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Geometry and Gâteaux smoothness in separable Banach spaces

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Abstract

It is a classical fact, due to Day, that every separable Banach space admits an equivalent Gâteaux smooth renorming. In fact, it admits an equivalent uniformly Gâteaux smooth norm, as was shown later by Šmulyan. It is therefore rather unexpected that the existence of Gâteaux smooth renormings satisfying various quantitative estimates on the directional derivative has rather strong structural and geometrical implications for the space. For example, by a result of Vanderwerff, if the directional derivatives satisfy a p-estimate, where p varies arbitrarily with respect to the point and the direction in question, then the Banach space must be an Asplund space. In the present survey paper, we discuss the interplay between various types of Gâteaux differentiability of norms and extreme points with the geometry of separable Banach spaces. In particular, we present various characterizations of Asplund, reflexive, superreflexive, and other classes of separable Banach spaces, via smooth as well as rotund renormings. We also include open problems of various levels of difficulty, with the hope of stimulating research in the area of smoothness and renormings of Banach spaces.

In nonlinear analysis, the differentiability of norms plays an important role. The most important type of differentiability is that of Fréchet differentiability. However, in many instances it suffices to use weaker forms of differentiability, i.e., variants of the Gâteaux differentiability (that are more often accessible). This happens especially when some convexity arguments can be combined with Baire category techniques. The present paper surveys some of these results and discusses several ideas and constructions in their proofs.

We focus on the interplay of these concepts with the geometry of separable spaces, for example with problems on containment of c_0 or ℓ_1, with superreflexivity, the Radon–Nikodým property, etc. Several open problems in this area are discussed.

We refer to, e.g., [Gode], [DGZb], [Fab], [AlKal06], [BoVa10], and [FHHMZ] for all unexplained notions and results used in this note.

1 Pointwise directional Hölder derivatives

Pisier proved in [Pisi75] that every Banach space that admits a uniformly Fréchet differentiable norm (i.e., a superreflexive space) can be renormed (by this we mean "equivalently

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renormed") by a norm with modulus of smoothness of power type. This is not true, even directionwise, for the case of the uniform Gâteaux differentiability.

Recall that a real valued function f is *Gâteaux differentiable* (or, *Gâteaux smooth*) at a point x of a Banach space X, if there exists an element g in the dual space X^* such that $\lim_{t\to 0} \frac{1}{t}(f(x+th)-f(x)) = g(h)$ for each $h \in X$.
A norm is called a *Gâteaux differentiable norm*, if it is Gâteaux differentiable at all nozero points in the space.
Recall that a norm $\|\cdot\|$ of a Banach space X is *uniformly Gâteaux differentiable on* X (UG, in short) if for each $h \in X$, the limit above is uniform in x in the unit sphere S_X of X, i.e. if, and only, if for each $h \in X$, $\lim_{t\to 0} \frac{1}{t}(\|x+th\| + \|x-th\| - 2\|x\|) = 0$ uniformly for $x \in S_X$.

It is well known that *every* separable Banach space can be renormed by a UG norm: Indeed, if $\{x_i\}$ is a dense sequence in the unit sphere S_X of a separable Banach space $(X, \|\cdot\|_0)$, let the norm $\|\cdot\|$ be defined for $f \in X^*$ by

$$\|f\|^2 = \|f\|_0^2 + \sum 2^{-i} f^2(x_i),$$

where $\|\cdot\|_0$ is the canonical dual norm of X^*. Then $\|\cdot\|$ is a dual equivalent norm on X^*, and standard convexity arguments give that $f_n - g_n \to 0$ in the weak* topology of X^* whenever $\|f_n\| = \|g_n\| = 1$ and $\|f_n + g_n\| \to 2$. This means, by the Šmulyan lemma (see, e.g., [FHHMZ, Corollary 7.22]), that the predual norm of $\|\cdot\|$ is UG.

The situation is different if we require that the derivative be pointwise directionally Hölder in the following sense: A function φ on a Banach space X is said to have a *directional Hölder derivative at* x_0 if for each $h \in B_X$ there are $K_h > 0$, $\delta_h > 0$ and $\alpha_h \in (0,1]$ such that

$$|\big(\varphi'(x_0+th) - \varphi'(x_0)\big)(th)| \le K_h t^{1+\alpha_h}$$

for all $0 \le t \le \delta_h$. In case that $\alpha_h = 0$ for all $h \in S_X$, φ' is said to be *directionally Lipschitz at* x_0. We will say that φ has *pointwise directional Hölder derivative on* X if for each $x \in X$, φ has directional Hölder derivative at x.
We say that a function φ has *uniform directional Hölder derivative* if for each $h \in S_X$ there are $C_h > 0$ and $\alpha_h > 0$ such that

$$|\big(\varphi'(x+th) - \varphi'(x)\big)(th)| \le C_h t^{1+\alpha_h}$$

for every $x \in X$.

These cases have already a strong impact on the structure of the space. This is seen in the following results.

Recall that a *bump function* on a Banach space X is a real valued function on X with bounded nonempty support.

Theorem 1 ([Vand93])**.** *Assume that X is a separable Banach space. Suppose that X admits a continuous bump function with pointwise directional Hölder derivative. Then X^* is separable.*

Note that not every Banach space with separable dual admits a bump as in Theorem 1. Indeed, the reflexive separable space $(\sum \ell_\infty^n)_2$ is not superreflexive and thus cannot admit such a bump by Theorem 2 below.

After the combined efforts of Davis, Huff, Maynard, Phelps and Rieffel (see references in [Diest, Chapter 6]), we know that a Banach space X has the *Radon-Nikodým property* (in

short, RNP) if and only it is *dentable*, i.e., if each bounded set in X has slices of arbitrarily small diameter (where a *slice* is the intersection of the set with an open halfspace). This happens if, and only if, each bounded closed convex set C in X is the closed convex hull of its strongly exposed points (see, e.g., [FHHMZ, Theorem 11.3]). A point $x_0 \in C$ is called *exposed* (by a functional $f \in X^*$) if $\{x \in C : \ f(x) = \sup_{x \in C} f(x)\} = \{x_0\}$, and *strongly exposed* (by f) if it is exposed by f and $\|x_n - x_0\| \to 0$ whenever $f(x_n) \to f(x_0)$.
There are examples of Banach space failing RNP and not containing either c_0 or L_1, see [Tal] and [BoRo2].

A Banach space X is called an *Asplund space* if every separable subspace of X has separable dual. A norm on a Banach space is said to be *strictly convex* (or *rotund*) if each point in its sphere is an extreme point of the ball. A norm $\|\cdot\|$ is *locally uniformly rotund* (LUR, in short) if $\|x_n - x\| \to 0$ whenever $x_n, x \in S_X$ are such that $\|x_n + x\| \to 2$.

The following couple of results summarize some of the known results in this area.

Theorem 2 ([DGZ93a], [MPVZ93], [Vand93][MaVa]). *Let X be a Banach space. Then each one of the following four conditions implies that X is superreflexive.*
(i) *The space X has the RNP property and admits a continuous bump function with pointwise directional Hölder derivative.*
(ii) *Both X and X^* admit continuous bump functions with pointwise directional Hölder derivative.*
(iii) *The space X admits an LUR norm with pointwise directional Hölder derivative on the sphere.*
(iv) *The space X admits a bounded bump function with uniformly directional Hölder derivative.*

Note that due to Pisier's theorem mentioned above, and due to further results in this area (cf. e.g. [FHHMZ, Chapter 9]), each of the conditions in Theorem 2 characterizes superreflexive spaces.

Note also that, using that the space c_0 admits a C^∞ smooth norm, Theorem 2 (iii) shows that the Asplund averaging procedure (cf. e.g. [DGZb, Chapter 3]) does not work for this kind of Gâteaux differentiability.
The Day norm on c_0 (cf. e.g. [DGZb, p. 69]) is an example of a LUR norm whose pointwise modulus of smoothness is of power type 2 for points of a dense subset. This norm is also an example of a function for which the set of points at which the derivative is pointwise Lipschitz is not a G_δ set (cf. [DGZ93a]).

Theorem 3 ([Vand93]). *Let X be a Banach space. Assume that both X and X^* admit continuous bump functions with pointwise directional Lipschitz derivative. Then X is isomorphic to a Hilbert space.*

Corollary 4 ([MPVZ93], [MaVa], [BoFa93], [DGHZ87]). (i) *A separable $C(K)$ space admits a continuous bump function with pointwise directional Hölder derivative if, and only if, K is countable.*
(ii) *If X is a Banach space, then the space of compact operators $K(X)$ on X admits a continuous bump function with pointwise directional Lipschitz derivative if, and only if, X is isomorphic to a Hilbert space.*

We will not give proofs of the results above in this note. We will present in the lemmas below only some of the main ideas and constructions used in these proofs.

One of the main tools in the proofs consists of the use of the Baire Category theorem in several ways:

First, in [Vand93], the Baire category theorem is used in the proof of Theorem 1 in connection with the Ekeland variational principle (see, e.g., [FHHMZ, Chapter 7]).
Second, the Baire Category theorem is used for the directions of differentiability in the convex case ([BoNo94]).
Third, by using the RNP and duality, the Baire Category Theorem is used to apply the Day method to reach the uniformity required ([DGZ93a], [MPVZ93], [Day43]).

If f is a real valued function on a Banach space X, $x \in X$ and $\eta > 0$, we put

$$\rho_x(\eta) = \sup\{|f(x+h) + f(x-h) - 2f(x)| : \|h\| \leq \eta\}$$

and call, for a fixed $x \in X$, the function ρ_x the *pointwise modulus of smoothness of f at x* ([Zyg59, p. 43]). We say that a real valued function f on X has *pointwise modulus of smoothness ρ_x of power type $p > 0$ at x* if

$$\limsup_{\eta \to 0} \rho_x(\eta)\eta^{-p} < \infty.$$

The *modulus of convexity* of the norm $\|\cdot\|$ is defined for $\varepsilon \in [0, 2]$ by

$$\delta(\varepsilon) = \inf\{1 - \|(x+y)/2\| : x, y \in S_X, \|x - y\| \geq \varepsilon\}.$$

The norm is *uniformly convex* (or *uniformly rotund*, UR in short) if, and only if, $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$.
We say that the norm $\|\cdot\|$ *has modulus of convexity of power type $p > 0$* if there is a constant $C > 0$ such that

$$\delta(\varepsilon) \geq K\varepsilon^p$$

for every $\varepsilon \in (0, 2]$.
The *modulus of smoothness* of the norm $\|\cdot\|$ is defined for $\tau > 0$ by

$$\rho(\tau) = \sup\{\frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \|x\| = \|y\| = 1\}.$$

The norm $\|\cdot\|$ is *uniformly Fréchet differentiable* (or *uniformly Fréchet smooth*) if, and only if,

$$\lim_{\tau \to 0} \rho(\tau)/\tau = 0.$$

We say that the norm $\|\cdot\|$ *has modulus of smoothness of power type $p > 1$* if there is a constant $C > 0$ such that

$$\rho(\tau) \leq K\tau^p$$

for every $\tau > 0$.

We will use the fact that the the derivative $\|\cdot\|'$ is α-Hölder ($\alpha \in (0, 1]$) if, and only if, the norm $\|\cdot\|$ has modulus of smoothness of power type $1 + \alpha$ (see, e.g., [DGZb, p. 204]).

The norm $\|\cdot\|$ is *locally uniformly rotund on X* if for every $x \in S_X$ and every $\varepsilon \in (0, 2]$,

$$0 < \delta_x(\varepsilon) := \inf\{1 - \|x + y\|/2; y \in S_X, \|y - x\| \geq \varepsilon\}.$$

For given $x \in S_X$, we call the function $\delta_x(\varepsilon)$ *the pointwise modulus of rotundity at x*. If there is a $p > 0$ and a constant $C > 0$ such that $\delta_x(\varepsilon) \geq C\varepsilon^p$, for each $\varepsilon \in (0, 2]$, we say that $\|\cdot\|$ has *pointwise modulus of rotundity of power type p at x*.

Lemma 5 ([BoNo94]). *Let a Banach space X have a norm with pointwise directional Lipschitz derivative on the sphere. Then X admits a norm with pointwise modulus of smoothness of power type 2.*

Proof (Sketch) Fix $x_0 \in S_X$. Define F_n by

$$F_n = \{h \in B_X : \|x_0 + th\| + \|x_0 - th\| - 2\|x_0\| \leq n\|th\|^2, \text{ for all } 0 < t \leq 1\}.$$

Then F_n is closed for each n, and $\bigcup_n F_n = B_X$. By the Baire Category theorem, there is a neighborhood V of some point $h_0 \neq 0$ in the interior of B_X and a positive integer n such that

$$\|x_0 + th\| + \|x_0 - th\| - 2\|x_0\| \leq n\|th\|^2, \text{ for all } h \in V,\ 0 < t \leq 1.$$

Consider the cone generated by taking the convex hull of $-h_0$ and V. This cone contains B_r for some $r > 0$. For some $k \geq n$, the convexity gives

$$\|x_0 + th\| + \|x_0 - th\| - 2\|x_0 \leq k\|th\|^2, \text{ for all } \|h\| \leq r,\ 0 < t \leq 1.$$

□

Lemma 6 ([DGHZ87]). *Let X be a Banach space. Assume that for each $x \in S_X$ there is $p(x) > 0$ such that $\|\cdot\|$ has pointwise modulus of rotundity of power type $p(x)$ at x. Then X is superreflexive.*

Proof. For positive integers N and p, we let

$$F_{N,p} = \{x \in S_X : \delta_x(\varepsilon) \geq N^{-1}\varepsilon^p, \text{ for all } \varepsilon \in (0, 2]\}.$$

It is not difficult to show that each $F_{N,p}$ is closed. Our assumption means that

$$S_X = \bigcup_{N,p} F_{N,p}.$$

By the Baire Category theorem, there exist positive integers N_1 and p_1 such that F_{N_1,p_1} has a nonempty interior in S_X. Hence there is an open set O in X such that $O \cap S_X \neq \emptyset$, and

$$x, y \in O \cup S_X,\ \|x - y\| \geq \varepsilon \quad \text{implies} \quad \|x + y\| \leq 2 - 2N_1^{-1}\varepsilon^{p_1}.$$

Now, a result of Day ([Day73, Theorem 1]) provides a uniformly rotund norm of power type p_1. □

Lemma 7 ([DGZ93a]). *Let a Banach space X with the RNP admit a Lipschitz bump function φ with pointwise Lipschitz derivative. Then X admits a norm with modulus of smoothness of power type 2.*

The main idea of the proof
Put $\psi(x) = \varphi^{-2}(x)$ if $\varphi(x) \neq 0$, and $+\infty$ otherwise. Let ψ^* be the Fenchel conjugate of ψ, i.e. for $f \in X^*$ put

$$\psi^*(f) = \sup\{f(x) - \psi(x) :\ x \in X\}.$$

As X has the RNP, ψ^* is Fréchet differentiable on a norm dense set of points Ω in X^* ([Coll76]) with derivative in X. Let $\widetilde{\psi}$ denote the Fenchel conjugate of ψ^* in X. The derivatives of ψ^* give rise to the epigraph of $\widetilde{\psi}$, and since they are strongly exposed points,

they are actually in the epigraph of ψ. It is not hard to check that these strongly exposed points are points where the derivative of $\widetilde{\psi}$ is directionally Lipschitz and, by the previous lemma, pointwise Lipschitz, so they represents points with pointwise modulus of smoothness of power type 2. By passing to the dual, we get points with pointwise rotundity behavior of power type 2, which give, by the method of the proof of previous lemma, a norm on X^* of modulus of rotundity of power type 2. We then finish the proof by taking the predual of this norm. □

Sketch of main ideas of the proof of Theorem 2.
(i) We use the idea in the proof of Lemma 7, see [MPVZ93].
(ii) The space X^* is Asplund (Theorem 1) and thus X has the RNP (see e.g. [FHHMZ, Chapter 11]). Hence (ii) follows from (i).
(iii) We use the general method of passing from local uniform differentiability to the uniform differentiability (see e.g. [DGZb, p. 188]) combined with the method of the proof of Lemma 5 and Lemma 6.
(iv) We use the general method of constructing uniformly differentiable norms from uniformly differentiable bumps as explained e.g. in [FHHMZ, Chapter 9]. The complete proof is given in [MaVa].

□

Sketch of the main idea in the proof of Theorem 3.
Both X and X^* are Asplund spaces by Theorem 1. Hence both have the RNP (cf. e.g. [FHHMZ, Chapter 11]). Thus both X and X^* admit norms with modulus of smoothness of power type 2 and therefore, by Kwapien's theorem (see e.g. [AlKal06, Chapter 7.4]), X is isomorphic to a Hilbert space.

□

Sketch of the main ideas in the proof of Corollary 4.
(i) follows from Theorem 1, since if $C(K)$ is separable Asplund then K is scattered and thus countable (it is metrizable). If K is countable, then $C(K)$ admits a C^∞ Fréchet differentiable norm (see, e.g., [FHHMZ, Corollary 10.14]).
(ii) The "if" part follows [Tom74]. For the "only if" part, use the fact that $K(X)$ contains copies of X and X^*.
Thus both X and X^* are Asplund spaces and thus both have the RNP. Thus both have norms with modulus of smoothness of power type 2 and thus, by Kwapien theorem (cf. e.g, [AlKal06]), X is isomorphic to a Hilbert space.

□

Problem 1. ([DGZ93a]) *Can the assumption of the RNP in Theorem 2 be replaced by the weaker condition that X does not contain an isomorphic copy of c_0?*

2 Second order Gâteaux differentiability

We will say that a function $\varphi : X \to \mathbb{R}$ is *twice Gâteaux differentiable at $x \in X$* provided that the Gâteaux derivative $\varphi'(y)$ exists for y in a neighborhood of x, the limit

$$\varphi''(x)(h,k) := \lim_{t\downarrow 0}(\frac{1}{t}(\varphi'(x+tk) - \varphi(x)))(h)$$

exists for each $h, k \in X$, and that $\varphi''(\cdot,\cdot)$ is a continuous symmetric bilinear form.

The following result shows that twice Gâteaux differentiable norms are quite easily accessible in some separable superreflexive spaces.

Theorem 8 ([FWZ]). *Assume that X is a separable Banach space with the RNP. Then X admits an equivalent twice Gâteaux differentiable norm if, and only if, X admits a norm with modulus of smoothness of power type 2.*

Proof. First if X admits a twice Gâteaux differentiable norm then X admits a norm with modulus of smoothness of power type 2 by Lemma 7.
We give only roughly, the main idea of the proof of the other implication and refer to [FWZ] for the details.
Select a dense sequence $\{h_i\}$ in S_X. Find a C^∞ smooth function $\varphi_0 : \mathbb{R} \to \mathbb{R}$ such that φ_0 is nonnegative and even, vanishes outside $[-1, 1]$, and satisfies $\int_{\mathbb{R}} \varphi_0 = 1$. Put $f_0 = \|x\|^2$ and $\varphi_n(t) = 2^n \varphi_0(2^n t)$ for $t \in \mathbb{R}$, $n \geq 1$. Define a sequence of functions $\{f_n : X \to \mathbb{R}\}$ by

$$f_n(x) = \int_{\mathbb{R}^{n+1}} f_0 \left(x - \sum_{i=0}^{n} t_i h_i \right) \prod_{i=0}^{n} \varphi_i(t_i)\, dt_0\, dt_1 ... dt_n$$

Then f_n converge uniformly on bounded sets, to a twice Gâteaux differentiable function g which gives rise, via Minkowski functional of the set $\{x : g(x) \leq C\}$ for some C, to the desired norm on X. □

Remark. Since ℓ_p, $p \in [1, 2)$, does not admit any norm of modulus of smoothness of power type 2 (cf. e.g. [DGZb, p. 222]), it does not admit any twice Gâteaux differentiable norm by Theorem 8. On the other hand the space $(\sum \ell_4^n)_2$ has a norm with modulus of smoothness of power type 2, and admits no twice Fréchet differentiable norm [DGJ]; however, by Theorem 8, it admits a twice Gâteaux differentiable norm. It is proved in [Troy] that, for p odd, ℓ_p space admits p times Gâteaux differentiable norm.

The following is a corollary of the proof of Theorem 8

Theorem 9 ([MPVZ93]). *Assume that X is a separable Banach space with a norm of modulus of smoothness of power type 2. Then*
(i) *Every convex function which is bounded on bounded subsets of X can be approximated uniformly on bounded sets by twice Gâteaux differentiable convex functions whose first derivatives are Lipschitz.*
(ii) *The space X admits a twice Gâteaux smooth UR norm.*

The following result is a corollary of Theorem 3.

Theorem 10 ([Vand93]). *Let X be a Banach space. Assume that both X and X^* admit continuous bump functions the restriction of which to each line in X, respectively in X^*, is twice differentiable. Then X is isomorphic to a Hilbert space.*

Theorem 10 improves on Meškov's result that X is isomorphic to a Hilbert space if both X and X^* admit Fréchet C^2 smooth bumps (cf. e.g. [DGZb, Chapter 5], [FHHMZ, Chapter 10]).

3 Preserved extreme points

The classical Krein–Milman theorem (see, e.g., [FHHMZ, Theorem 3.65]) says that every nonempty convex compact subset C of a locally convex space has an extreme point —hence C is the closed convex hull of the set of its extreme points. If C is a nonempty bounded closed convex subset of a Banach space, its w^*-closure $\overline{C}^{w^*}$ in X^{**} is a w^*-compact convex subset of X^{**}, hence it is the w^*-closed convex hull of the set of its extreme points.

Definition 11. *Let C be a nonempty bounded closed convex subset of a Banach space X. The elements in* $\operatorname{Ext}\overline{C}^{w^*} \cap \operatorname{Ext} C$ *are called* preserved extreme points of C.

Preserved extreme points are called *weak*-extreme* in [Mat]. Extreme points of C which are not preserved extreme points of C will be called *unpreserved*.

A straightforward consequence of James' weak compactness theorem is that a bounded closed convex subset C of a Banach space is weakly compact if (and only if) $\operatorname{Ext} C = \operatorname{Ext}\overline{C}^{w^*}$. Theorem 24 below gives a renorming characterization of reflexivity in terms of preserved extreme points.
The following simple observation will be used several times in this note: Given a non-empty subset A of a Banach space X, a finite subset $\{f_i : \ i = 1, 2, \dots, n\}$ of X^*, and numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ in $\mathbb{R}$,

$$\begin{aligned} \{x^{**} \in \overline{A}^{w^*} : \ f_i(x^{**}) > \alpha_i, \ i = 1, 2, \dots, n\} \\ \subset \overline{\{x \in A : \ f_i(x) > \alpha_i, \ i = 1, 2, \dots, n\}}^{w^*} \\ \subset \{x^{**} \in \overline{A}^{w^*} : \ f_i(x^{**}) \geq \alpha_i, \ i = 1, 2, \dots, n\} \end{aligned} \tag{3.1}$$

The following lemma characterizes preserved extreme points of bounded closed convex subsets of a Banach space.

Lemma 12 (Rosenthal, see [LLT1])**.** *Assume that C is a bounded closed convex set in a Banach space X. A point e in C is a preserved extreme point of C if, and only if, the slices of C containing e form a neighborhood base of the restricted weak topology on C at the point e.*

Proof. If e is a preserved extreme point, the result is a consequence of Choquet's lemma (see, e.g., [FHHMZ, Lemma 3.69]). For the other direction, assume that $2e = y^{**} + z^{**}$, where $y^{**}, z^{**} \in \overline{C}^{w^*}$, $y^{**} \neq z^{**}$, and let U be a closed neighborhood of e in (X^{**}, w^*) missing both y^{**} and z^{**}. By the assumption, $C \cap U$ contains a set of the form $C \cap S$, where $S := \{x^{**} : \ x^{**} \in X^{**}, \ f(x^{**}) > \alpha\}$ for some $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(e) > \alpha$. Using (3.1) we get $\overline{C}^{w^*} \cap S \subset U$; however, $\overline{C}^{w^*} \cap S$ must contain either y^{**} or z^{**} (or both), a contradiction. □

3.1 Preserved extreme points, RNP and KMP

Recall that a Banach space X is said to have the *Krein–Milman property* (KMP, in short) if every bounded closed convex set in X has an extreme point. Equivalently, every bounded closed convex set in X is the closed convex hull of the set of all extreme points (the equivalence is due to Lindenstrauss, cf., e.g., [FHHMZ, Exer. 7.57]). Note that every space with the RNP has the KMP (see e.g. [FHHMZ, Chapter 11]). It is an open problem (see Problem 3) whether every space with the KMP has the RNP. This problem has been solved in the

positive in dual spaces in [HuMo], and, for the notion of preserved extreme points, in the following result (see also Theorem 19 below). Here, $\operatorname{dist}(A,B) := \inf\{\|a-b\| :\ a\in A,\ b\in B\}$ for two non-empty subsets A and B of a Banach space X.

Theorem 13. *Let X be a Banach space. Then, the following conditions are equivalent:*
(i) *X fails the RNP.*
(ii) [SchSeWe] *For every $\varepsilon > 0$ there exists an equivalent norm $\|\cdot\|$ on X such that* $\operatorname{dist}(\operatorname{Ext}(B_{(X^{**}\|\cdot\|)}), X) \geq 1-\varepsilon$.
(iii) [Bou1], [Ste2] *There exists an equivalent norm $\|\cdot\|$ on X such that each extreme point of $B_{(X,\|\cdot\|)}$ is unpreserved (and* $\operatorname{dist}(\operatorname{Ext}(B_{(X^{**}\|\cdot\|)}), X) > 0$*).*

We shall not prove this theorem. We only note that (ii)⇒(iii) is obvious, and that (iii)⇒(i) follows from the fact that, if $\|\cdot\|$ is an equivalent norm in X, and X has the RNP, then $B_{(X,\|\cdot\|)}$ has a strongly exposed point. Clearly, such a point is a preserved extreme point of $B_{(X,\|\cdot\|)}$.

Remark. A simple consequence of (i)⇔(ii) in Theorem 13 is the following: If a Banach space X fails the RNP then, for every $\varepsilon > 0$, there exists an equivalent norm $\|\cdot\|$ on X such that every slice S of $B_{(X,\|\cdot\|)}$ has $\|\cdot\|$-diameter greater than $1-\varepsilon$. Indeed, Let $\|\cdot\|$ be the norm associated to ε given by Theorem 13. Let $S(f,\delta) := \{x\in B_{(X,\|\cdot\|)} :\ f(x) > 1-\delta\}$ be an arbitrary slice of $B_{(X,\|\cdot\|)}$, where $f\in S_{(X^*,\|\cdot\|)}$ and $\delta > 0$. By the Bishop–Phelps theorem, there exists $g\in S_{(X^*,\|\cdot\|)}$ close enough to f that attains its supremum on $B_{(X,\|\cdot\|)}$. We can find $\delta' > 0$ small enough so that $S(g,\delta')\subset S(f,\delta)$. It follows that $\{x^{**}\in B_{(X^{**},\|\cdot\|)} : g(x^{**}) = 1\}$ contains both an extreme point of $B_{(X^{**},\|\cdot\|)}$ and an element $x\in S_{(X,\|\cdot\|)}$, so $\|\cdot\|\text{-diam}\,\overline{S(f,\delta)}^{w^*} \geq 1-\varepsilon$. This implies that $\operatorname{diam} S(f,\delta)\geq 1-\varepsilon$. In connection with this result, let us mention the following open problem.

Problem 2 ([SchSeWe])**.** *Assume that a Banach space X fails the RNP property and $\varepsilon > 0$ is given. Can X be renormed so that all the slices of the new ball have diameter greater than or equal to $2-\varepsilon$?*

A result of Collier [Coll76, Theorem 4] says that *a Banach space $(X,\|\cdot\|)$ has the RNP if, and only if, each dual equivalent norm on X^* is Fréchet differentiable somewhere.* The necessity is a consequence of the fact, mentioned earlier, that if X has the RNP, then every nonempty closed convex and bounded subset of X —in particular the closed unit ball $B_{(X,|||\cdot|||)}$ of an equivalent norm $|||\cdot|||$ on X— is the closed convex hull of the set of its strongly exposed points. So $B_{(X,|||\cdot|||)}$ has a strongly exposed point, say x (exposed by $f\in S_{(X^{**},|||\cdot|||)}$). Then, by the Šmulyan's lemma, $|||\cdot|||$ is Fréchet differentiable at f. The sufficiency follows from the Remark after Theorem 13, or, if we wish, from Theorem 14 below, itself a consequence of Theorem 13. Note too that, again by the Šmulyan's lemma, if a dual equivalent norm $\|\cdot\|$ on X^* is Fréchet differentiable at some $f\in S_{X^*}$, then $\|\cdot\|'(f)\in X$.

Theorem 14 ([BaDa], [Gil02])**.** *Let X be a Banach space. Assume that for each equivalent dual norm on X^*, there is a point where the norm is Gâteaux differentiable with the derivative lying in X. Then X has the RNP.*

Proof. Assume that X does not have the RNP. Then, by Theorem 13, there is an equivalent norm $\|\cdot\|$ on X such that none of the points in $S_{(X,\|\cdot\|)}$ is an extreme point of $B_{(X^{**},\|\cdot\|)}$. Let $f\in S_{(X^*,\|\cdot\|)}$ where $\|\cdot\|$ on X^* would be Gâteaux differentiable with the derivative x in $S_{(X,\|\cdot\|)}$. Then, $\{x^{**}\in X^{**} :\ f(x^{**}) = 1\}\cap B_{(X^{**},\|\cdot\|)} = \{x\}$, so x is an extreme point of $B_{(X^{**},\|\cdot\|)}$, a contradiction. □

Recall that a Banach space X is called *weakly sequentially complete* if every weakly Cauchy sequence in X is weakly convergent in X. As a straightforward consequence of Theorem 14 and the Šmulyan's lemma, we get the following statement.

Theorem 15 ([BaDa]). *Assume that X is a weakly sequentially complete Banach space. Assume that every equivalent dual norm on X^* is Gâteaux differentiable at some point. Then X has the RNP.*

A point x_0 in a closed bounded convex subset C of a Banach space X is called a *weakly exposed point of C* (by some $f \in X^*$) if x_0 is exposed by f and $x_n \xrightarrow{w} x_0$ whenever $f(x_n) \to f(x_0)$. It is easy to prove that, in C, every strongly exposed point is weakly exposed, that every weakly exposed point is exposed and a preserved extreme point, and that every exposed point is extreme. Note that the renorming in Theorem 26 has the property that all the points on its new unit sphere are exposed points of the new unit ball but none of them is a weakly exposed point of this ball.

Theorem 13 gives also a proof of the sufficient condition in the following result.

Theorem 16 ([Bou2], Theorem I.4). *A Banach space X has the RNP if, and only if, each nonempty bounded closed convex set in X has a weakly exposed point.*

Proof. We already mentioned that if X has RNP, then every bounded closed convex subset of X is the closed convex hull of the set of its strongly exposed points. Each of them is, certainly, a weakly exposed point. On the other side, if X has not RNP, the closed unit ball of the equivalent norm $|||\cdot|||$ given by Theorem 13 has no weakly exposed point. Indeed, assume that $e \in S_{(X,|||\cdot|||)}$ is a weakly exposed point of $B_{(X,|||\cdot|||)}$ (exposed by $f \in S_{(X^*,|||\cdot|||)}$). Since e is not extreme in $B_{X^{**}}$, $2e = x_1^{**} + x_2^{**}$ for some x_1^{**} and x_2^{**} in $B_{(X^{**},\|\cdot\|)}$, where $x_1^{**} \neq x_2^{**}$. Find $g \in S_{(X^*,|||\cdot|||)}$ such that $g(x_1^{**}) > g(e)$, and a sequence $\{x_n\}$ in $B_{(X,|||\cdot|||)}$ such that $g(x_n) \to g(x_1^{**})$ and $f(x_n) \to f(x_1^{**})$ $(= 1 = f(e))$. It follows that $x_n \xrightarrow{w} e$, a contradiction. □

There is an interplay between the notion of extreme points and the convex points of continuity properties. We first provide a definition.

Definition 17. *A Banach space X is said to have the* Convex Point of Continuity Property *(CPCP, in short) if every closed bounded convex subset C of X has a point at which the relative weak and norm topologies on C coincide.*
The space X^* has the weak*-convex point of continuity property *(C*PCP, in short) if every weak* compact convex subset C of X^* has a point at which the relative weak* and norm topologies on C coincide.*

If X is separable and has the property that X^* has the C*PCP, then X does not contain a copy of ℓ_1 (cf. e.g. [DGZb, Chapter 3]).
The RNP property implies the CPCP property of the space and the RNP property of the dual space X^* implies the C*PCP property of X^* (cf. e.g. [FHHMZ, Chapter 11]).
The predual of the James tree space (see e.g. [AlKal06], [FHHMZ, Chapter 4]), JT_*, has the CPCP property and not the RNP property (cf. e.g. [EW]).
The non-Asplund space JT has the property that JT^* has the C*PCP property ([GMS]).
There is a separable space X not containing ℓ_1 and yet, X^* does not have the C*PCP ([GMS]).

In the direction of the interplay between CPCP and rotundity properties, let us mention, first of all, that Troyanski showed that if a Banach space X has a strictly convex norm on the sphere of which the weak and norm topology coincide, then X can be renormed by a locally uniformly rotund norm (see, e.g. [DGZb, chapter 4] and Raja's geometric proof of it in [FHHMZ, Exercise in Chapter 8]).

Then we have the following result.

Theorem 18 ([LLT]). *Let X be a Banach space, C be a non-empty closed convex and bounded subset of X, and e be an extreme point of C at which the relative weak and norm topologies on C coincide. Then the slices of C containing the point e form a neighborhood base at e of the norm topology on C.*

Proof. First, we show that e is then a preserved extreme point of C. Indeed, assume that $e = \frac{1}{2}(y^{**} + z^{**})$, where $y^{**}, z^{**} \in \overline{C}^{w^*}$, $y^{**} \neq z^{**}$. From the coincidence of the topologies at e follows the coincidence of the relative w^* and norm topologies on $\overline{C}^{w^*}$ at e (use (3.1)). Then, by a standard cone argument, we get that the same must hold for both y^{**} and z^{**}. Thus, both y^{**} and z^{**} are in C, a contradiction with the fact that e is an extreme point of C.
Given $r > 0$, $e + rB_{X^{**}}$ contains a w^*-neighborhood U of e in X^{**}. The Choquet's Lemma applied to the w^*-compact set $\overline{C}^{w^*}$ gives a w^*-slice S of $\overline{C}^{w^*}$ such that $e \in S \subset e + rB_{X^{**}}$. It follows that $e \in S \cap C \subset e + rB_X$. □

For similar results we refer also to [LLT1].

Schachermayer proved in [Sch] the following theorem.

Theorem 19 ([Sch]). *Assume that a Banach space X has both the KMP and the CPCP. Then X has the RNP.*

Note, then, that a Banach space has the RNP if, and only if, it has, simultaneously, the KMP and the CPCP. Related to this, we already mentioned that the following is a well-known long-standing open problem in this area.

Problem 3. *Assume that every closed convex bounded set in a Banach space X has an extreme point (in other words, assume that X has KMP). Does X have the RNP?*

It is worth to mention here a geometric characterization of RNP in terms of extreme points due to Bourgain [Bou1]: *A Banach space X has the RNP if, and only, if every nonempty weakly closed bounded subset of X has an extreme point.*

Remark. We do not know the answer to Problem 3 even if we replace the word "extreme" by the word "exposed".

Theorem 20 below was proved in [DGHZ87]; compare it with the renorming characterization of the RNP given in [Diest, Corollary 1, page 219], saying that *a Banach space X has the RNP, if and only if, the closed unit ball of every equivalent norm is dentable.* It follows from this characterization that *a Banach space X has the RNP if, and only if, the closed unit ball of every equivalent norm is the closed convex hull of its strongly exposed points.*

Theorem 20 ([DGHZ87]). *If X is a separable Banach space, then X has the $CPCP$ property if, and only if, the closed unit ball of every strictly convex norm is the closed convex hull of its strongly exposed points.*

Talagrand proved the following theorem

Theorem 21 (Talagrand, see [SchSeWe]). *If a separable Banach space X contains an isomorphic copy of ℓ_1, then X^* contains a weak* compact convex norm non dentable subset.*

Yet, the following problem seems to be open.

Problem 4. *Let X be separable. Is it true that X does not contain a copy of ℓ_1 if, and only if, every weak* compact set in X^* is norm dentable?*

Related to Theorem 21 and Problem 4, recall that the James tree space JT is a separable space not containing isomorphic copies of ℓ_1, having a nonseparable dual, and being saturated with copies of ℓ_2. The following result is due to Stegall.

Theorem 22 ([Ste]). *Let X denote the James tree space JT. Then*
(i) *On the unit sphere of X^*, the weak and norm topologies coincide.*
(ii) *If C is an arbitrary weak* compact convex set in X^*, then the functionals which strongly expose C form a dense G_δ subset of X^{**}.*

3.2 Preserved extreme points and reflexivity

Theorem 24 below (that should be compared with Theorem 13) shows how preserved extreme points of balls can be used to characterize reflexive spaces. We slightly modified the original proof in order to use the same technique in proving Theorem 26. The following standard fact will be used.

Lemma 23. *Let $(X, \|\cdot\|)$ and $(Y, |\cdot|)$ be Banach spaces. Let $T : X \to Y$ be a continuous linear mapping. Define*

$$|||x||| = \|x\| + |Tx|, \text{ for } x \in X. \tag{3.2}$$

Then,
(i) *$|||\cdot|||$ is an equivalent norm on X.*
(ii) *The corresponding norm on $(X, \|\cdot\|)^{**}$ is given by*

$$|||x^{**}||| = \|x^{**}\| + |T^{**}x^{**}|. \tag{3.3}$$

(iii) *If $|\cdot|$ is strictly convex and T is one-to-one, then $|||\cdot|||$ is strictly convex.*

Theorem 24. *Let $(X, \|\cdot\|)$ be a Banach space. Then, the following are equivalent:*
(i) *The space X fails to be reflexive.*
(ii) [Godu85] *There exists an equivalent norm $|||\cdot|||$ on X and an extreme point of $B_{(X,|||\cdot|||)}$ that is unpreserved.*

Proof. Obviously, only (i)⇒(ii) needs a proof. Assume first that X is separable (and not reflexive). Fix $x_0^{**} \in X^{**} \setminus X$, and let $N := \operatorname{Ker} x_0^{**}$ ($\subset X^*$), a norming subspace of X^*, so $\|x\|_N := \sup\{|x^*(x)| : x^* \in N,\ \|x^*\| = 1\}$, for $x \in X$, defines an equivalent norm on X (whose higher dual norms are denoted, as usual, again by $\|\cdot\|_N$; note that $\|\cdot\|_N$ induces on N the norm $\|\cdot\|$). Let $\{x_n^* : n \in \mathbb{N}\}$ be a subset of $S_{(N,\|\cdot\|_N)}$ such that $x_n^* \xrightarrow{w^*} 0$, and $\overline{\operatorname{span}}^{w^*}\{x_n^* : n \in \mathbb{N}\} = X^*$ (find, for example, a Markushevich basis $\{x_n; x_n^*\}$ in $X \times X^*$ such that $x_n^* \in S_{(N,\|\cdot\|_N)}$ for all $n \in \mathbb{N}$, see, e.g., [FHHMZ]). We may now define two one-to-one continuous linear operators $S : X \to c_0$ and $T : X \to \ell_2$ by

$$S(x) = \big(x_n^*(x)\big)_{n=1}^{\infty}, \qquad T(x) = \left(\frac{1}{2^n} x_n^*(x)\right)_{n=1}^{\infty}, \qquad \text{for all } x \in X. \tag{3.4}$$

Put

$$A_k = \{x \in X : \|x\|_N \le k\|S(x)\|_\infty\}, \text{ for } k \in \mathbb{N}. \tag{3.5}$$

We obtain an increasing sequence $\{A_k\}_{k=1}^{\infty}$ of (homogeneous) subsets of X and, certainly, $A_2 \neq \emptyset$ (indeed, $x \in A_2$ for every $x \in S_{(N,\|\cdot\|_N)}$ such that $\langle x_n^*, x\rangle > 1/2$ for some $n \in \mathbb{N}$)[1]. Put then $k = 2$ and define a new equivalent norm $|||\cdot|||$ on X by

$$|||x||| = \max\left\{\frac{1}{2k}\|x\|_N, \|S(x)\|_\infty\right\} + \|T(x)\|_2, \text{ for all } x \in X. \tag{3.6}$$

[1] We define this, seemingly, artificial sequence $\{A_k\}$ to allow further manipulations in subsequent arguments, although, strictly speaking, we need only here the nonempty set A_2.

Certainly, $|||\cdot|||$ is strictly convex in X. According to Lemma 23, the bidual norm $|||\cdot|||$ on X^{**} is given by

$$|||x^{**}||| = \max\left\{\frac{1}{2k}\|x^{**}\|_N, \|S^{**}(x^{**})\|_\infty\right\} + \|T^{**}(x^{**})\|_2, \text{ for all } x^{**} \in X^{**}. \tag{3.7}$$

Fix $x \in A_k$ (so $\|x\|_N \leq k\|S(x)\|_\infty$) such that $|||x||| = 1$. Since $|||\cdot|||$ is strictly convex, x is an extreme point of $B_{(X,|||\cdot|||)}$. Fix $\delta > 0$ so small that $\|x \pm \delta x_0^{**}\|_N < 2\|x\|_N$. Note that $S^{**}(x \pm \delta x_0^{**}) = S(x)$ and $T^{**}(x \pm \delta x_0^{**}) = T(x)$. Then

$$\begin{aligned}
&|||x \pm \delta x_0^{**}||| \\
&\quad = \max\left\{\frac{1}{2k}\|x \pm \delta x_0^{**}\|_N, \|S^{**}(x \pm \delta x_0^{**})\|_\infty\right\} + \|T^{**}(x \pm \delta x_0^{**})\|_2 \\
&\quad \leq \max\left\{\frac{1}{2k}2\|x\|_N, \|S(x)\|_\infty\right\} + \|T(x)\|_2 \\
&\quad = \|S(x)\|_\infty + \|T(x)\|_2 \leq |||x||| \ (= 1),
\end{aligned}$$

hence x is not an extreme point of $B_{(X^{**},|||\cdot|||)}$.
If X is not separable, it is enough to apply a separable-reduction argument by using the next lemma. □

Lemma 25. *Let $(X, \|\cdot\|)$ be a Banach space, and let Y be a closed linear subspace. Then, every equivalent norm on Y can be extended to an equivalent norm on X in such a way that, if some extreme point y_0 of $B_{(Y,|||\cdot|||)}$ is unpreserved, then y_0 is an extreme point of $B_{(X,|||\cdot|||)}$ that is unpreserved.*

Proof. Assume that $\|y\| \leq |||y||| \leq c\|y\|$ for some $c > 0$ and for all $y \in Y$. Let $|\cdot|$ be the Minkowski functional of the set $\overline{\text{conv}}\,(B_{(Y,|||\cdot|||)} \cup \frac{1}{c}B_{(X,\|\cdot\|)})$ (an equivalent norm on X that induces the norm $|||\cdot|||$ on Y). Finally, put $|||x||| := |x| + \text{dist}_{\,|\cdot|}(x, Y)$ for $x \in X$. This is again an equivalent norm in X that induces on Y the norm $|||\cdot|||$. We shall prove first that y_0 is an extreme point of $B_{(X,|||\cdot|||)}$. Indeed, assume that $2y_0 = x_1 + x_2$, where x_1 and x_2 are elements in X such that $|||x_1||| = |||x_2||| = 1$. If $x_1 \in Y$ and $x_2 \in Y$, we get $x_1 = x_2$. Otherwise,

$$\begin{aligned}
1 = |||y_0||| = |y_0| &\leq \frac{1}{2}(|x_1| + |x_2|) \\
&< \frac{1}{2}(|x_1| + \text{dist}_{\,|\cdot|}(x_1, Y) + |x_2| + \text{dist}_{\,|\cdot|}(y_2, Y)) = \frac{1}{2}(|||x_1||| + |||x_2|||) = 1,
\end{aligned}$$

a contradiction.
Since y_0 is not an extreme point of $B_{(Y^{**},|||\cdot|||)}$, and the space $(Y^{**}, |||\cdot|||)$ is isometrically isomorphic to a subspace of $(X^{**}, |||\cdot|||)$, it follows that y_0 is not an extreme point of $B_{(X^{**},|||\cdot|||)}$, as claimed. □

Remark. The proof of (i) ⇒ (ii) in the previous theorem shows that, in a *separable* Banach space X, the nonreflexivity is equivalent to the existence of a *strictly convex* norm $|||\cdot|||$ on X such that some (extreme) point of $S_{(X,|||\cdot|||)}$ is unpreserved.

The proof we provide here of the following result —which should be compared with Theorem 13— relies on the technique used for proving Theorem 24.

Theorem 26 ([Morr83])**.** *Assume that $(X, \|\cdot\|)$ is a separable Banach space that contains an isomorphic copy of c_0. Then X admits an equivalent strictly convex norm $|||\cdot|||$ such that each point in $S_{(X,|||\cdot|||)}$ is not a preserved extreme point of $B_{(X,|||\cdot|||)}$.*

Proof. (i) We shall prove it first for the space $(c_0, \|\cdot\|_\infty)$ itself. This follows from the proof of Theorem 24 as we shall show presently. Keep the notation there, letting $x_0^{**} = (1, 1, 1, \ldots)$ $(\in \ell_\infty \setminus c_0)$. In this case, the subspace N is 1-norming, so $\|\cdot\|_N = \|\cdot\|_\infty$ on c_0 (and $\|\cdot\|_N = \|\cdot\|_1$ on $c_0^* = \ell_1$). If, for $n \in \mathbb{N}$, the symbol e_n^* denotes the n-th canonical unit vector of ℓ_1, the countable set $\Gamma := \{(1/2)(e_n^* - e_m^*) : \ n, m \in \mathbb{N}, \ n < m\}$ $(\subset \ell_1)$ is in $S_{(N, \|\cdot\|_N)}$ and $\overline{\text{span}}^{w^*}(\Gamma) = \ell_1$. The mapping S defined in the proof of Theorem 24 for the set 2Γ is a one-to-one continuous linear operator from X into the c_0-sum of countably many copies of $(c, \|\cdot\|_\infty)$, i.e., the space $Z := c_0(c \oplus c \oplus c \oplus \ldots)$ endowed with the supremum norm (a space linearly isomorphic to $(c_0, \|\cdot\|_\infty)$). Precisely, put

$$S(x) := \big((\langle e_1^* - e_m^*, x\rangle)_{m>1}, (\langle e_2^* - e_m^*, x\rangle)_{m>2}, \ldots\big), \text{ for } x \in c_0.$$

Obviously $2\|x\|_\infty \geq \|S(x)\|_\infty \geq \|x\|_\infty$ for every $x \in c_0$, so $c_0 = A_1$ (see formula (3.5)). The norm $|||\cdot|||$ defined by (3.6) in the proof of Theorem 24 for $k = 1$ satisfies the requirements (even more: we found that a single direction in ℓ_∞ is enough to check that no element $x \in S_{(c_0, |||\cdot|||)}$ is a preserved extreme point).
(ii) Assume now that X contains an isomorphic copy of c_0. By Sobczyk's theorem (see, e.g., [FHHMPZ, Theorem 5.14], [FHHMZ, Theorem 5.11]), this copy is complemented in X, i.e., X is isomorphic to $(G \oplus c_0, |\cdot|)$, where $|(g, x)| = \max\{\|g\|, \|x\|_\infty\}$, $g \in G$ $(\subset X)$, and $x \in c_0$. Let $S : G \to \ell_2$ be a one-to-one linear and continuous operator, and let $T : c_0 \to \ell_2$ be the operator defined in the proof of Theorem 24. Let $U : G \oplus c_0 \to \ell_2 \oplus \ell_2$ be defined by $U(g, x) = (Sg, Tx)$, $g \in G$, $x \in c_0$. Then U is a one-to-one continuous linear operator from $G \oplus c_0$ into $\ell_2 \oplus \ell_2$. Put

$$|||(g, x)||| := |(g, x)| + \|U(g, x)\|_2, \text{ for } (g, x) \in G \oplus c_0. \tag{3.8}$$

As before, the space $(G \oplus c_0, |||\cdot|||)$ is strictly convex. We shall prove that no element $(g, x) \in S_{(G \oplus c_0, |||\cdot|||)}$ is an extreme point of the bidual space of $(G \oplus c_0, |||\cdot|||)$. Choose, as in (i), $x^{**} \neq 0$ in ℓ_∞ such that $T^{**}x^{**} = 0$ and

$$\|x \pm x^{**}\| = \|x\| \ (\leq |(g, x)|). \tag{3.9}$$

We have

$$\begin{aligned} |||(g, x) \pm (0, x^{**})||| &= \|(g, x \pm x^{**})\| + \|U^{**}(g, x \pm x^{**})\|_2 \\ &= \max(\|g\|, \|x \pm x^{**}\|) + \|U(g, x)\|_2 \\ &\leq \max(\|g\|, \|x\|_\infty) + \|U(g, x)\|_2 = |||(g, x)||| = 1. \end{aligned}$$

It follows that (g, x) is not extreme. □

Problem 5. ([Morr83]) *Which spaces can be renormed to be strictly convex but to have unpreserved extreme points?*

4 Strongly Gâteaux differentiable norms

Recall that a multivalued map M from a topological space $\mathcal{X}$ into a topological space $\mathcal{Y}$ is called *upper semicontinuous* if for every open set U in $\mathcal{Y}$ the set $\{x \in \mathcal{X} : \ M(x) \subset U\}$ is open.

Definition 27. *We will say that the norm $\|\cdot\|$ of a Banach space X is* strongly Gâteaux differentiable *at a point $x_0 \in S_X$ if it is Gâteaux differentiable at x_0 and the (multivalued) map $x \to \partial\|\cdot\|(x)$ from S_X into X^* is norm-to-weak upper semicontinuous at x_0.*
A norm on a Banach space will be called strongly Gâteaux differentiable *if it is strongly Gâteaux differentiable at each point of S_X.*

In the literature this concept appears also under the name *very smooth*. Note that the notion of strong Gâteaux differentiability obviously coincides with that of Gâteaux differentiability for reflexive spaces. The Šmulyan lemma (see, e.g. [FHHMZ, Chapter 7]) implies that Fréchet differentiable norms are strongly Gâteaux differentiable.

Lemma 28. *Let $(X, \|\cdot\|)$ be a Banach space and let $x \in S_X$. Assume that $\|\cdot\|$ is Gâteaux differentiable at x. Then the following are equivalent.*
(i) *The norm $\|\cdot\|$ of X is strongly Gâteaux differentiable at x.*
(ii) *The norm $\|\cdot\|$ of X^{**} is Gâteaux differentiable at x.*
(iii) *Every sequence (x_n^*) in B_{X^*} such that $x_n^*(x) \to 1$ is weakly convergent to $x^* := \|\cdot\|'(x)$.*
(iv) *The functional $x^* := \|\cdot\|'(x)$ is a point of continuity for the identity mapping $I : (B_{X^*}, w^*) \to (B_{X^*}, w)$.*

Proof. First of all, (ii) and (iii) are equivalent by the Šmulyan lemma (see, e.g., [FHHMZ, Corollary 7.22 (iv)]).
(i) $\Rightarrow$ (iii): Use the Brøndsted-Rockafellar theorem (see, e.g. [Ph93, page 48]) to choose $y_n \in S_X$ and $y_n^* \in S_{X^*}$ such that $y_n^*(y_n) = 1$, $\|y_n - x\| \le \varepsilon_n/\sqrt{\varepsilon_n}$ and $\|y_n^* - x_n^*\| \le \sqrt{\varepsilon_n}$. Since $\varepsilon_n \to 0$, it follows that $y_n \to x$. By (i) we get $y_n^* \xrightarrow{w} \|\cdot\|'(x)$, so $x_n^* \xrightarrow{w} \|\cdot\|'(x)$.
(iii) $\Rightarrow$ (iv) is obvious.
(iv) $\Rightarrow$ (i): Assume that (i) fails. Then there exists a w-neighborhood U of x^*, a sequence $\{x_n\}$ in X, and elements $x_n^* \in \partial\|\cdot\|(x_n)$, such that $x_n \to x$ and $x_n^* \notin U$ for every $n \in \mathbb{N}$. Since the duality mapping is $\|\cdot\|$-w^*-upper semicontinuous, we get that $x_n^* \xrightarrow{w^*} x^*$, hence $x_n^* \xrightarrow{w} x^*$, a contradiction. □

Theorem 29 ([Day73], [Gil74], [Rain])**.** *Assume the dual norm of X^* is strongly Gâteaux differentiable. Then X is reflexive.*

Proof Let $F \in S_{X^{**}}$ attain its norm at $f \in S_{X^*}$. Let $x_n \in S_X$ be such that $f(x_n) \to 1$. By the strong Gâteaux differentiability of the dual norm at f, we get that $x_n \to F$ in the weak topology of X^{**}, and thus F is in the norm closed linear hull of $\{x_n\}$ by the Mazur theorem. Thus $F \in X$. Hence $X^{**} \subset X$ by the Bishop–Phelps theorem (cf. e.g. [FHHMZ, Theorem 7.41]). □

Theorem 29 has the following corollary.

Corollary 30 ([Dix])**.** *Assume that the norm of the fourth dual of a Banach space is strictly convex. Then X is reflexive.*

In this direction, let us mention the following well-known and easy-to-proof result (see e.g. [DGZb, p. 51], and compare with Theorem 15).

Theorem 31. *Let X be a weakly sequentially complete Banach space whose dual norm is Gâteaux differentiable. Then X is reflexive.*

Theorem 32 ([Sing75])**.** *Assume the norm of X is strongly Gâteaux differentiable. Then X is an Asplund space.*

Proof. We shall show that X^* is separable if X is separable. To this end, let $\{x_n\}$ be a dense sequence in S_X. Let $f \in S_{X^*}$ attain its norm at $x \in S_X$. Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$.
Then f is the weak limit of the sequence $\{\|\cdot\|'(x_{n_k})\}$. Thus f is in the norm-closed linear hull of $\{\|\cdot\|'(x_{n_k})\}$, by the Mazur theorem (cf. e.g. [FHHMZ, Theorem 3.45]). The separability

of X^* then follows from the Bishop–Phelps theorem on the density of the norm attaining functionals (cf. e.g. [FHHMZ, Theorem 7.41]). □

Remark. Note that the notion of strong Gâteaux differentiability of the norm coincides with the Fréchet differentiability for the $C(K)$ spaces, since it gives the Asplund property by Theorem 32 and thus the isomorphism of $C(K)^*$ to ℓ_1. Then, it is enough to use the Schur property of ℓ_1 (cf. e.g., [FHHMZ, Theorem 5.36, 14.24 and 14.25]).

We do not know the answer to the following problem.

Problem 6. *Which separable spaces admit norms that are nowhere strongly Gâteaux differentiable?*

The following result is due to Godun, who provided an ad hoc argument in [Godu85]. The proof we present here, although closely related to the original one, shows that, after a reduction to the Asplund setting, this result is dual to Theorem 24.

Theorem 33 ([Godu85b])**.** *Let $(X, \|\cdot\|)$ be a separable Banach space. Then X is reflexive if, and only if, every Gâteaux differentiable equivalent norm on X is strongly Gâteaux differentiable.*

Proof. Obviously, only the sufficient condition must be proved. So assume that X is not reflexive. If X is not Asplund, the result follows from Theorem 32. Indeed, it is then enough to renorm X by a Gâteaux differentiable norm and use Theorem 32. If X, on the contrary, is Asplund, then X^* is separable. We shall see that the construction in the proof of Theorem 24 carried on X^* gives, if starting conveniently, a dual (rotund) norm whose predual norm is the sought one. Indeed (and we use the notation there), it is enough to choose an element $(0 \neq)\ x_0^{***} \in X^\perp\ (\subset X^{***})$. Then $X \subset \operatorname{Ker} x_0^{***} \subset X^{**}$, and the norm $\|\cdot\|_N$ in X^* defined by the norming subspace $N := \operatorname{Ker} x_0^{***}$ is, in fact, $\|\cdot\|$. We choose a w-null sequence $\{x_n\}$ *in* $S_{(X,\|\cdot\|)}$ such that $\overline{\operatorname{span}}^{w^*}\{x_n :\ n \in \mathbb{N}\} = X^{**}$, and define $S : X^* \to c_0$ and $T : X^* \to \ell_2$ as in (3.4) by using $\{x_n\}$. Then the norm $|||\cdot|||$ defined on X^* by (3.6) is an equivalent dual rotund norm. Put $|||\cdot|||$ for its predual norm in X, an equivalent Gâteaux differentiable norm on X. The sets $A_k := \{x^* \in X^* :\ \|x^*\| \leq k\|Sx^*\|_\infty\}$, $k = 1, 2, \ldots$, are $\|\cdot\|$-closed, and $\bigcup_{k=1}^\infty A_k = X^*$, hence there is $k \in \mathbb{N}$ such that A_k has nonempty interior. Recall that every element x^* in $A_k \cap S_{(X^*,|||\cdot|||)}$ is an extreme point of $B_{(X^*,|||\cdot|||)}$ that is not preserved in X^{***}, precisely because for some $\delta > 0$ we have $|||x^* \pm \delta x_0^{***}||| \leq 1$. Use the Bishop-Phelps theorem to ensure that $A_k \cap S_{(X^{***},|||\cdot|||)}$ contains an element x^* that attains its norm at some element x in $B_{(X,|||\cdot|||)}$. Then x is a smooth point of $|||\cdot|||$; however, it is not a very smooth point due to the fact that $x^* \pm \delta x_0^{***} \in B_{(X^{***},|||\cdot|||)}$ and $\langle x^* \pm \delta x_0^{***}, x\rangle = \langle x^*, x\rangle$.□

Let $(X, \|\cdot\|)$ be a Banach space. A norm $|||\cdot|||$ on X^* is called *weak* uniformly rotund* (W*UR in short) if $f_n - g_n \xrightarrow{w^*} 0$ whenever $f_n, g_n \in S_{(X^*,|||\cdot|||)}$ are such that $|||f_n + g_n||| \to 2$. It is well known (see, e.g., [DGZb, Theorem II.6.7]), that $\|\cdot\|$ on X is UG if, and only if, $\|\cdot\|$ on X^* is W*UR.

Theorem 34 ([BoFa93])**.** *Let X be an infinite-dimensional separable Banach space. Then there is a (uniformly) Gâteaux differentiable norm on X that is somewhere not Fréchet differentiable.*

Proof. Without loss of generality, we may assume that $\|\cdot\|$ on X satisfies $\|(y,t)\| = \|y\| + |t|$ for $Y \oplus \mathbb{R} = X$, where Y is a closed hyperplane of X. Then the norm $\|\cdot\|$ on $Y^* \oplus \mathbb{R}$ $(= X^*)$ satisfies $\|(\zeta, r)\| = \max\{\|\zeta\|, |r|\}$, for $(\zeta, r) \in Y^* \oplus \mathbb{R}$. Let $\{y_n\}$ be a dense sequence in the unit ball of Y and define a compact operator $T : \ell_2 \to Y$ by

$$T((\lambda_n)) = \sum 2^{-n}\lambda_n y_n,\ (\lambda_n) \in \ell_2.$$

Define a norm $|||\cdot|||$ on $Y^* \oplus \mathbb{R}$ by

$$|||(\zeta, r)|||^2 = \max\{\|\zeta\|^2, r^2\} + \|T^*\zeta\|_2^2 + |r|^2, \quad (\zeta, r) \in Y^* \oplus \mathbb{R}.$$

Observe that $|||\cdot|||$ is an equivalent dual norm on $Y^* \oplus \mathbb{R}$, dual to a norm on $Y \oplus \mathbb{R}$ denoted again by $|||\cdot|||$.

The Josefson–Niszenweig theorem allows to choose a sequence $\{\zeta_n\}$ in S_{Y^*} such that $\zeta_n \xrightarrow{w^*} 0$. Then $(\zeta_n, 1) \to (0, 1)$ in the weak* topology of $Y^* \oplus \mathbb{R}$, and $|||(0, 1)||| = \sqrt{2}$.

Now, since T^* is also a compact operator, we have $T^*\zeta_n \xrightarrow{\|\cdot\|_2} 0$. Therefore $|||(\zeta_n, 1)||| \to \sqrt{2}$. Put $\xi_n = (\zeta_n, 1)/|||(\zeta_n, 1)|||$ for $n \in \mathbb{N}$, and $\xi = (0, \frac{\sqrt{2}}{2})$. We have $|||\xi_n||| = |||\xi||| = 1$ and $\xi_n \xrightarrow{w^*} \xi$.

Put $x = (0, \sqrt{2}) \in Y \oplus \mathbb{R}$. Note that $|||x||| = 1$. Indeed, $\langle (0, \frac{\sqrt{2}}{2}), x\rangle = 1$, and

$$\begin{aligned} |||x||| &= \sup\{\langle(\zeta, r), (0, \sqrt{2})\rangle : \ (\zeta, r) \in Y^* \oplus \mathbb{R}, \ |||(\zeta, r)||| \le 1\} \\ &= \sup\{\sqrt{2}r : \ (\zeta, r) \in Y^* \oplus \mathbb{R}, \ \max\{\|\zeta\|^2, r^2\} + \|T^*\zeta\|^2 + r^2 \le 1\} \\ &\le \sup\{\sqrt{2}r : \ r \in \mathbb{R}, \ 2r^2 \le 1\} = 1 \end{aligned}$$

Then $|||\xi_n||| = |||\xi||| = 1 = \langle \xi, x\rangle = 1$, $\xi_n \xrightarrow{w^*} \xi$, and an easy computation shows that $|||\xi_n - \xi||| \to \frac{\sqrt{2}}{2}$. Thus, by the Šmulyan lemma, $|||\cdot|||$ is not Fréchet differentiable at x.

It remains to show that $|||\cdot|||$ on X is UG. For it we show that its dual norm $|||\cdot|||$ on X^* is W*UR. Indeed, if $w_n := (\zeta_n^1, r_n^1) \in Y^* \oplus \mathbb{R}$ and $z_n := (\zeta_n^2, r_n^2) \in Y^* \oplus \mathbb{R}$, $n \in \mathbb{N}$, are such that $\{w_n\}$ is bounded, and $2|||w_n|||^2 + 2|||z_n|||^2 - |||w_n + z_n|||^2 \to 0$, then, by a standard convexity argument, using the second and third term in the definition of the norm $|||\cdot|||$ on X^*, we have that $r_n^1 - r_n^2 \to 0$ and $T^*(\zeta_n^1 - \zeta_n^2) \to 0$, which gives $\zeta_n^1 - \zeta_n^2 \xrightarrow{w^*} 0$ as T^* is a weak*-to-weak* isomorphism of the dual ball of Y^* onto its image in ℓ_2, since T^* is one-to-one. Therefore the predual norm $|||\cdot|||$ on X is UG. □

In [BoFa93], the following results are proved.

Theorem 35 ([BoFa93])**.** *Let X^* be separable. Then X admits a norm that is everywhere (outside the origin) Gâteaux differentiable but is Fréchet differentiable exactly at each point of $X \setminus \operatorname{span}\{x_0\}$, where x_0 is a fixed nonzero point.*

Theorem 36 ([BoFa93])**.** *If X is an infinite dimensional Banach space. Then X admits a norm $\|\cdot\|$ and $x_0 \in X$ such that $\|\cdot\|$ is Gâteaux differentiable at x_0 but not Fréchet differentiable at x_0.*

Remark. Note that Theorem 36 is equivalent to the Josefson–Niszenweig theorem. Indeed, given an infinite dimensional Banach space X, let $\|\cdot\|$ in X be the norm defined in Theorem 36. Let $x_0 \in S_{(X,\|\cdot\|)}$ be a point where $\|\cdot\|$ is Gâteaux and not Fréchet differentiable, with $\|\cdot\|'(x_0) = x_0^*$. Use the Šmulyan lemma to find a sequence $\{x_n^*\}$ in $B_{(X^*,\|\cdot\|)}$ such that $\langle x_n^*, x_0\rangle \to 1$ and $\|x_n^* - x_0^*\| \ge \varepsilon$, for all $n \in \mathbb{N}$ and for some $\varepsilon > 0$. Then $x_n^* \xrightarrow{w^*} x_0^*$. The sequence $\{(x_n^* - x_0^*)/\|(x_n^* - x_0^*)\|\}$ in in $S_{(X^*,\|\cdot\|)}$ and is w^*-null.

The norm $\|\cdot\|$ on a separable Banach space is called *octahedral* (see e.g [DGZb, Chapter 3]) if there is $u \in X^{**} \setminus 0$ such that $\|u + x\| = \|u\| + \|x\|$ for all $x \in X$. Recall that X *contains an isomorphic copy of ℓ_1 if, and only if, X admits an equivalent octahedral norm* (see e.g. [DGZb, Chapter 3]).

Problem 7 ([Tan96]). *Assume that X is a separable Banach space that contains an isomorphic copy of ℓ_1. Does X admit an equivalent Gâteaux smooth octahedral norm?*

Note that an example of a separable space with Gâteaux differentiable octahedral norm is, e.g., in [DGZb, p. 120].

Theorem 37 ([Tan96]). *Assume that a separable Banach space X contains an isomorphic copy of ℓ_1. Then X admits an equivalent (uniformly) Gâteaux differentiable norm that is nowhere strongly Gâteaux differentiable.*

Proof. There is an equivalent octahedral norm $\|\cdot\|$ on X (see e.g. [DGZb])), which implies the existence of an element $x_0^{**} \in X^{**}$ such that $x_0^{**}\big|_{(B_{X^*,\|\cdot\|})}$ has no point of continuity on $(B_{(X^*,\|\cdot\|)}, w^*)$. Let $\{x_n : n \in \mathbb{N}\}$ be a countable dense subset of $S_{(X,\|\cdot\|)}$, and define a continuous linear operator $T : X^* \to \ell_2$ by $Tx^* = \left(\frac{1}{2^n} x^*(x_n)\right)_{n=1}^{\infty}$, for $x^* \in X^*$. Then put

$$|x^*| := \|x^*\| + \|Tx^*\|_2, \text{ for } x^* \in X^*. \tag{4.1}$$

This is an equivalent w^*-uniformly rotund dual norm on X^*, so its predual norm $|\cdot|$ in X is UG. Fix $x \in S_{(X,|\cdot|)}$, and let $x^* = |\cdot|'(x)$ $(\in S_{(X^*,|\cdot|)})$. We may find a sequence $\{x_n^*\}$ in $\|x^*\|B_{(X^*,\|\cdot\|)}$ such that $x_n^* \xrightarrow{w^*} x^*$, while $x_0^{**}(x_n^*) \not\to x_0^{**}(x^*)$. Note that $|x_n^*| \to |x^*|$. Indeed, $\|Tx_n^*\| \to \|Tx^*\|$ because $x^* \to \|Tx^*\|_2$ is a w^*-continuous mapping, and $\|f_n\| \to \|f\|$ as $\|\cdot\|$ is w^*-lower semicontinuous. Hence, according to the Šmulyan lemma, the dual norm of $|\cdot|$ on X^{**} is not Gâteaux differentiable at x. □

Remark. For the particular case $X := \ell_1$, we do not need to rely on the concept of octahedrality of a norm (incidentally, the canonical norm of ℓ_1 is indeed octahedral). It is enough to use the following lemma, since the existence of an element in the bidual space with a restriction to the dual unit ball that has no point of w^*-continuity is what matters.

Lemma 38. *Fix a non-principal ultrafilter $\mathcal{U}$ in $\mathbb{N}$, and let $u = \lim_{\mathcal{U}} e_n$ $(\in \ell_\infty^*)$ be the w^*-limit along $\mathcal{U}$ of the sequence $\{e_n\}_{n\in\mathbb{N}}$ consisting of the canonical unit vectors in ℓ_1. Then $u\big|_{B_{\ell_\infty}}$ has no point of continuity as a mapping from (B_{ℓ_∞}, w^*) into $\mathbb{R}$.*

Proof. Fix $x^* = (x_n) \in B_{\ell_\infty}$. Let $l := \lim_{\mathcal{U}} x_n$, and choose $k \in [-1, 1]$ such that $k \neq l$. For $n \in \mathbb{N}$, put $s_n^* := (x_1, x_2, \ldots, x_n, k, k, \ldots)$ $(\in B_{\ell_\infty})$. Then $s_n^* \xrightarrow{w^*} x^*$, although $u(s_n^*) = k$ for all $n \in \mathbb{N}$, and $u(x^*) = l$. □

Problem 8 ([Tan96]). *Which separable spaces can be renormed by a Gâteaux differentiable norm that is nowhere strongly Gâteaux differentiable?*

Answering a question of Mazur in [Mazur33], Phelps proved in [Ph60] that there is an equivalent Gâteaux differentiable norm on ℓ_1 that is nowhere Fréchet differentiable. This motivated the following result.

Theorem 39 ([DGHZ87]). *Assume that X is a separable Banach space. Then X admits an equivalent Gâteaux differentiable norm that is nowhere Fréchet differentiable if, and only if, X^* does not have the C^*PCP.*

Proof. We will show only one implication, namely that if X^* does not have C^*PCP, then X admits an equivalent Gâteaux differentiable norm that is nowhere Fréchet differentiable

Claim. Let A, B be subsets of X^* such that for every nonempty w^*-open subset O of B, we have $\operatorname{diam}(O) > \varepsilon$. Then the same conclusion holds for $(A+B)$.
Indeed, let $f \in V$, where V is a w^*-open subset of $(A+B)$. We write $f = g + f'$ with $g \in A$ and $f' \in B$. By continuity, there is a w^*-neighbourhood O of f' in B such that $(g+O) \subset V$. From the assumption, there are $f_1, f_2 \in O$ with $\|f_1 - f_2\| > \varepsilon$. Then $(g+f_i) \in V$ $(i = 1, 2)$ and

$$\|(g+f_1) - (g+f_2)\| = \|f_1 - f_2\| > \varepsilon,$$

which shows that $\operatorname{diam}(V) > \varepsilon$.
We now return to the proof of Theorem 39. Assume that there is a convex weak* compact subset K of X^* and $\varepsilon > 0$ such that for every nonempty weak* open subset O of K we have $\operatorname{diam}(O) > \varepsilon$. Applying our claim twice shows that $C := B_{X^*} + (K) + (-K)$ shares the property of K. Clearly, C is the dual unit ball of an equivalent norm, denoted by $|||\cdot|||$.
Let $\{x_i;\ i \geq 1\}$ be a norm-dense sequence in the unit ball of $(X, |||\cdot|||)$. We define

$$|f|^* = |||f||| + \big(\sum_{i=1}^{\infty} 2^{-i} f(x_i)^2\big)^{1/2}.$$

Note that $|f|^* \geq |||f||| \geq |f|^*/2$ for any $f \in X^*$. It is easily seen that $|\cdot|^*$ is a dual strictly convex norm and therefore its predual norm $|\cdot|$ is Gâteaux differentiable. For showing that $|\cdot|$ is nowhere Fréchet differentiable, it suffices to show that any nonempty w^*-open subset O of $\{|f|^* \leq 1\}$ has $\|\cdot\|$-norm-diameter at least $\varepsilon/2$.
Clearly, $O \cap \{|f|^* = 1\} \neq \emptyset$. Hence we can pick $f \in O$ with $1 = |f|^* \geq |||f||| \geq 1/2$.
There exist sequences f_n, g_n such that $|||f_n||| \leq |||f|||$, $|||g_n||| \leq |||f|||$ for every n, satisfying $\|f_n - g_n\| \geq \varepsilon$ and

$$w^* - \lim f_n = w^* \lim g_n = f.$$

By elementary rules on interchanging the limit and the summation, it follows that

$$\lim_n \sum 2^{-i} f_n^2(x_i) = \sum 2^{-i} f^2(x_i).$$

If for some subsequence n_k and $\delta \in (0,1)$,

$$|||f_{n_k}||| + (\sum 2^{-i} f_{n_k}^2(x_i))^{1/2} \leq \delta |||f||| + \delta (\sum 2^{-i} f^2(x_i))^{1/2},$$

then by using the w^*-lower semicontinuity of $|||\cdot|||$, we obtain

$$1 = |||f||| + (\sum 2^{-i} f^2(x_i))^{1/2} \leq \delta |||f||| + \delta (\sum 2^{-i} f^2(x_i))^{1/2} = \delta,$$

a contradiction which shows that $\lim |f_n|^* = |f|^*$. Similarly we obtain that $\lim |g_n|^* = |f|^*$. Because $\|f_n - g_n\| \geq \varepsilon$ for each n, we have $\|f_n/|f_n|^* - g_n/|g_n|^*\| \geq \varepsilon/2$, for n large enough. □

We will show now an easier variant of Theorem 20. Recall that the norm $\|\cdot\|$ of a Banach space X is called *weakly uniformly rotund* (WUR in short) if $x_n - y_n \to 0$ in the weak topology of X whenever $x_n, y_n \in S_X$ are such that $\|x_n + y_n\| \to 2$.

Theorem 40 ([DGHZ87]). *Let X^* be separable. Then X has the CPCP if, and only if, the closed unit ball of every equivalent WUR norm in X is dentable.*

Proof If X does not have the CPCP, then a variation on the proof of Theorem 39 shows that X admits an equivalent WUR norm whose closed unit ball is not dentable. On the other hand, if X has the CPCP and $\|\cdot\|$ is a WUR norm on X, then its dual norm is

Gâteaux differentiable by the Šmulyan lemma. The closed unit ball B_X of $(X, \|\cdot\|)$ has a point where the identity map from (B_X, w) into $(B_X, \|\cdot\|)$ has a point of continuity x. Clearly x lies on the unit sphere S_X of $(X, \|\cdot\|)$. Let $f \in S_{X^*}$ be such that $f(x) = 1$. If $x_n \in B_X$ are such that $f(x_n) \to 1$, then $x_n \xrightarrow{w} x$ by the Šmulyan lemma, as the dual norm is Gâteaux differentiable at f. Since the weak and norm topologies on B_X coincide at x, we have that f is a point of Fréchet differentiability of the dual norm and x is thus a strongly exposed point of B_X. □

If a separable Banach space X contains a copy of ℓ_1, then $C[0,1]$ is a quotient of X by a result of Pełczyński (see e.g. [FHHMZ, Corollary 5.33]). Therefore X^* contains a copy of the nonseparable space $\ell_1(\Gamma)$, that does not admit any equivalent Gâteaux differentiable norm (see e.g. [FHHMZ, Exercise 7.65]).

5 Uniformities of rotund norms

We have seen in the previous sections how various notions of rotundity are intimately linked with smoothness, structure, and geometry of a Banach space. Let us now give a systematic description of variants of uniformities for rotund norms.

Definition 41. *Let $(X, \|\cdot\|)$ be a Banach space (respectively in some cases a dual Banach space equipped with a dual norm). Consider the following conditions.*

If the relation $\lim_{n\to\infty} \|x_n + y_n\| = 2$, for some $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subset B_X$, implies

$$\lim_{n\to\infty} (x_n - y_n) = 0, \quad \textit{then } \|\cdot\| \textit{ is called } \text{uniformly rotund (UR).}$$

$$w\text{-}\lim_{n\to\infty} (x_n - y_n) = 0, \quad \textit{then } \|\cdot\| \textit{ is called } \text{weakly uniformly rotund (WUR).}$$

$$w^*\text{-}\lim_{n\to\infty} (x_n - y_n) = 0, \quad \textit{then } \|\cdot\| \textit{ is called } \text{weakly}^* \text{ uniformly rotund (W}^*\text{UR).}$$

If the relation $\lim_{n,m\to\infty} \|x_n + y_m\| = 2$, for some $\{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \subset B_X$, implies

$$\lim_{n,m\to\infty} (x_n - y_m) = 0, \quad \textit{then } \|\cdot\| \textit{ is called } \text{2-uniformly rotund (2UR).}$$

$$w\text{-}\lim_{n,m\to\infty} (x_n - y_m) = 0, \quad \textit{then } \|\cdot\| \textit{ is called } \text{2-weakly uniformly rotund (2WUR).}$$

$$w^*\text{-}\lim_{n,m\to\infty} (x_n - y_m) = 0, \quad \textit{then } \|\cdot\| \textit{ is called } \text{2-weakly}^* \text{ uniformly rotund (2W}^*\text{UR).}$$

If the relation $\lim_{n,m\to\infty} \|x_n + x_m\| = 2$, for some $\{x_n\}_{n=1}^\infty \subset B_X$, implies

$$\lim_{n\to\infty} x_n = x, \quad \textit{for some } x \in X, \quad \textit{then } \|\cdot\| \textit{ is called } \text{2-rotund (2R).}$$

$$w\text{-}\lim_{n,m\to\infty} (x_n - x_m) = 0, \quad \textit{then } \|\cdot\| \textit{ is called } \text{weakly Cauchy rotund (WCR).}$$

$$w\text{-}\lim_{n\to\infty} x_n = x, \quad \textit{for some } x \in X, \quad \textit{then } \|\cdot\| \textit{ is called } \text{weakly 2-rotund (W2R).}$$

$$w^*\text{-}\lim_{n,m\to\infty} (x_n - x_m) = 0, \quad \textit{then } \|\cdot\| \textit{ is called } \text{weakly}^* \text{ Cauchy rotund (W}^*\text{CR).}$$

$$w^*\text{-}\lim_{n\to\infty} x_n = x, \quad \textit{for some } x \in X, \quad \textit{then } \|\cdot\| \textit{ is called } \text{weakly}^* \text{ 2- rotund (W}^*\text{2R).}$$

Let us pass to simple properties of the above notions. Note that it can be easily shown that if a norm $\|\cdot\|$ on X has any one of the above properties then it is a rotund norm on X. For a given norm we have the easy implications: UR⇒2UR⇒2R, WUR⇒2WUR⇒WCR⇐W2R, and analogously W*UR⇒2W*UR⇒W*CR⇐W*2R. It is known (see below) that JT admits a 2WUR renorming but it has no equivalent WUR norm. Also, W2R is strictly stronger than WUR. As regards the rest, we pose the following problem.

Problem 9. *Which of the implications above can be reversed?*

The first three notions UR, WUR, and W*UR are classical and have been discussed in previous sections. Recall that a Banach space X admits a UR renorming if, and only if, X is superreflexive (Enflo [Enf]). If X admits a WUR norm, then X is an Asplund space ([Haj96]). If X is moreover separable, then WUR renormability is equivalent to being an Asplund space. Next, X^* has a W*UR renorming if, and only if, B_{X^*} in its w^*-topology is a uniform Eberlein compact [FGZ]. Regarding the duality with smoothness, a norm $\|\cdot\|$ on X is uniformly Fréchet differentiable (resp. UG) if, and only if, $\|\cdot\|^*$ is UR (resp. W*UR). Finally, $\|\cdot\|$ is WUR if, and only if, $\|\cdot\|^{**}$ is W*UR. The proofs of all these results can be found in [DGZb].
The notion 2WUR was introduced in [HaRy]. The main result of this paper is that the James tree space JT has an equivalent 2WUR renorming.
Let us explain the situation in more detail. In [BGV] the authors investigate the properties of the Clarke subdifferential of a typical Lipschitz function on a given Banach space. They call a Banach space $(X, \|.\|)$ *Lipschitz separated*, if for every closed convex set $C \subset X$ and every bounded 1-Lipschitz real valued function f on C and $x \notin C$, there exist 1-Lipschitz extensions of f on the whole X, say f_1, f_2, satisfying $f_1(x) \neq f_2(x)$. This property depends heavily on the norm $\|.\|$. In [BGV] the following characterization is proved:

Theorem 42. *For a given Banach space $(X, \|\cdot\|)$ the following are equivalent:*

(1) *X is Lipschitz separated.*

(2) *For every pair of sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subset B_X$ such that $\lim_{n,m\to\infty} \|x_n + y_m\| = 2$, there is no $\phi \in X^*$ such that $\limsup_{n\to\infty} \phi(x_n) < 0 < \liminf_{n\to\infty} \phi(y_n)$.*

It is observed in [BGV] that the WUR property of $\|\cdot\|$ implies (2) (and so does 2WUR by a similar argument), and on the other hand (2) implies that $\|\cdot\|^{**}$ is rotund. The last fact implies that ℓ_1 is not isomorphic to any subspace of X. Indeed, if $\ell_1 \hookrightarrow X$, then $\ell_\infty \hookrightarrow X^*$, so in particular $\ell_1(c) \hookrightarrow X^*$. Thus $\ell_\infty(c) \hookrightarrow X^{**}$. For these classical results see, e.g., [HMVZ]. On the other hand, there exists no rotund renorming of $\ell_\infty(c)$ by Day's result (e.g., [DGZb, Corollary II.7.13]). Recall that, by [Haj96], the space JH of Hagler from [Hag], which also does not contain ℓ_1, does not admit an equivalent norm $\|\cdot\|$ such that $\|\cdot\|^{**}$ is rotund. Therefore JH does not admit a Lipschitz separated renorming. Thus separable spaces with 2-WUR renorming (or Lipschitz separating renorming) cut in between Asplund spaces and spaces not containing ℓ_1.

Problem 10. *Is every Lipschitz separated separable Banach space 2WUR renormable?*

The notions 2UR and 2W*UR seem to be new. Since 2UR implies 2R, it can hold only for reflexive spaces (see below).

Problem 11. *Study the notions 2UR and 2W*UR with respect to duality with some notions of smoothness. Characterize spaces sharing these properties and find the connections with other notions above.*

We are getting to the last set of notions. Milman, in [Mil], introduced the notions of 2R and W2R and suggested the problem whether they characterize reflexivity. This was solved in [HaJo] for W2R, and for 2R in [OdSch], in the separable case.

Theorem 43 ([HaJo],[OdSch])**.** *Let X be a Banach space. Then X is reflexive if, and only if, it admits an equivalent W2R norm. If X is a separable Banach space, then X is reflexive if, and only if, it admits an equivalent 2R norm.*

Problem 12. *Let X be a nonseparable reflexive Banach space. Is there an equivalent 2R renorming of X?*

The remaining notions are again new and have not yet been studied. Therefore we suggest to study them in some detail.

Problem 13. *Study the notions WCR, W*CR, 2WUR and 2W*UR with respect to duality with some notions of smoothness. Characterize spaces sharing these properties and find connections with other notions in the present note.*

In particular, since $\ell_1 \not\hookrightarrow X$ if X has 2WUR norm, check whether $\|\cdot\|$ is 2WUR if, and only if, $\|\cdot\|^{**}$ is 2W*UR, and find their dual notion of smoothness. Similarly,

Problem 14. *Is it true that $\|\cdot\|$ is WCR if, and only if, $\|\cdot\|^{**}$ is W*CR?*

Problem 15. *Is there an equivalent renorming characterization (perhaps WCR) of a separable Banach space not containing a copy of ℓ_1?*

We point out that there is an equivalent characterization of Banach spaces which do contain a copy of ℓ_1 by means of octahedrality, [DGZb].

Problem 16. *Study the notions analogous to Definition 41 where sequences are replaced by nets.*

Problem 17 (Godefroy)**.** *Suppose that w^*-convergent sequences on S_{X^*} are norm convergent. Is X an Asplund space?*

6 Two more constructions of smooth norms

The following result gives a quite general geometric method of construction of smooth norms whose dual norms *are not* strictly convex.

Theorem 44 ([Klee59])**.** *Every infinite dimensional separable nonreflexive Banach space admits a Gâteaux differentiable norm the dual norm of which is not strictly convex.*

Sketch of the Proof. (See Figures 1 and 2.) Let L, J, and H be closed linear subspaces of X such that $L \subset J \subset H$, J a hyperplane of H, and H a hyperplane of X. Let $p \in H$ such that $\operatorname{dist}(p, J) \geq 2$ and $q \in X$ such that $\operatorname{dist}(q, H) \geq 1$. Let Q_1 and Q_2 be the closed half-spaces of X containing 0 and bounded by the translated hyperplanes determined by $\operatorname{span}(J+q)\cup\{-p\}$ and $\operatorname{span}(J+q)\cup\{p\}$, respectively. We shall produce a smooth absolutely convex body B in X such that $B \subset Q_1 \cap Q_2$, $B \cap (L+q) = \emptyset$, but $\operatorname{dist}(B, L+q) = 0$. Then, if $|\cdot|_B$ denotes the Minkowski functional of B in X (an equivalent norm in X), as well as the corresponding norm in X/L and the dual norm in X^*, the closed unit ball of the space $(X/L, |\cdot|_B)$ admits two distinct supporting hyperplanes at the point $q + L$. This implies, in particular, that $|\cdot|_B$ in X^* is not strictly convex.
To construct such B, let C_0 be the closed unit ball of $(H, \|\cdot\|)$. There exists a decreasing sequence $\{C_n\}$ of bounded closed convex in L whose intersection is empty (note that L is

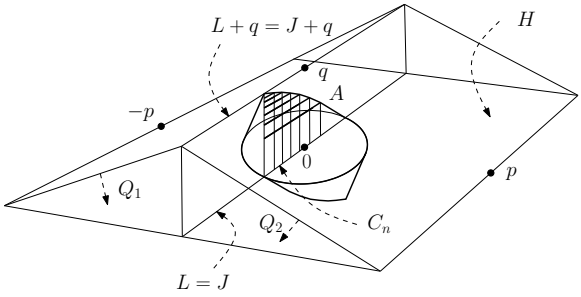

Figure 1: Construction of the set A (for simplicity, we assumed $L = J$)

not reflexive). Consider $A := \overline{\Gamma}\big(\bigcup_{n=0}^{\infty}(C_n + (1-2^{-n})q\big)$, where $\Gamma(S)$ denotes the convex and balanced hull of a set S. Then $A \subset X \setminus (L+q)$ and $\operatorname{dist}(A, L+q) = 0$.
There exists a compact absolutely convex smooth subset K that is contained in the open unit ball in $(X, \|\cdot\|)$ (for example $T^*B_{\ell_2}$ where T is a compact one-to-one operator from X into ℓ_2). For each $t \in (-1,1)$, let $A_t = A \cap (H + tq)$. Finally, let $B = \bigcup_{t\in(-1,1)} \big(A_t + (1-t)K\big)$. This set has the required properties. □

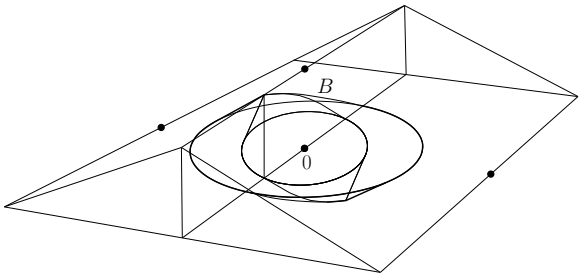

Figure 2: Construction of the set B

Another example of a separable Banach space whose norm is Gâteaux differentiable and its dual norm is *not* strictly convex was given by Troyanski in [Troy1].

Talagrand proved that the nonseparable space $C[0,\omega_1]$, where ω_1 is the first uncountable ordinal, admits a Fréchet differentiable norm but admits no norm whose dual norm is strictly convex (see e.g. [DGZb, Chapter 7]).

Problem 18. *Assume X^* is separable. Can a modified Klee's construction in Theorem 44 produce a Fréchet differentiable norm the dual of which is not strictly convex?*

The following theorem should be compared with Corollary 30. For a description of the James space J see, e.g., [FHHMZ, Definition 4.43].

Theorem 45 ([Sm76])**.** *The James space J admits a norm whose third dual is strictly convex.*

Proof. Let B denote the James space J renormed by a norm $\|\cdot\|$ such that its dual is at the same time LUR and W*UR (cf. e.g. [FHHMZ, Chapter 8]). Write $B^{***} = B^* \oplus \operatorname{span}\{b^{***}\}$, where $b^{***} \in B^{\perp}$. We claim that B^{***} is rotund. To show this suppose that x^{***} and y^{***}

are norm 1 elements in B^{***} such that $\|x^{***}+y^{***}\|=2$. We are to show that $x^{***}=y^{***}$. Write $x^{***}=x^*+\alpha b^{***}$ and $y^{***}=y^*+\beta b^{***}$, where x^* and y^* are in B^* and α and β are real numbers.
If $x^*\neq y^*$, then there exists an $x\in S_B$ such that $(x^*-y^*)(x)\neq 0$. By the Principle of Local Reflexivity, there is a sequence of linear maps $T_n:\text{span}\{x^{***},y^{***}\}\to B^*$ such that, for each $n\in\mathbb{N}$,

$$(T_n(x^{***}-y^{***}))(x)=(x^{***}-y^{***})(x)=(x^*-y^*)(x)$$

and

$$(1-\varepsilon_n)\|z\|\leq\|T_n(z)\|\leq(1+\varepsilon_n)\|z\|$$

for all $z\in\text{span}\{x^{***},y^{***}\}$, where $\{\varepsilon_n\}$ is a positive sequence of real numbers decreasing to 0 (use the fact that $b^{***}(x)=0$).
For $n\in\mathbb{N}$, let $x_n^*=T_n(x^{***})$ and $y_n^*=T_n(y^{***})$. Then we have $\|x_n^*\|\to 1$, $\|y_n^*\|\to 1$, and $\|x_n^*+y_n^*\|\to 2$. Thus by the W*UR property of the dual norm, $(x_n^*-y_n^*)(x)\to 0$. However, $(x_n^*-y_n^*)(x)=(x^*-y^*)(x)$ for each n, a contradiction. Thus $x^*=y^*$.
Choose $f\in S_{B^4}$ such that $f(\frac{1}{2}(x^{***}+y^{***}))=1$. Then $f(x^{***})=f(y^{***})=1$. Then we have

$$0=f(x^{***}-y^{***})=(\alpha-\beta)f(b^{***}).$$

If $f(b^{***})\neq 0$, then $\alpha=\beta$ and thus $x^{***}=y^{***}$ and the proof is finished.
If $f(b^{***})=0$, then

$$f(x^*)=f(x^*+\alpha b^{***})=f(x^{***})=1.$$

From this and since $\|x^*\|\leq\|x^*+\alpha b^{***}\|=1$, it follows that $\|x^*\|=1$ and $\|x^*+x^{***}\|=2$.
If $x^{***}\neq x^*$, then by the priciple of Local Reflexivity, there exists a sequence of linear maps $T_n:\text{span}\{x^*,x^{***}\}\to B^*$ such that $T_n(x^*)=x^*$ for each n and

$$(1-\varepsilon_n)\|z\|\leq\|T_n(z)\|\leq(1+\varepsilon_n)\|z\|$$

for all $z\in\text{span}\{x^*,x^{***}\}$ and for all $n\in\mathbb{N}$, where $\{\varepsilon_n\}$ is a sequence of positive real numbers decreasing to 0.
Let $x_n^*=T_n(x^{***})$. Then we have $\|x^*\|=1$, $\|x_n^*\|\to 1$ and $\|x^*+x_n^*\|\to 2$. By the LUR property of the dual norm, we thus have that $\|x^*-x_n^*\|\to 0$. However, $\|x^*-x_n^*\|\geq(1-\varepsilon_n)\|x^*-x^{***}\|$. Thus $x^{***}=x^*$. Similarly we can show that $y^{***}=y^*=x^*$. Thus $x^{***}=y^{***}$ and the proof is completed. □

It is shown in [Haj96] that the James tree space JT admits a norm whose second dual is strictly convex. Thus its predual JT_* has a norm whose third dual is strictly convex. It is also shown in [Haj96] that the separable Hagler's space JH (that also does not contain a copy of ℓ_1 and JH^* is nonseparable (cf. [Hag])) admits no equivalent norm whose second dual is strictly convex.

Yet, the following problem seems to be open.

Problem 19 ([Haj96])**.** *Is it true that if X is separable and does not contain a copy of ℓ_1, then X^* admits an equivalent Gâteaux differentiable norm?*

Remark We do not know if appropriate versions of many results discussed above hold true in the nonseparable setting.

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