamma)} \quad \textit{if}\ p' > N, \tag{3.3}$$

then problem (3.1) has a unique problem $v \in W_0^{1,p}(\mathbb{R}_+^N))/\mathcal{P}_{[1-\frac{N}{p}]}$.
Moreover if $g \in W_0^{1-1/p,p}(\Gamma)$ *then there exists a unique solution* $v \in W_1^{2,p}(\mathbb{R}_+^N)$.

Remark 3.2
In case $p' = N$ Remark 3.1 holds with $g \in W_1^{1-1/p,p}(\Gamma)$.

3.1 Mapping T_t

We consider the general case of constructing the transformation T_t. Let D be a domain in $\mathbb{R}^N$ with the boundary ∂D piecewise C^k for a given integer $k \geq 0$. Let T_t be a one - to - one mapping from $\bar{D}$ onto $\bar{D}$ such that

$$T_t \text{ and } T_t^{-1} \text{ belong to } C^k(\bar{D}; \mathbb{R}^N) \tag{$*$}$$

and

$$t \to T_t(x),\ t \to T_t^{-1} \in C([0, \varepsilon)),\ \forall x \in \bar{D} \tag{$**$}$$

thus $(t,x) \to T_t(x) \in C([0,\epsilon); C^k(\bar{D};\mathbb{R}^N)) = C(0,\varepsilon; C^k(\bar{D};\mathbb{R}^N))$. For any $X \in \bar{D}$ and $t > 0$ the point $x(t) = T_t(X)$ moves along the trajectory $x(.)$ with the velocity

$$\|\frac{d}{dt}x(t)\|_{\mathbb{R}^N} = \|\frac{\partial}{\partial t}T_t(X)\|_{\mathbb{R}^N}. \tag{3.5}$$

$$V(t,x) = (\frac{\partial}{\partial t}T_t) \circ T_t^{-1}(x). \tag{3.6}$$

It is obvious that $V(t,x)$ takes the form

$$V(t,x) = (\frac{\partial}{\partial t}T_t) \circ T_t^{-1}(x). \tag{3.7}$$

From (3.5) and (3.6) the vector field $V(t)$, defined as $V(t)(x) = V(t,x)$, satisfies the relation

$$V \in C(0,\varepsilon; C^k(\bar{D};\mathbb{R}^N)). \tag{3.7}$$

If V is a vector field such that (3.7) holds, then the transformation T_t depending on V , and such that conditions (3.5) and (3.6) are satisfied, is defined by (1.2).

Theorem 3.3. *Let D be a bounded domain in $\mathbb{R}^N$ with the piecewise smooth boundary ∂D , and $V \in C(0,\varepsilon; C^k(\bar{D},\mathbb{R}^N))$ be a given vector field which satisfies*

$$V(t,x) \cdot n(x) = 0 \textit{ for a.e. } x \in \partial D, \tag{3.8}$$

and

$$\textit{if } n = n(x) \textit{ is not defined as a singular point } x \in \partial D \textit{ we set } V(t,x) = 0. \tag{3.9}$$

Then there exists an interval I, $0 \in I$, and the one - to - one transformation $T_t(V) : \mathbb{R}^N \to \mathbb{R}^N$ such that $T_t(V)$ maps $\bar{D}$ onto $\bar{D}$. Furthermore $T_t(V)$ satisfies conditions (3.2),(3.3),(1.2). In particular the vector field V can be written in the form

$$V = \partial_t T_t(V) \circ T_t(V)^{-1}.$$

On the other hand, if T_t is a transformation of $\bar{D}$, T_t satisfies (3.2),(3.3) and V is defined by the formula

$$V = \partial_t T_t \circ T_t^{-1},$$

then (3.10) holds for V. Furthermore $V \in C(0,\varepsilon; C^k(\bar{D},\mathbb{R}^N))$ and the transformation $T_t(X) = x(t,X)$ is defined as the local solution to the system of ordinary differential equations (1.2), that $T_t = T_t(V)$.

Now, we are interested in the case of unbounded domains D.

Definition 3.4. *Let D be a domain in $\mathbb{R}^N$ whose boundary ∂D is piecewise C^k, $k \geq 1$. It is supposed that the outward unit normal field n exists a.e. on ∂D, i.e. except for singular points $\bar{x}$ of ∂D. The following notation is used*

$V^k(D) = \{V \in D^k(\mathbb{R}^N;\mathbb{R}^N) | \langle V, n\rangle_{R^N} = 0$ *on* ∂D *except for the singular points* $\bar{x}$ *of* $\partial D, V(\bar{x}) = 0$ *for all singular points* $\bar{x}\}$.

$V^k(D)$ *is equipped with the topology induced by* $D^k(\mathbb{R}^N;\mathbb{R}^N)$.

So, if $V \in C(0,\varepsilon;V^k(D))$, then there exists a compact set $\bar{O}$ in $\mathbb{R}^N$ such that the support of $V(t)$ is included in $\bar{O}$ for all $0 \leq t \leq \varepsilon$. So, we have the following theorem

Theorem 3.5. *Let D be a bounded domain in $\mathbb{R}^N$ with the piecewise smooth boundary ∂D, and $V \in C(0,\epsilon;V^k(D))$ be a vector field. Then there exists an interval $I = [0,\delta), 0 < \delta \leq \varepsilon$ and a one - to - one transformation $T_t(V)$ for each $t \in I$ which maps $\bar{D}$ onto $\bar{D}$ and satisfies all properties of Theorem 3.3.*

3.2 Sobolev spaces and boundary value problems. Transported and Perturbed Problems.

We already introduced the Sobolev weighted spaces in fixed domain. Now, we are interested in the definition of Sobolev spaces with corresponding weights in perturbed domain. The most important property is definition of the traces and that the theorem of the traces should be satisfied. Since our mapping is C^k we can define the Sobolev spaces with weights on perturbed domain through the Sobolev spaces on fixed domain.

Definition 3.6. *We say that $u_t \in W_0^{1,2}(\Omega_t)$ iff $u^t = u_t \circ T_t \in W_0^{1,2}(\Omega)$, where the corresponding seminorm is defined by*

$$\Big(\int_\Omega \|\nabla u^t\|^2 dx\Big)^{1/2} = \Big(\int_{\Omega_t} \|DT_t \circ T_t^{-1} \cdot \nabla u_t\|^2 |\gamma(t)|^{-1} dx_t\Big)^{1/2}$$

and the corresponding norm is given by

$$\begin{aligned} \|u_t\|_{W_0^{1,2}(\Omega_t)} = \{ & \textstyle\int_{\Omega_t} \|DT_t \circ T_t^{-1} \cdot \nabla u_t\|^2 |\gamma(t)|^{-1} \; dx_t + \\ & + \textstyle\int_{\Omega_t} \|u_t \circ T_t^{-1}\|^2 (\rho \circ T_t^{-1})^{-2} |\gamma(t)|^{-1} dx_t \}^{1/2}. \end{aligned} \tag{3.11}$$

Now, we want to define the traces.

Definition 3.7. We say that $g_t \in W_0^{1/2,2}(\Gamma_t)$ iff $g_t \circ T_t = g^t \in W_0^{1/2,2}(\Gamma)$, with the norm defined by

$$\int_{\Gamma_t \times \Gamma_t} \frac{1}{w^2(t)} \frac{|g_t(x'_t) - g_t(y'(t)|^2}{|S_t x'_t - S_t y'_t|^3} dx'_t dy'_t = \int_{\Gamma \times \Gamma} \frac{|g^t(x') - g^t(y')|^2}{|x' - y'|^3} dx' dy' < \infty,$$

where $w(t) = |det(DT_t)^*| \|DT^{-1} \cdot n\|_{\mathbb{R}^N}$, $x'_t = T_t x'$, $y'_t = T_t y'$, $M(T_t) = det(DT_t)^* DT^{-1}$ is the cofactor matrix of the Jacobian matrix DT_t.

Remark 3.3 For description of change of variables in boundary integral see [9], page 77.

Then we can give the definition of the dual spaces.

Definition 3.8. We define $W_0^{-1,2}(\Omega_t)$ and $W_0^{-1/2,2}(\Gamma_t)$ by the following way:

$$W_0^{-1,2}(\Omega_t) = (\mathring{W}_0^{1,2}(\Omega_t))^*$$

and

$$W_0^{-1/2,2}(\Gamma_t) = (W_0^{1/2,2}(\Gamma_t))^*.$$

Very important is the property of a weak differentiability of T_t with respect to t. We repeat here the part of the proof from our previous work see [3].

We denote by D the following set

$$D = \{(x', x_N) \in \mathbb{R}^N, \; x_N > -a\}, \quad \text{fixed } a > 0, \text{ a sufficient large.}$$

Proposition 3.2. *Let $N \geq 3$, $f \in W_1^{0,2}(D) \subset W_0^{-1,2}(D)$. Let $V \in C(0, \varepsilon, \mathcal{D}^k(\mathbb{R}_+^N)$ be given, $k \geq 1$, then the mapping $t \to f \circ T_t$ is weakly differentiable in the space $W_0^{-1,2}(D)$.*

Proof. Let $\varphi \in \mathring{W}_0^{1,2}(D) \subset W_{-1}^{0,2}(D)$ be given and we denote $S_t = T_t^{-1}$,

$$\lambda(t) = \gamma(t)^{-1} \circ T_t^{-1} = \gamma(t)^{-1} \circ S_t.$$

We have

$$\frac{1}{t}\int_D (f \circ T_t - f)\varphi dx = \frac{1}{t}\int_D f(\lambda(t)\varphi \circ S_t - \varphi)dx.$$

Furthermore

$$\frac{1}{t}(\lambda(t)\varphi \circ S_t - \varphi) = \lambda(t)\frac{1}{t}(\varphi \circ S_t - \varphi) + \frac{1}{t}(\lambda(t) - 1)\varphi),$$

the right-hand side of this equality converges to

$$-\nabla\varphi \cdot V(0) + \lambda'(0)\varphi$$

strongly in $W_{-1}^{0,2}(D)$ as $t \to 0$. Moreover, it is evident that $\lambda'(0) = -\text{ div } V(0)$.

Since S_t is associated with the speed vector field $-V_t$, therefore

$$\int_D \frac{1}{t}(f \circ T_t - f)\varphi dx \to -\int_D f \text{ div } (\varphi V(0))dx = \langle f \cdot V(0), \varphi\rangle_{W_0^{-1,2}(D) \times \mathring{W}_0^{1,2}(D)}$$

as $t \to 0$; this proves the proposition. □

Definition 3.9. *Let $h_t \in W_0^{-1,2}(\Omega_t)$, $\varphi \in \mathring{W}_0^{1,2}(\Omega)$ then we define the following form*

$$\langle \tau h_t, \varphi\rangle_{W_0^{-1,2}(\Omega) \times \mathring{W}_0^{1,2}(\Omega)} = \langle h_t, \gamma^{-1}(t)\varphi \circ T_t^{-1}\rangle_{W_0^{-1,2}(\Omega_t) \times \mathring{W}_0^{1,2}(\Omega_t)}, \qquad (3.12)$$

where by τh_t we mean $h_t \circ T_t$.

Remark 3.4 Let $h_t \in W_1^{0,2}(\Omega_t)$, then for all $\varphi \in \mathring{W}_0^{1,2}(\Omega), \varphi \circ T_t^{-1} \in \mathring{W}_0^{1,2}(\Omega_t)$ and we have

$$\langle h_t, \gamma(t)^{-1}\varphi \circ T_t^{-1}\rangle_{W_0^{-1,2}(\Omega_t)\times \mathring{W}_0^{1,2}(\Omega_t)} = \int_{\Omega_t} h_t(x)\cdot \gamma(t)^{-1}\varphi \circ T_t^{-1}(x)dx = \int_\Omega h_t \circ T_t(X)\varphi(X)dX = \langle h_t \circ T_t, \varphi\rangle_{W_0^{-1,2}(\Omega)\times \mathring{W}_0^{1,2}(\Omega)},$$

where $\tau h_t = h_t \circ T_t$ and $h_t \in W_0^{0,2}(\Omega_t)$.

Definition 3.10. *Let $g_t \in W_0^{-1/2,2}(\Gamma_t)$, $\varphi \in W_0^{1/2,2}(\Gamma)$ then we define the following form*

$$\langle \tau g_t, \varphi \ >_{W_0^{-1/2,2}(\Gamma)\times W_0^{1/2,2}(\Gamma)} \ = \ \langle g_t, w(t)\varphi \circ T_t\rangle_{W_0^{-1/2,2}(\Gamma_t)\times W_0^{1/2,2}(\Gamma_t)}.$$

Proposition 3.11. *(i) Let $h_t \in W_0^{-1,2}(\Omega_t)$ then*

$$h_t = \text{div } \mathbf{F},\ \mathbf{F} = (f_1, ..., f_N)$$

with $f_i \in L^2(\Omega_t), i = 1, ..., N$.
(ii) $h^t = \gamma(t)^{-1}$ div $(DT_t^{-1}\mathbf{F} \circ T_t)$ and
(iii) In particular if $h \in W_0^{-1,2}(D)$, where $\Omega \subset D, \Omega_t \subset D$ then $h^t \in W_0^{-1,2}(D)$ and

$$\frac{h - h_t}{t} \rightarrow \dot{h} \text{ weakly in } W_0^{-1,2}(D).$$

Proof. It was proved in our previous work see [3]. □

Proposition 3.12. *Let there be given a vector field $V \in C(0, \varepsilon; D^k(\mathbb{R}^N; \mathbb{R}^N))$, $k \geq 1$ and an element $f \in W^{2,1}(\mathbb{R}^N)$. Then*

$$\frac{1}{t}[f \circ T_t - f] \rightarrow \nabla f \cdot fV(0) \text{ strongly in } W^{1,1}(\mathbb{R}^N)$$

as $t \rightarrow 0$, where $W^{2,1}(\mathbb{R}^N)$ and $W^{1,1}(\mathbb{R}^N)$ are classical Sobolev spaces.

Proof. see Proposition 2.37 in Sokolowski, Zolesio, [9] page 71. □

Proposition 3.13. Existence of strong differentiability of $\mathcal{H}_\xi$.
Let $f \in L^2(\mathbb{R}^N)$, $\theta \in C(0,\epsilon;\mathcal{D}^k(\mathbb{R}^N,\mathbb{R}^N))$ be given, then the mapping $\xi\theta \to \frac{1}{|\xi|}[f \circ \mathcal{H}_\xi - f]$ is strongly differentiable in the space $W^{-2,2}(\mathbb{R}^N)$.

Proof. Applying the Proposition 3.12 and Proposition 3.2 we obtain that

$$\|q_\xi \phi \circ \mathcal{H}_\xi - \phi - \text{div } (\phi\xi\theta)\| \le c|\xi|^2\|\theta\|^2,$$

for all $\phi \in W^{2,2}(\mathbb{R}^N)$.

□

By an application of the transport technique to our problem (1.1) defined in Ω_t , we get for all $\psi \in \mathring{W}_0^{1,2}(\Omega_t)$

$$\langle \Delta u_t, \psi\rangle_{W_0^{-1,2}(\Omega_t)\times \mathring{W}_0^{1,2}(\Omega_t)} = \langle \text{ div } (A(t)\nabla u^t\ ), \psi \circ T_t\rangle_{W_0^{-1,2}(\Omega)\times \mathring{W}_0^{1,2}(\Omega)}.$$

Let $\varphi = \psi \circ T_t \in \mathring{W}_0^{1,2}(\Omega)$, then

$$\langle \Delta u_t, \varphi \circ T_t^{-1}\rangle_{W_0^{-1,2}(\Omega_t)\times \mathring{W}_0^{1,2}(\Omega_t)} = \langle \text{ div } (A(t)\nabla u^t), \varphi\rangle_{W_0^{-1,2}(\Omega)\times \mathring{W}_0^{1,2}(\Omega)}$$

provided that

$$-\Delta u_t = h_t \text{ in } \Omega_t$$

and

$$-\text{ div } (A(t)\nabla u^t) = \gamma(t)h^t \text{ in } \Omega.$$

For problem (PP)

$$-\Delta u_t = f_t \text{ in } \Omega_t$$

thus

$$-\text{ div } (A(t)\nabla u^t) = \gamma(t)f^t \text{ in } \Omega.$$

Therefore, we will get the perturbed problem

$$-\Delta u_t = f_t \text{ in } \Omega_t\,,\ \frac{\partial u_t}{\partial n_t} = g_t \text{ on } \Gamma_t$$

and also transported problem

$$-\text{ div } (A(t)\nabla u^t) = \gamma(t)f^t \text{ in } \Omega\,,\ \nabla u^t \cdot \eta^t = g^t \text{ on } \Gamma,$$

where $g^t = g_t \circ T_t$.

4 Weak material derivatives

4.1 Transported problem(TRP)

We investigate the existence of the transported problem in the fixed domain Ω satisfying the equations

$$\begin{array}{rcl} -\frac{1}{\beta} \text{ div } (A(t)\nabla u^t) & = & f^t \text{ in } \Omega \\ \nabla u^t \cdot \eta^t & = & g^t \text{ on } \Gamma. \end{array} \tag{4.1}$$

where

$$A(t) = det(DT_t)^* DT_t^{-1} DT_t^{-1}. \tag{4.2}$$

Theorem 4.1. *Let $N \geq 3$, suppose $f^t \in W_1^{0,2}(\mathbb{R}^N_+)$ and $g^t \in W_0^{-1/2,2}(\Gamma)$ then problem (TRP) has a unique solution $u^t \in W_0^{1,2}(\mathbb{R}^N_+)$ satisfying the following estimate*

$$\|u^t\|_{W_0^{1,2}(\mathbb{R}^N_+)} \leq c\Big(\|f^t\|_{W_1^{0,2}(\mathbb{R}^N_+)} + \|g^t\|_{W_0^{-1/2,2}(\Gamma)}\Big).$$

Moreover if $g^t \in W_1^{1/2,2}(\Gamma)$ then $u^t \in W_1^{2,2}(\mathbb{R}^N_+)$ and the following estimates

$$\|u^t\|_{W_1^{2,2}(\mathbb{R}^N_+)} \leq c\Big(\|f^t\|_{W_1^{0,2}(\mathbb{R}^N_+)} + \|g^t\|_{W_1^{1/2,2}(\Gamma)}\Big)$$

holds.

Proof. We define the bilinear form

$$B(u^t, v^t) = \int_\Omega A(t)\nabla u^t \nabla v^t.$$

Since $\gamma(0) = 1$ then for sufficiently small δ we have $\gamma(t) > 1/2$ for all $t \in (-\delta, \delta)$ and the bilinear form B is uniformly elliptic, i.e.

$$B(u^t, u^t) \geq c\|\nabla u^t\|_2^2$$

for some positive constant $c > 0$ which implies the uniform ellipticity of the form B. Then applying Theorem 3.2 with $p = 2$ we get the existence of solution.

□

4.2 Perturbed problem (PP)

Proposition 4.2. *Let* $f \in W_1^{0,2}(\Omega_t)$, $g \in W_1^{1/2,2}(\Gamma)$ *then*

$$\frac{F^t - f}{t} \to \text{div } Vf + \dot{f} \text{weakly in } W_0^{-1,2}(\Omega_t).$$

Proof. We have the following equalities and the convergence

$$\frac{1}{t}\langle F^t - f, \varphi\rangle - \langle \dot{f} - \text{ div } Vf, \varphi\rangle =$$

$$= \frac{1}{t} < \gamma(t) f^t - f, \varphi\rangle =$$

$$= \frac{1}{t}\langle - \text{ div } (A(t)\nabla u^t) + \text{ div } \nabla u, \varphi\rangle + \langle \text{ div } Vf, \varphi\rangle =$$

$$= \frac{1}{t}\langle (A(t) - I)\nabla u^t, \nabla\varphi\rangle - \frac{1}{t}\langle \nabla(u^t - u), \nabla\varphi\rangle + \langle \text{ div } Vf, \varphi\rangle$$

$$\to \langle A'(0)\nabla u, \nabla\varphi\rangle - \langle \nabla\dot{u}, \nabla\varphi\rangle + \langle \text{ div } Vf, \varphi\rangle = \langle \dot{f}, \varphi\rangle + \langle \text{ div } Vf, \varphi\rangle.$$

For $f \in W_1^{0,2}(\mathbb{R}_+^N)$ it follows that $\dot{f} \in W_0^{-1,2}(\Omega_t)$ and also

$$\frac{f^t - f}{t} - \dot{f} \to 0 \text{ weakly in } W_0^{-1,2}(\Omega_t).$$

For $g \in W_1^{\frac{1}{2},2}(\Gamma)$, with $\dot{g} \in W_0^{-1/2,2}(\Gamma)$ it follows that

$$\frac{g^t - g}{t} - \dot{g} \to 0 \text{ weakly in } W_0^{-1/2,2}(\Gamma).$$

□

Theorem 4.3. *Let* $N \geq 3$, *suppose* $f_t \in W_1^{0,2}(\Omega_t)$ *and* $g_t \in W_0^{-1/2,2}(\Gamma_t)$ *then problem* (PP) *has a unique solution* $u_t \in W_0^{1,2}(\Omega_t)$ *satisfying the following estimate*

$$\|u_t\|_{W_0^{1,2}(\Omega_t)} \leq c\Big(\|f_t\|_{W_1^{0,2}(\Omega_t)} + \|g_t\|_{W^{-1/2,2}(\Gamma_t)}\Big).$$

Moreover if $g_t \in W_1^{1/2,2}(\Gamma_t)$ *then* $u_t \in W_1^{2,2}(\Omega_t)$.

Proof:

We have for $\psi \in_0^{1,2} (\Omega_t)$

$$\langle f_t, \psi\rangle_{W_0^{-1,2}(\Omega_t)\times \mathring{W}_0^{1,2}(\Omega_t)} = \langle \gamma(t) f_t, \gamma^{-1}\varphi \circ T_t^{-1}\rangle_{W_0^{-1,2}(\Omega_t)\times \mathring{W}_0^{1,2}(\Omega_t)} =$$

$$= \langle \gamma(t)\Delta u_t, \gamma^{-1}(t)\varphi \circ T_t^{-1}\rangle_{W_0^{-1,2}(\Omega_t)\times \mathring{W}_0^{1,2}(\Omega_t)} =$$

$$\langle \text{ div } (A(t)\nabla u^t, \varphi\rangle_{W_0^{-1,2}(\Omega)\times \mathring{W}_0^{1,2}(\Omega)} = \langle f^t, \varphi\rangle_{W_0^{-1,2}(\Omega)\times \mathring{W}_0^{1,2}(\Omega)},$$

where $\varphi = \psi \circ T_t$. Applying Theorem 4.1 it follows the existence of a unique solution of (1.1) on the perturbed domain.

4.3 Proof of Main theorem 1

The aim of this section is to show the existence of material derivative as a weak limit of

$$\frac{u^t - u}{t} \to \dot{u} \in W_0^{1,2}(\Omega) \tag{4.3}$$

Denoting

$$w^t = \frac{u^t - u}{t} - \dot{u}$$

we obtain the following equation

$$\begin{array}{rcll} -\Delta w^t &=& \text{div } \left[\frac{A(t)-I}{t}\nabla u^t - A'(0)\nabla u\right] + \frac{f^t - f}{t} - \dot{f}] & \text{in } \Omega \\ \frac{\partial w^t}{\partial n^t} &=& \frac{g^t - g}{t} - \dot{g} - \nabla u \cdot DV \cdot n - \nabla u \cdot \dot{n} & \text{on } \Gamma. \end{array} \tag{4.4}$$

The weak formulation of (4.4) is the following

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^N_+} \nabla w_t \cdot \nabla \phi = \\ &\textstyle\int_{\mathbb{R}^N_+} [\frac{A(t)-I}{t}\nabla u^t \cdot \nabla \phi - A'(0)\nabla u \nabla \phi] + \int_{\mathbb{R}^N_+} [\frac{f^t - f}{t} - \dot{f}]\ \phi\ dx + \\ &\textstyle+ \int_\Gamma \left(\frac{g^t - g}{t} - \dot{g} - \nabla u \cdot V \cdot n - \nabla u \cdot \dot{n}\right)\phi d\sigma + \int_\Gamma \left(\frac{A(t)-I}{t} g^t + A'(0) g\right)\phi,\ \forall \phi \in W_0^{1,2}(\mathbb{R}^N_+) \end{aligned} \tag{4.4$'$}$$

where $A'(0) = -DV - {}^*DV + \text{div}VI$.

The goal of this section is to prove the following convergence

$$w^t = \frac{u^t - u}{t} - \dot{u} \to 0 \text{ as } t \to 0,\ \text{weakly in } W_0^{1,2}(\mathbb{R}^N_+)$$

and

$$\frac{\partial w^t}{\partial n^t} = \frac{g^t - g}{t} - \dot{g} \to 0 \text{ as } t \to 0,\ t \to 0,\ \text{weakly in } W_0^{-1/2,2}(\mathbb{R}^{N-1}).$$

To get the assertion it is sufficient to prove the weak convergence of the following terms

$$\frac{A(t) - I}{t}\nabla u^t - A^{'}(0)\nabla u \to 0,\ t \to 0,\ \text{weakly in } L^2(\mathbb{R}^N_+)^N \qquad (4.5)$$

and

$$\frac{A(t) - I}{t} g^t - A'(0)g - \nabla u \cdot DV \cdot n - \nabla u \cdot \dot{n} \to 0 \text{ weakly in } W_0^{-1/2,2}(\mathbb{R}^{N-1}_+) \qquad (4.5)'.$$

since for the right hand side we have by our assumptions

$$\frac{f^t - f}{t} - \dot{f} \to 0 \text{ with } t \to 0 \text{ weakly in } \ W_0^{-1,2}(\mathbb{R}^N_+), \qquad (4.6)$$

and

$$\frac{g^t - g}{t} - \dot{g} \to 0 \text{ with } t \to 0 \text{ weakly in } \ W_0^{-1/2,2}(\mathbb{R}^{N-1}). \qquad (4.6)'$$

Let $\varphi = u^t - u$ be a test function in variational formulation, hence

$$\int_{\mathbb{R}^N_+} A(t)|\nabla(u^t - u)|^2 - (A(t) - I)\nabla u \cdot \nabla(u^t - u) +$$
$$+ \int_\Gamma A(t)(g^t - g)(u^t - u) + (A(t) - I)g(u^t - u) = \langle f^t - f, u^t - u\rangle + \langle g^t - g, u^t - u\rangle.$$

Since the field V is compactly supported in $\mathbb{R}^N$, it follows that

$$\int_{\mathbb{R}^N_+} A(t)|\nabla(u^t - u)|^2 \leq \int_{\mathbb{R}^N_+} |(A(t) - I)\nabla u \cdot ||\nabla(u^t - u)| +$$

$$+ \|f^t - f\|_{W_0^{-1,2}} \|u^t - u\|_{W_0^{1,2}} +$$

$$\int_\Gamma A(t)(g^t - g)(u^t - u) + (A(t) - I)g(u^t - u) + \|g^t - g\|_{W_0^{-1/2,2}} \|u^t - u\|_{W_0^{1,2}} \leq$$

$$\leq c(t)\|\nabla u\|_{L^2}\|\nabla(u^t - u)\|_{L^2} + \|f^t - f\|_{W_0^{-1,2}}\|u^t - u\|_{W_0^{1,2}} +$$

$$c(t)\|g^t - g\|_{W_0^{-1/2,2}}\|u^t - u\|_{W_0^{1,2}} + c(t)\|u^t - u\|_{W_0^{1,2}}\|g\|_{W_0^{-1/2,2}}, \qquad (4.7)$$

where $c(t)$ we obtain from the estimation of $A(t)$:

$$\frac{1}{2}\|u^t - u\|_{W_0^{1,2}} \leq c(t)\|\nabla u\|_{L^2} + c\|f^t - f\|_{W_0^{-1,2}} + \|g\|_{W_0^{-1/2,2}} + c\|g^t - g\|_{W_0^{-1/2,2}}.$$

Since $f \in W_0^{-1,2}$ and we have shown that f^t is strongly continuous with respect to t i.e. $f^t \to f$ in $W_0^{-1,2}$, $g^t \to g$ in $W_0^{-1/2,2}$, which implies $u^t \to u$ in $W_1^{1,2}(\mathbb{R}^3_+)$.

Since V is compactly supported it means supp $V \subset B(R)$ for some R, thus the first term in right hand side of (4.7) takes the form

$$\begin{aligned} &\textstyle\int_\Omega (A(t) - I)\nabla u \cdot \nabla\varphi = \int_{B(R)} (A(t) - I)\nabla u \cdot (\nabla u^t - \nabla u) \leq \\ &\quad \leq c(t)\|\nabla u\|_{L^2}\|\nabla(u^t - u)\|_{W_0^{-1,2}} \end{aligned}$$

and it follows, in view of the properties of the mapping $T_t(V)$, that $c(t) \to 0$, which implies that (4.5) hold, similarly (4.5)'.
Therefore, we obtain the existence of material derivative $\dot{u} \in W_1^{1,2}(\mathbb{R}^N_+)$ which is given by a unique solution to problem (1.10).

5 Fréchet material derivatives

5.1 Transported problem(TRP)

For the convenience of reader we repeat the results in the language of perturbations of identity technique. Let us fix $\theta \in \Theta_k$ and let $\xi \in (-\delta, \delta)$, consider $\mathcal{H}_\xi = I + \xi\theta$. We investigate the existence of the transported problem in the fixed domain Ω satisfying the equations

$$\begin{aligned} -\frac{1}{\beta} \text{ div } (A(\xi)\nabla u^\xi) &= f^\xi \text{ in } \Omega \\ \nabla u^\xi \cdot \eta^\xi &= g^\xi \text{ on } \Gamma. \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} A(\xi) &= q_\xi^* D\mathcal{H}_\xi^{-1} D\mathcal{H}_\xi^{-1}, \\ \eta^\xi &= \mathcal{H}_\xi^{-1} \cdot n^\xi. \end{aligned} \tag{5.2}$$

Theorem 5.1. *Let $N \geq 3$, suppose $f^\xi \in W_1^{0,2}(\mathbb{R}^N_+)$ and $g \in W_1^{1/2,2}(\Gamma)$ then problem (TRP) has a unique solution $u^\xi \in W_1^{2,2}(\mathbb{R}^N_+)$ and*

$$\|u^\xi\|_{W_1^{2,2}(\mathbb{R}^N_+)} \leq C(\|f^\xi\|_{W_1^{0,2}(\mathbb{R}^N_+)} + \|g^\xi\|_{W_1^{1/2,2}(\Gamma)}).$$

Proof. Again we define the bilinear form

$$B(u^\xi, v^\xi) = \int_\Omega A_\xi \nabla u^\xi \nabla v^\xi$$

is the uniformly elliptic then applying Theorem 3.2 we get the existence of the solution u^ξ. □

5.2 Perturbed problem (PP)

Proposition 5.2. *Let* $N \geq 3, f \in W_1^{0,2}(\Omega_\xi)$, *then for* $|\xi| \to 0$

$$\frac{F^\xi - f}{|\xi|} \to \text{div } \theta f + \dot{f} \text{strongly in } W_0^{-1,2}(\Omega_\xi).$$

Theorem 5.3. *Let* $N \geq 3$, *suppose* $f_\xi \in W_1^{0,2}(\Omega_\xi)$, $g_\xi \in W_1^{1/2,2}(\Gamma_\xi)$ *then problem* (PP) *has a unique solution* $u_\xi \in W_1^{2,2}(\Omega_\xi)$.

Proof. We have for all $\psi \in_0^{1,2} (\Omega_\xi)$

$$\langle f_\xi, \psi \rangle_{W_0^{-1,2}(\Omega_\xi) \times \mathring{W}_0^{1,2}(\Omega_\xi)} = \langle \gamma(t) f_\xi, q_\xi^{-1} \varphi \circ \mathcal{H}_\xi^{-1} \rangle_{W_0^{-1,2}(\Omega_\xi) \times \mathring{W}_0^{1,2}(\Omega_\xi)} =$$

$$= \langle q_\xi \Delta u_\xi, q_\xi^{-1} \varphi \circ \mathcal{H}_\xi^{-1} \rangle_{W_0^{-1,2}(\Omega_\xi) \times \mathring{W}_0^{1,2}(\Omega_\xi)} =$$

$$\langle \text{ div } (A_\xi \nabla u^\xi, \varphi \rangle_{W_0^{-1,2}(\Omega) \times \mathring{W}_0^{1,2}(\Omega)} = \langle f^\xi, \varphi \rangle_{W_0^{-1,2}(\Omega) \times \mathring{W}_0^{1,2}(\Omega)}.$$

Applying Theorem 5.1 we get the existence of solution of perturbed problem. □

5.3 Proof of the Main theorem 2

The aim of this section is to prove the existence material derivative as a weak limit of

$$\frac{u^\xi - u}{|\xi|} \to \dot{u} \in W_0^{1,2}(\Omega) \tag{5.3}$$

Denoting

$$w^\xi = \frac{u^\xi - u}{|\xi|} - \dot{u}$$

we obtain the following equation

$$-\Delta w^\xi = \text{div } \left[\tfrac{A(\xi)-I}{|\xi|}\nabla u^t - A'(0)\nabla u\right] + \tfrac{f^\xi - f}{|\xi|} - \dot{f}] \quad \text{in } \Omega. \tag{5.4}$$

and

$$\frac{\partial w^\xi}{\partial n^\xi} = \frac{g^\xi - g}{|\xi|} - \dot{g} - \nabla u \cdot DV \cdot n - \nabla u \cdot \dot{n} \quad \text{on } \Gamma.$$

The weak formulation of (5.4) is the following

$$\begin{aligned}
&\textstyle\int_{\mathbb{R}^N_+} \nabla w_\xi \cdot \nabla\phi = \\
&\textstyle\int_{\mathbb{R}^N_+} [\frac{A(\xi)-I}{|\xi|}\nabla u^\xi \cdot \nabla\phi - A'(0)\nabla u \nabla\phi] + \int_{\mathbb{R}^N_+} [\frac{f^\xi - f}{|\xi|} - \dot{f}]\phi dx + \\
&\textstyle + \int_\Gamma \left(\frac{g^\xi - g}{|\xi|} - \dot{g} - \nabla u \cdot DV \cdot n - \nabla u \cdot \dot{n}\right)\phi d\sigma + \int_\Gamma \left(\frac{A(\xi)-I}{|\xi|} g^\xi + A'(0) g\right)\phi, \ \forall \phi \in W_0^{1,2}(\mathbb{R}^N_+)
\end{aligned} \tag{5.5}$$

The goal of this section is to prove the following convergence

$$w^\xi = \frac{u^\xi - u}{|\xi|} - \dot{u} \to 0 \text{ as } |\xi| \to 0, \text{ strongly in } W_0^{1,2}(\mathbb{R}^N_+),$$

and

$$\frac{\partial w^\xi}{\partial n^\xi} = \frac{g^\xi - g}{|\xi|} - \dot{g} \to 0 \text{ as } |\xi| \to 0, \quad \text{weakly in } W_0^{1/2,2}(\mathbb{R}^{N-1}).$$

To get the assertion it is sufficient to prove the strong convergence of the following terms

$$\frac{A(\xi) - I}{|\xi|}\nabla u^\xi - A^{'}(0)\nabla u \to 0, \ |\xi| \to 0, \text{ strongly in } W_0^{0,2}(\mathbb{R}^N_+)^N, \tag{5.6}$$

where $A'(0) = \text{div } \theta I - {}^*D\theta - D\theta$ and

$$\frac{A(\xi) - I}{t} g^\xi - A'(0)g - \nabla u \cdot DV \cdot n - \nabla u \cdot \dot{n} \to 0 \text{ weakly in } W_0^{-1/2,2}(\mathbb{R}^{N-1}_+) \tag*{(5.6)$'$.}$$

We assume that

$$\frac{f^\xi - f}{|\xi|} - \dot{f} \to 0 \text{ with } |\xi| \to 0 \text{ strongly in } \ W_0^{-1,2}(\mathbb{R}^N_+), \tag{5.7}$$

and

$$\frac{g^\xi - f}{|\xi|} - \dot{g} \to 0 \text{ with } |\xi| \to 0 \text{ strongly in } W_0^{-1/2,2}(\mathbb{R}^{N-1}). \qquad (5.7)'$$

Let $\varphi = u^\xi - u$ be a test function in the variational formulation, hence

$$\int_{\mathbb{R}^N_+} A(\xi)|\nabla(u^\xi - u)|^2 - (A(\xi) - I)\nabla u \cdot \nabla(u^\xi - u)$$
$$+ \int_\Gamma A(t)(g^\xi - g)(u^\xi - u) + (A(\xi) - I)g(u^\xi - u) = \langle f^\xi - f, u^\xi - u\rangle + \langle g^\xi - g, u^\xi - u\rangle.$$

From the properties of vector field θ it follows that

$$\int_{\mathbb{R}^N_+} A(\xi)|\nabla(u^\xi - u)|^2 \leq$$

$$\leq \int_{\mathbb{R}^N_+} |(A(\xi) - I)\nabla u||\nabla(u^\xi - u)| + \|f^\xi - f\|_{W_0^{0,2}}\|u^\xi - u\|_{W_0^{1,2}} \leq$$

$$\int_\Gamma A(\xi)(g^\xi - g)(u^\xi - u) + (A(\xi) - I)g(u^\xi - u) + \|g^\xi - g\|_{W_1^{1/2,2}}\|u^\xi - u\|_{W_0^{1,2}}$$

$$\leq c(\xi)\|\nabla u\|_{W^{0,2}}\|\nabla(u^\xi - u)\|_{W^{0,2}} + \|f^\xi - f\|_{W_1^{0,2}}\|u^\xi - u\|_{W_0^{1,2}} +$$

$$c(\xi)\|g^\xi - g\|_{W_1^{1/2,2}}\|u^\xi - u\|_{W_0^{1,2}} + c(\xi)\|u^\xi - u\|_{W_0^{1,2}}\|g\|_{W_1^{1/2,2}}. \qquad (5.8)$$

From the properties of $A(\xi)$ we have

$$\tfrac{1}{2}\|u^\xi - u\|_{W_0^{1,2}} \leq c(\xi)\|\nabla u\|_{L^2} + c\|f^\xi - f\|_{W_1^{0,2}} + c(\xi)\|g\|_{W_1^{1/2,2}} + c\|g^t - g\|_{W_1^{1/2,2}}.$$

Since $f \in W_1^{0,2} \subset W_0^{-1,2}$ and we consider that that f^ξ is strongly continuous with respect to ξ i.e. $f^\xi \to f$ in $W_0^{0,2}$, which implies $u^\xi \to u$ in $W_0^{1,2}(R^3_+)$ strongly.
Since θ is compactly supported supp $\theta \subset B(R)$ for some R, thus the first term in right hand side of (5.8) takes the form

$$\int_\Omega (A(\xi){-}I)\nabla u{\cdot}\nabla\varphi = \int_{B(R)} (A(\xi){-}I)\nabla u{\cdot}(\nabla u^\xi{-}\nabla u) \leq c(\xi)\|\nabla u\|_{L^2}\|\nabla(u^\xi{-}u)\|_{L^2}$$

and it follows, in view of the properties of the mapping $\mathcal{H}_\xi(\xi)$, that $c(\xi) \to 0$, which implies that (5.6) hold.

Therefore, we get the material derivative $\dot{u} \in W_1^{1,2}(\mathbb{R}_+^N)$ which is given by a unique solution to problem (1.10) which is same as before, but the strong convergence in the energy space.

Remark 5.1 It is not difficult to extend our result to L^p theory.

6 References

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