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POLYNOMIALS AND IDENTITIES ON REAL BANACH SPACES

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1. Introduction

In our present paper we study the duality theory and linear identities for real polynomials and functions on Banach spaces, which allows for a unified treatment and generalization of some classical results in the area. The basic idea is to exploit point evaluations of polynomials, as e.g. in [Rez93]. As a by-product we also obtain a curious generalization of the well-known Hilbert lemma on the representation of the even powers of the Hilbert norm as sums of powers of functionals (Corollary 2.14). In Theorems 2.16 and 2.20 (generalizing [Wil18] and [Rez79]) we prove that identities derived from pairwise linearly independent point evaluations can be satisfied only by polynomials. We apply the Lagrange interpolation theory in order to create a machinery allowing the creation of linear identities which characterize spaces of polynomials of prescribed degrees (Theorem 2.18, Theorem 3.2). We elucidate the special situation when all the evaluation points are collinear (Corollary 2.24 and Theorem 3.4). Our work is based on (and generalizes) the theory of functional equations in the complex plane due to Wilson [Wil18] and Reznick (in the homogeneous case) [Rez78], [Rez79], the classical characterizations of polynomials due to Fréchet [Fr09], [Fr09b], and Mazur and W. Orlicz, [MO1], [MO2] which can be summarized in the following theorem.

Theorem 1.1. *Let X, Y be real Banach spaces, $f : X \to Y$ be continuous. TFAE*

(i) $f \in \mathcal{P}^n(X;Y)$.

(ii) $\Delta^{n+1} f(x, h_1, \dots, h_n) \equiv 0$ *for all* $x, h_i \in X$.

(iii) $f \restriction_E$ *is a polynomial of degree at most n for every affine one-dimensional subspace E of X.*

(iv)

$$\sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} P(x + kh) = 0 \ \text{ for all } \ x, h \in X. \tag{1}$$

Here we use the higher order differences defined as follows.

$$\Delta^k f(x_0; h_1, \dots, h_k) = \sum_{j=0}^{k} \sum_{A \subset \{1,\dots,k\}, |A|=j} (-1)^{k-j} f(x_0 + \sum_{l \in A} h_l). \tag{2}$$

In particular,

$$\Delta^k f(x_0; h, \dots, h) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x_0 + jh), \tag{3}$$

The theory of linear identities for Banach space norms was developed by many authors. Its first and well-known result is a theorem of Jordan and von Neumann.

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Theorem 1.2. [JvN] *Let $(X, \|\cdot\|)$ be a Banach space such that*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2, x, y \in X. \tag{4}$$

Then X is isometric to a Hilbert space.

This theorem has been the basis of subsequent development with the aim of using similar identities in order to characterize the Hilbert space, or the classes of Banach spaces allowing the polynomial norms, e.g. Carlsson [Car], Day [Day47], [Day59], Giles [Gile67], G.G. Johnson [Joh-G], Koehler [Koe], [Koe72], Lorch [Lor48], Reznick [Rez78], [Rez79] and Senechalle [Sen68]. This theory is closely related to the isometric Banach space theory, see e.g. Koldobsky and Konig [KolKon] and references therein. In our paper, we develop an abstract approach to the theory of linear identities, generalizing Wilson's and Reznick's work. The novelty lies in giving a new functional-analytic meaning to these identities, finding the link to the Lagrange interpolation, and finding a general method for establishing new identitites with prescribed properties.

2. General theory

We begin developing our abstract framework. Let X, Y be Banach spaces. We denote by $\mathcal{P}(^d X; Y)$ (resp. $\mathcal{P}^d(X; Y)$) the Banach space of continuous d-homogeneous polynomials from X to Y (resp. continuous polynomials of degree at most d). Using Theorem 1.1 one proves easily the next useful fact.

Fact 2.1. *Let X, Y be real Banach spaces, $P : X \to Y$ be a continuous mapping such that $\phi \circ P$ is a polynomial of degree at most d, for every $\phi \in Y^*$. Then $P \in \mathcal{P}^d(X; Y)$.*

Let $n \in \mathbb{N}$. We are going to use some notation and results in [Rez93]. We have a natural identification $(\mathbb{R}^n)^* = \mathbb{R}^n$, using the dot product. For simplicity of notation, we put $F_{n,d} = \mathcal{P}(^d \mathbb{R}^n; \mathbb{R})$. Denote the set of multiindices by

$$\mathcal{I}(n,d) = \{\alpha : \alpha : \{1, 2, \ldots, n\} \to \{0, \ldots, d\}, |\alpha| = \sum_{i=1}^{n} \alpha(i) = d\}. \tag{5}$$

One gets $\dim F_{n,d} = |\mathcal{I}(n,d)| = \binom{n+d-1}{n-1}$. For simplicity of notation, we put $\Pi_{n,d} = \mathcal{P}^d(\mathbb{R}^n)$, $\mathcal{J}(n,d) = \cup_{l=0}^{d} \mathcal{I}(n,l)$ is the set of all multiindices of degree at most d on $\mathbb{R}^n$. Clearly, for every $P \in \Pi_{n,d}$ there exist a uniquely determined representation $P(x) = \sum_{\alpha \in \mathcal{J}(n,d)} a_\alpha x^\alpha$.

Fact 2.2.

$$\mathit{dim}\, \Pi_{n,d} = \sum_{l=0}^{d} \binom{n+l-1}{n-1} = \binom{n+d}{n} = \mathit{dim}\, F_{n+1,d}. \tag{6}$$

Moreover, there is a natural linear isomorphism $i : \Pi_{n,d} \to F_{n+1,d}$, given by the restriction $i(P) = P\restriction_E$, where $E = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 1\}$ is an affine hyperplane. In other words, performing i on a d-homogeneous polynomial means replacing $n+1$-st coordinate by constant 1.

Let $C(\mathbb{R}^n)$ be the space of all continuous functions on $\mathbb{R}^n$. Point evaluations at $x \in \mathbb{R}^n$ belong to the linear dual of $C(\mathbb{R}^n)$. Point evaluations separate elements of $C(\mathbb{R}^n)$. For $z \in \mathbb{R}^n$ we are going to use the notation $\mathbf{z} = 1z \in C(\mathbb{R}^n)^*$ where $\mathbf{z}(f) = f(z), f \in C(\mathbb{R}^n)$, and we will call these evaluation functionals nodes. To simplify the language, we will occasionally identify $z \in \mathbb{R}^n$ with its corresponding node $\mathbf{z}$, calling the elements of $\mathbb{R}^n$ themselves nodes. We are going to introduce an abstract formalism suitable for working with nodes and their linear combinations.

Consider the linear space $\mathcal{F}(\mathbb{R}^n)$ of all formal finite linear combinations of nodes. It is important to note that a linear multiple $\xi\mathbf{z}$ of the node $\mathbf{z}$ is not the same element as the node corresponding to the point $\xi z \in \mathbb{R}^n$. Informally, whenever we write ξz as an element of $\mathcal{F}(\mathbb{R}^n)$, it is understood that we are dealing with the element $\xi\mathbf{z}$. In order to distinguish the usual vector summation from the space $\mathbb{R}^n$ from the formal summation of the nodes we will introduce the new summation symbol $\boxplus$. So for every $\mathbf{x} \in \mathcal{F}(\mathbb{R}^n)$ there exist $a_i \in \mathbb{R}, x_i \in \mathbb{R}^n$ so that

$$\mathbf{x} = a_1x_1 \boxplus \cdots \boxplus a_kx_k = \boxplus - \sum_{i=1}^{k} a_ix_i \tag{7}$$

The previous expression is unique if x_i are assumed pairwise distinct and $a_i \neq 0, i = 1, \dots, k$.

The operation $\boxplus$ formally acts on $\mathbf{x} = \boxplus - \sum_{i=1}^{k} a_ix_i$ and $\mathbf{y} = \boxplus - \sum_{i=1}^{l} b_iy_i$ as

$$\mathbf{x} \boxplus \mathbf{y} = (\boxplus - \sum_{i=1}^{k} a_ix_i) \boxplus (\boxplus - \sum_{i=1}^{l} b_iy_i), \tag{8}$$

Similarly, we define the left scalar multiplication of $\xi \in \mathbb{R}$ and $\mathbf{x}$ as

$$\xi\mathbf{x} = (\boxplus - \sum_{i=1}^{k} (\xi a_i)x_i). \tag{9}$$

With these operations $\mathcal{F}(\mathbb{R}^n)$ is a linear space. Then $\langle C(\mathbb{R}^n), \mathcal{F}(\mathbb{R}^n)\rangle$ form a dual pair ([FHHMZ]) with the evaluation

$$\langle f, \mathbf{x}\rangle = \sum_{i=1}^{k} a_i f(x_i). \tag{10}$$

Restricting this dual pairing to subspaces $F_{n,d}$ (resp. $\Pi_{n,d}$) of $C(\mathbb{R}^n)$ leads to a dual factorization of the action of $\boxplus$ on $\mathcal{F}(\mathbb{R}^n)$ so that $\mathbf{x}_d = \boxplus_d - \sum_{i=1}^{k} a_ix_i$ (resp. $\mathbf{x}^d = \boxplus^d - \sum_{i=1}^{k} a_ix_i$) and

$$\mathbf{x}_d = \boxplus_d - \sum_{i=1}^{k} a_ix_i = \mathbf{y}_d = \boxplus_d - \sum_{i=1}^{l} b_iy_i \tag{11}$$

iff

$$\langle f, \mathbf{x}_d\rangle = \langle f, \mathbf{y}_d\rangle \text{ holds for all } f \in F_{n,d}. \tag{12}$$

(and the resp. case of $\Pi_{n,d}$).

Thus we have a (non-unique) representation of the elements of $F^*_{n,d}$ (resp. $\Pi^*_{n,d}$) as elements in $\mathcal{F}(\mathbb{R}^n)$, given by

$$\langle P, \mathbf{x}\rangle = \langle P, \boxplus - \sum_{i=1}^{k} a_ix_i\rangle = \sum_{i=1}^{k} a_iP(x_i). \tag{13}$$

$P \in \Pi_{n,d}$, $\mathbf{x} = \boxplus - \sum_{i=1}^{k} a_ix_i$. We let $\mathcal{K}_d \hookrightarrow \mathcal{F}(\mathbb{R}^n)$ be the subspace consisting of all elements for which

$$\langle P, \boxplus - \sum_{i=1}^{k} a_ix_i\rangle = 0 \text{ holds for all } P \in \Pi_{n,d}. \tag{14}$$

Then $\Pi^*_{n,d} = \mathcal{F}(\mathbb{R}^n)/\mathcal{K}_d$. Suppose $A = \{y_1, \dots, y_r\} \subset \mathbb{R}^n$. We say that the corresponding set of nodes $\mathbf{A} = \{\mathbf{y_1}, \dots, \mathbf{y_r}\}$ is $F_{n,d}$-independent if the nodes are linearly independent as elements of $F^*_{n,d}$. For simplicity, if the space $F_{n,d}$ is understood, we will often drop the boldface notation and say that A is a set of nodes, and that A is $F_{n,d}$-independent. It is clear form basic linear algebra that A is $F_{n,d}$-independent iff there exist dual elements $\{h_1, \ldots, h_r\} \subset F_{n,d}$ so that $h_j(y_k) = \delta_j^k$. If $\{y_1, ..., y_r\}$ are $F_{n,d}$-independent then $r \leq |\mathcal{I}(n,d)|$. In case of $r = |\mathcal{I}(n,d)|$, $F^*_{n,d} = \text{span}(\{y_k\}_{k=1}^r)$ and we call $\{y_k\}_{k=1}^r$ a basic set of nodes for $F_{n,d}$. A classical example of a basic set of nodes for $F_{n,d}$ is the set $\mathcal{I}(n,d)$ (Biermann, see [Rez93]). The following result is immediate.

Proposition 2.3. *Let $r = |\mathcal{I}(n,d)|$. If $\{y_k\}_{k=1}^r$ is a basic set of nodes for $F_{n,d}$ and $\{h_k\}_{k=1}^r \subset F_{n,d}$ is its dual basis, then for all $P \in F_{n,d}$*

$$P(x) = \sum_{k=1}^{r} P(y_k) h_k(x). \tag{15}$$

The following is a general characterization of basic sets of nodes [Lor], [Rez93].

Theorem 2.4. *Let $r = |\mathcal{I}(n,d)|$, $\mathcal{I}(n,d) = \{\alpha_1, \ldots, \alpha_r\}$. Let $\{y_k\}_{k=1}^r \subset I\!R^n$. Then $\{y_k\}_{k=1}^r$ is a basic set of nodes for $F_{n,d}$ iff it holds*

$$det \begin{pmatrix} y_1^{\alpha_1} y_1^{\alpha_2} \ldots y_1^{\alpha_r} \\ y_2^{\alpha_1} y_2^{\alpha_2} \ldots y_2^{\alpha_r} \\ \ldots \\ y_r^{\alpha_1} y_r^{\alpha_2} \ldots y_r^{\alpha_r} \end{pmatrix} \neq 0. \tag{16}$$

Moreover, if $\{y_k\}_{k=1}^r$ is a basic set of nodes for $F_{n,d}$ then every $P \in F_{n,d}$ can be written uniquely as $P(x) = \sum_{k=1}^r a_k \langle y_k, x \rangle^d$.

The same notation and terminology applies to the case of $\Pi_{n,d}$ spaces. Analogously, for $r = |\mathcal{J}(n,d)|$, we say that $\{y_k\}_{k=1}^r \subset I\!R^n$ is a basic set of nodes for $\Pi_{n,d}$ if these elements form a linear basis of $\Pi^*_{n,d}$. Observe that basic sets of nodes exist, as the pointwise evaluations form a separating set of functionals for $\Pi_{n,d}$. The following is a general characterization of basic sets of nodes for $\Pi_{n,d}$, analogous to Theorem 2.4, [Lor].

Theorem 2.5. *Let $r = |\mathcal{J}(n,d)|$, $\mathcal{J}(n,d) = \{\alpha_1, \ldots, \alpha_r\}$. Let $\{y_k\}_{k=1}^r \subset I\!R^n$. Then $\{y_k\}_{k=1}^r$ is a basic set of nodes for $\Pi_{n,d}$ iff it holds*

$$det \begin{pmatrix} y_1^{\alpha_1} y_1^{\alpha_2} \ldots y_1^{\alpha_r} \\ y_2^{\alpha_1} y_2^{\alpha_2} \ldots y_2^{\alpha_r} \\ \ldots \\ y_r^{\alpha_1} y_r^{\alpha_2} \ldots y_r^{\alpha_r} \end{pmatrix} \neq 0. \tag{17}$$

Moreover, if $\{y_k\}_{k=1}^r$ is a basic set of nodes then every node $y \in I\!R^n \hookrightarrow \Pi^*_{n,d}$ can be written uniquely as a linear combination of the elements in $\{y_k\}_{k=1}^r$. More precisely, $\mathbf{y} = \boxplus^d - \sum_{k=1}^r a_k y_k$ iff $\{a_k\}_{k=1}^r$ form a solution of the system of linear equations

$$\sum_{k=1}^{r} a_k y_k^{\alpha} = y^{\alpha}, \ \alpha \in \mathcal{J}(n,d). \tag{18}$$

The Generalized Lagrange formula is an expression of linear dependence of nodes in the dual of $\Pi_{n,d}$.

Theorem 2.6. *(Generalized Lagrange Formula) Let $r = |\mathcal{J}(n,d)|$, $\{y_k\}_{k=1}^r$ be a basic set of nodes for $\Pi_{n,d}$. Then for every $z \in I\!R^n \setminus \{y_k\}_{k=1}^r$ there exist a unique set of coefficients $a_k(z) \in I\!R$ such that $\mathbf{z} = \boxplus^d - \sum_{k=1}^r a_k(z) y_k$. The functions $z \to a_k(z)$ are polynomials of degree at most d, given by the formula*

$$a_k(z) = \frac{det\begin{pmatrix} y_1^{\alpha_1} y_1^{\alpha_2} \dots y_1^{\alpha_r} \\ y_2^{\alpha_1} y_2^{\alpha_2} \dots y_2^{\alpha_r} \\ \dots \\ z^{\alpha_1} z^{\alpha_2} \dots z^{\alpha_r} \\ \dots \\ y_r^{\alpha_1} y_r^{\alpha_2} \dots y_r^{\alpha_r} \end{pmatrix}}{det\begin{pmatrix} y_1^{\alpha_1} y_1^{\alpha_2} \dots y_1^{\alpha_r} \\ y_2^{\alpha_1} y_2^{\alpha_2} \dots y_2^{\alpha_r} \\ \dots \\ y_k^{\alpha_1} y_k^{\alpha_2} \dots y_k^{\alpha_r} \\ \dots \\ y_r^{\alpha_1} y_r^{\alpha_2} \dots y_r^{\alpha_r} \end{pmatrix}}. \tag{19}$$

Then $\{a_k, y_k\}_{k=1}^r$ is a biorthogonal system in $\Pi_{n,d} \times \Pi_{n,d}^*$ and the formula

$$P(z) = \sum_{k=1}^r a_k(z) P(y_k) \tag{20}$$

is valid for $P \in \Pi_{n,d}$.

We remark that the problem of characterizing geometrically basic sets of nodes for $\Pi_{n,d}$, when $n \geq 2$, is open, and it is important for approximation theory and its applications in numerical mathematics. We refer to [Lor], [ChuY], [BooRo] for more results and references.

Let $L \in \mathcal{L}(I\!R^N; I\!R^M)$. We let $\tilde{L} \in \mathcal{L}(\mathcal{F}(I\!R^N); \mathcal{F}(I\!R^M))$ be defined as

$$\tilde{L}(\boxplus - \sum_{i=1}^k a_i x_i) = \boxplus - \sum_{i=1}^k a_i L(x_i). \tag{21}$$

We introduce a partial ordering for elements of $\cup_{n=1}^\infty \mathcal{F}(I\!R^n)$ by setting for $\mathbf{x} = a_1 x_1 \boxplus \cdots \boxplus a_n x_n \in \mathcal{F}(I\!R^N)$ and $\mathbf{y} = b_1 y_1 \boxplus \cdots \boxplus b_m y_m \in \mathcal{F}(I\!R^M)$

$$\mathbf{x} \succ \mathbf{y} \ \text{ iff } \ \tilde{L}\mathbf{x} = \mathbf{y} \ \text{ for some } \ L \in \mathcal{L}(I\!R^N; I\!R^M). \tag{22}$$

Definition 2.7. *We say that a polynomial $P \in \Pi_{n,d}$ is compatible with $\mathbf{x} \in \mathcal{F}(I\!R^m)$ if*

$$\langle P \circ L, \mathbf{x} \rangle = \langle P, \tilde{L}\mathbf{x} \rangle = 0 \ \text{ for all } L \in \mathcal{L}(I\!R^m; I\!R^n). \tag{23}$$

Let X, Y are Banach spaces and $f : X \to Y$ is a continuous mapping, then we say that f is compatible with $\mathbf{x} \in \mathcal{F}(I\!R^m)$, $\mathbf{x} = \sum_{i=1}^k a_i x_i$ if

$$\langle f \circ L, \mathbf{x} \rangle = \sum_{i=1}^k a_i f(L x_i) = 0 \ \text{ for all } L \in \mathcal{L}(I\!R^m; X). \tag{24}$$

Remark 2.8. Clearly, if X, Y are Banach spaces, then a continuous mapping $f : X \to Y$ is compatible with $\mathbf{x} = a_1 x_1 \boxplus \cdots \boxplus a_n x_n \in \mathcal{F}(I\!R^m)$, where $x_k = (x_k^1, \dots, x_k^m)$, if and only if

$$\sum_{k=1}^n a_k f(\sum_{i=1}^m x_k^i z_i) = 0 \ \text{ for every } z_1, \dots, z_m \in X. \tag{25}$$

The expression (25) is called a linear identity. In particular, Fréchet theorem 1.1 is equivalent to saying that f is a polynomial of degree at most n iff f is compatible with an element $\mathbf{x_M} \in \mathcal{F}(I\!R^{k+1})$ (resp. $\mathbf{x_F} \in \mathcal{F}(I\!R^2)$) where

$$\mathbf{x_M} = \boxplus - \sum_{j=0}^k \sum_{A \subset \{1,\dots,k\}, |A|=j} (-1)^{k-j} (e_0 + \sum_{l \in A} e_l). \tag{26}$$

$$\mathbf{x_F} = \boxplus - \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} (1,k) \in \mathcal{F}(I\!R^2). \tag{27}$$

Moreover, the linear operator $L : I\!R^{k+1} \to I\!R^2$, $L(x_0, x_1, \ldots, x_k) = (x_0, \sum_{i=1}^k x_i)$ satisfies $\tilde{L}(\mathbf{x_M}) = \mathbf{x_F}$, so in particular $\mathbf{x_M} \succ \mathbf{x_F}$. It is easy to see that $L : I\!R^N \to I\!R^M$ leads to a linear mapping $L^* : \Pi_{M,d} \to \Pi_{N,d}$ defined as $L^*(P) = P \circ L$. The adjoint linear operator $L^{**} : \Pi^*_{N,d} \to \Pi^*_{M,d}$ coincides with $\tilde{L}$ (if the duals are represented using the canonical evaluations). The following is a simple consequence of the definitions.

Fact 2.9. *Let $\mathbf{x} \in \mathcal{F}(I\!R^m), \mathbf{y} \in \mathcal{F}(I\!R^n)$, X, Y be Banach spaces and $f : X \to Y$ be continuous. Suppose that $\mathbf{x} \succ \mathbf{y}$. Then the compatibility of f with $\mathbf{x}$ implies the compatibility of f with $\mathbf{y}$. Consequently, if $\tilde{L}\mathbf{x} = \mathbf{y}$ for some bijection $L \in \mathcal{L}(I\!R^m)$, then f is compatible with $\mathbf{x}$ if and only if f is compatible with $\mathbf{y}$.*

Lemma 2.10. *Let $\mathbf{x} \in \mathcal{F}(I\!R^m)$. TFAE*

(i) For every Banach spaces X, Y every $P \in \mathcal{P}(^d X; Y)$ is compatible with $\mathbf{x}$.
(ii) Every $P \in F_{m,d}$ is compatible with $\mathbf{x}$.
(iii) $\langle P, \mathbf{x} \rangle = 0$ for every $P \in F_{m,d}$.

Proof. The implications (i)⇒(ii)⇒(iii) are clear.

(iii)⇒(ii): Suppose that (iii) holds, and let $P \in F_{m,d}$. If $L \in \mathcal{L}(I\!R^m)$, then $P \circ L \in F_{m,d}$, hence $\langle P \circ L, \mathbf{x} \rangle = 0$, and therefore P is compatible with $\mathbf{x}$.

(ii)⇒(i): Suppose that every $P \in F_{m,d}$ is compatible with $\mathbf{x}$. Let X, Y be Banach spaces and $P \in \mathcal{P}(^d X; Y)$. Let $L \in \mathcal{L}(I\!R^m; X)$ and choose $\varphi \in Y^*$ arbitrary. Then $\varphi \circ P \circ L \in F_{m,d}$, and therefore

$$0 = \langle \varphi \circ P \circ L, \mathbf{x} \rangle = \varphi(\langle P \circ L, \mathbf{x} \rangle). \tag{28}$$

Since φ was arbitrary, we conclude using Fact 2.1 that $\langle P \circ L, \mathbf{x} \rangle = 0$. □

Lemma 2.11. *Let X, Y be Banach spaces and let $P = \sum_{k=0}^d P_k \in \mathcal{P}^d(X; Y)$ where $P_k \in \mathcal{P}(^k X; Y)$ are k-homogeneous summands. If P is compatible with $\mathbf{x} \in \mathcal{F}(I\!R^m)$, then each nonzero summand P_k is compatible with $\mathbf{x}$.*

Proof. By assumption,

$$\langle P \circ L, \mathbf{x} \rangle = \sum_{k=0}^d \langle P_k \circ L, \mathbf{x} \rangle = 0 \text{ for all } L \in \mathcal{L}(I\!R^m; X). \tag{29}$$

In particular, fixing L, composing $L \circ (tId_{I\!R^m})$, and using the homogenity of P_k we obtain

$$0 = \langle P \circ (L \circ (tId_{I\!R^m})), \mathbf{x} \rangle = \sum_{k=0}^d t^k \langle P_k \circ L, \mathbf{x} \rangle \text{ for all } L \in \mathcal{L}(I\!R^m; X). \tag{30}$$

The right hand side, for a fixed L, is an X-valued polynomial in t. Thus each $\langle P_k \circ L, \mathbf{x} \rangle = 0$, otherwise for some t the total value could not be zero. □

The following result was proved by Reznick. We give a proof using our formalism.

Lemma 2.12. *Let X, Y be Banach spaces and let $0 \neq P \in \mathcal{P}(^d X; Y)$, $\mathbf{x} \in \mathcal{F}(I\!R^m)$. Then P is compatible with $\mathbf{x}$ iff the polynomial $t \to t^d$ from $\mathcal{P}(^d I\!R)$ is compatible with $\mathbf{x}$.*

Proof. On one hand, there exists a one dimensional subspace $E \hookrightarrow X$ such that $P \restriction_E = at^d, a \neq 0$. So for every $L : \mathbb{R}^m \to E$ we have that $\langle P \circ L, \mathbf{x} \rangle = 0$. Consequently, t^d is compatible with $\mathbf{x}$. On the other hand, if t^d is compatible with $\mathbf{x}$, then so is every $\phi^d(y)$, where $\phi \in (\mathbb{R}^m)^*$. Indeed, $\phi^d(y)$ is a composition of a linear projection of $\mathbb{R}^m$ onto E, and the polynomial t^d defined on $E = \mathbb{R}$. If $Q \in F_{m,d}$, then by Theorem 2.4 $Q(y) = \sum a_k \phi_k^d(y)$, so Q is compatible with $\mathbf{x}$, being a sum of finitely many polynomials compatible with $\mathbf{x}$. Lemma 2.10 then finishes the proof. □

Corollary 2.13. *An element* $\mathbf{x} = a_1 x_1 \boxplus \cdots \boxplus a_n x_n \in \mathcal{F}(\mathbb{R}^n)$ *is compatible with* $t \to t^d$ *(or any other nonzero d-homogeneous polynomial) iff* $a_1 x_1 \boxplus_d \cdots \boxplus_d a_n x_n = 0$ *in* $F^*_{n,d}$.

Corollary 2.14. *Let* $0 \neq P \in F_{n,d}$. *Then for any* $Q \in F_{n,d}$ *there exist a finite collection of linear* $L_k \in \mathcal{L}(\mathbb{R}^n)$ *such that* $Q = \sum P \circ L_k$.

Proof. Suppose the contrary. Then the linear span $H = \operatorname{span}\{P \circ L : L \in \mathcal{L}(\mathbb{R}^n)\}$ in the space $F_{n,d}$ is a proper subspace, i.e. there exists some $Q \in F_{n,d} \setminus H$ and a linear functional $\mathbf{x}$ which is zero on H and nonzero on Q. Thus P is compatible with $\mathbf{x}$, but Q is not. This contradicts Lemma 2.12. □

The above corollary is an intriguing generalization of the celebrated Hilbert lemma, which claims that for a given $l, n \in \mathbb{N}$ there exists a finite collection $\{\phi_1, \ldots, \phi_N\} \subset (\mathbb{R}^n)^*$ such that

$$\|x\|_{\ell_2}^{2l} = \sum_{i=1}^{N} \phi_i^{2l}(x), x \in \mathbb{R}^n. \tag{31}$$

Indeed, choose $Q(x) = \|x\|_{\ell_2}^{2l}, P(x) = \phi^{2l}(x), 0 \neq \phi \in (\mathbb{R}^{2l})^*$ and apply Corollary 2.14. We see that in fact the Hilbert lemma remains valid for arbitrary pair of nonzero polynomials P, Q.

Next, we investigate the properties of $\mathbf{x}$, which lead to compatibility. We restrict our attention to the case when x_i are pairwise linearly independent. (This assumption is natural, as there are easy examples of $\mathbf{x}$ failing this condition such that all d-homogeneous continuous functions are compatible with $\mathbf{x}$.) An interesting example in this direction is derived from the polarization formula below. The second part (33) is an easy observation of the present authors, which follows by inspection of the classical proof (e.g. [Din] p.8).

Proposition 2.15. [BH31], [MO1]*(Polarization formula) For every* $P \in \mathcal{P}(^n X; Y)$ *there exists a unique symmetric n-linear form* $\check{P} \in \mathcal{L}^s(^n X; Y)$ *such that* $P(x) = \check{P}(x, \ldots, x)$. *The following formula holds.*

$$\check{P}(x_1, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \ldots \varepsilon_n P(\sum_{i=1}^{n} \varepsilon_i x_i). \tag{32}$$

On the other hand, for every $0 \neq P \in \mathcal{P}(^k X; Y), k < n$, *or* $k - n$ *odd and positive the following formula holds.*

$$\sum_{\varepsilon_i = \pm 1} \varepsilon_1 \ldots \varepsilon_n P(\sum_{i=1}^{n} \varepsilon_i x_i) = 0, x_i \in X \tag{33}$$

It the remaining case when $k > n$ *and* $k - n$ *is even, there exists* $x \in X$ *such that the right hand side in* (33) *for* $x_i = x, i = 1, \ldots, k$ *is nonzero.*

Translated into our language, we see that $\mathbf{x_B} = \boxplus - \sum_{\varepsilon_i = \pm 1} (\varepsilon_1 \ldots \varepsilon_n)(\sum_{i=1}^n \varepsilon_i e_i)$ is compatible with k-homogeneous polynomials iff either $k < n$ or $k - n$ is a positive odd number. Note that summands involved in the definition of $\mathbf{x_B}$ contain many pairs of linearly dependent vectors. Under the assumption that x_i are pairwise linearly independent, we will prove that every continuous mapping which is compatible with $\mathbf{x} = a_0x_0 \boxplus \cdots \boxplus a_{n+1}x_{n+1} \in \mathcal{F}(I\!R^m)$ is necessarily a polynomial of degree at most n. In particular, the Jordan-von Neumann Theorem 1.2 follows immediately from this statement. The result was proved by Reznick under the assumption that the continuous function f is homogeneous.

Theorem 2.16. *Let X, Y be Banach spaces. Let $f : X \to Y$ be a continuous mapping and $\mathbf{x} = a_0x_0 \boxplus \cdots \boxplus a_{n+1}x_{n+1} \in \mathcal{F}(I\!R^m)$, $m \geq 2$, where $x_k = (x_k^1, \ldots, x_k^m), k = 0, \ldots, n+1$, are pairwise linearly independent vectors. If f is compatible with $\mathbf{x}$ then f is a polynomial of degree at most n.*

Proof. First let $m = 2$. By Fact 2.1 it suffices to prove the statement for real valued functions on $I\!R$. Wilson [Wil18] proved (in our language), using algebraic manipulations, that every continuous function g on $I\!R^2$, which is compatible with $\mathbf{x}$ is also compatible with $\mathbf{x_F}$ from (27). By Theorem 1.1 we may conclude that that f is a polynomial of degree at most n. The case $m > 2$. By Fact 2.9 it is enough to find a $\mathbf{y} \in \mathcal{F}(I\!R^2)$, such that $\mathbf{x} \succ \mathbf{y}$, and $\mathbf{y}$ satisfies the assumptions of the previous case. So we claim that there exists an affine linear operator $T : I\!R^m \to I\!R^2$ such that

$$T(x_0^1, \ldots, x_0^m), \ldots, T(x_{n+1}^1, \ldots, x_{n+1}^m) \in I\!R^2 \tag{34}$$

are pairwise linearly independent vectors. This is easily seen as follows. Let $E_{i,j} = [(x_i^1, \ldots, x_i^m), (x_j^1, \ldots, x_j^m)] \hookrightarrow I\!R^m$, $i, j \in \{0, \ldots, n+1\}, i \neq j$, be a system of 2-dimensional subspaces of $I\!R^m$. There exists a $(m-2)$-dimensional subspace $F \hookrightarrow I\!R^m$ such that $F \cap E_{i,j} = \{0\}, i, j \in \{0, \ldots, n+1\}, i \neq j$. (Equivalently, $F + E_{i,j} = I\!R^m$). Then the orthogonal projection T in $I\!R^m$, with kernel F and two dimensional range $F^\perp \hookrightarrow I\!R^m$, clearly satisfies the condition. Suppose that $T(y_1, \ldots, y_m) = (\sum_{j=1}^m \alpha_j y_j, \sum_{j=1}^m \beta_j y_j)$. Let $B_k^1 = \sum_{j=1}^m \alpha_j x_k^j, B_k^2 = \sum_{j=1}^m \beta_j x_k^j$, $k = 0, \ldots, n+1$. By assumption, $(B_k^1, B_k^2), k = 0, \ldots, n+1$, are pairwise linearly independent. Let $y_1, y_2 \in X$ be arbitrary vectors, denote $z_j = \alpha_j y_1 + \beta_j y_2, j = 1, \ldots, m$. Then

$$B_k^1 y_1 + B_k^2 y_2 = \sum_{j=1}^m \alpha_j x_k^j y_1 + \sum_{j=1}^m \beta_j x_k^j y_2 = \sum_{j=1}^m x_k^j z_j \;\text{ for all }\; y_1, y_2 \in X. \tag{35}$$

Hence it holds

$$\sum_{k=0}^{n+1} a_k f(B_k^1 y_1 + B_k^2 y_2) = 0 \;\text{ for all }\; y_1, y_2 \in X, \tag{36}$$

and we may we conclude that $f \in \mathcal{P}^n(X; Y)$. □

If $\mathbf{x} \in \mathcal{F}(I\!R^m)$ and X, Y are Banach spaces, then the set of all continuous mappings $f : X \to Y$ which are compatible with $\mathbf{x}$ is clearly a linear space. We are now ready to describe this space more precisely.

Theorem 2.17. *Let $\mathbf{x} = a_0x_0 \boxplus \cdots \boxplus a_{n+1}x_{n+1} \in \mathcal{F}(I\!R^m)$, where $x_0, \ldots, x_{n+1}$ are pairwise linearly independent vectors. Then there exists $A \subset \{0, \ldots, n\}$ such that if X, Y are Banach spaces and $f : X \to Y$ is a continuous mapping, then f is compatible with $\mathbf{x}$ if and only if $f = \sum_{k \in A} P_k$ for some $P_k \in \mathcal{P}({}^kX; Y)$ (if A is empty, the sum is understood to be equal to 0).*

Proof. Let A be the set of all $k \in \{0, \ldots, n\}$ for which there exist Banach spaces X, Y and a nonzero polynomial from $\mathcal{P}({}^kX; Y)$ which is compatible with $\mathbf{x}$. By Lemma 2.12, if $k \in A$, then for every Banach spaces X, Y every polynomial from $\mathcal{P}({}^kX; Y)$ is compatible with $\mathbf{x}$, and the same holds also for their linear combinations. Let now X, Y be Banach spaces and $P : X \to Y$ be a continuous mapping compatible with $\mathbf{x}$. By Theorem 2.16 the mapping P is a polynomial of degree at most n. Say $P = \sum_{k=0}^{n} P_k$, where $P_k \in \mathcal{P}(^k X; Y)$. If $P_k \not\equiv 0$ for some $k \in \{0, \dots, n\}$, then it follows from Lemma 2.11 that $k \in A$. Hence $P = \sum_{k \in A} P_k$. $\square$

Theorem 2.18. *Let $0 \leq d_1 < d_2 < \cdots < d_m \leq d$ be given integers. Then there exists $\mathbf{x} \in \mathcal{F}(I\!R^n)$ such that $\mathbf{x}$ is compatible with $t \to t^l, l \leq d$ iff $l \in \{d_1, d_2, \dots, d_m\}$.*

Proof. Consider the linear subspace E of $\Pi_{n,d}$ generated by the union of $\cup_{k=1}^{m} F_{n,d_k}$. Choose for every $l \notin \{d_1, d_2, \dots, d_m\}$ some nonzero l-homogeneous polynomial $P_l \in \mathcal{P}(^l I\!R^n)$. Then $\operatorname{span}\{P_l\} \cap E = \{0\}$, so there exists $\mathbf{x} \in \Pi_{n,d}^*$, such that $\langle E, \mathbf{x} \rangle = 0$ and $\langle P_l, \mathbf{x} \rangle \neq 0$. It is now clear that $\mathbf{x}$ is not compatible with P_l, so it is not compatible with any homogeneous polynomial of degree l, but it is compatible with members of E. $\square$

More can be said if the points $x_0, \dots, x_{n+1}$ lie in an affine hyperplane not containing 0.

Lemma 2.19. *Let $\mathbf{x} = a_0 x_0 \boxplus \cdots \boxplus a_{n+1} x_{n+1} \in \mathcal{F}(I\!R^m)$, where $x_0, \dots, x_{n+1}$ lie in an affine hyperplane not containing 0. If every polynomial from $F_{m,d}$ is compatible with $\mathbf{x}$, then the same holds for every polynomial from $\Pi_{m,d}$.*

Proof. Let H be an affine hyperplane in $\mathbb{R}^m$ which contains $x_0, \dots, x_{n+1}$ and does not contain 0. Suppose that every polynomial from $F_{m,d}$ is compatible with $\mathbf{x}$. If $P \in \Pi_{m,d}$, then it is clear that there exists $Q \in F_{m,d}$ such that $Q\restriction_H = P\restriction_H$. Since Q is compatible with $\mathbf{x}$, we see that $\langle P, \mathbf{x} \rangle = \langle Q, \mathbf{x} \rangle = 0$. By Lemma 2.10 every polynomial from $\Pi_{m,d}$ is compatible with $\mathbf{x}$. $\square$

Theorem 2.20. *Let $\mathbf{x} = a_0 x_0 \boxplus \cdots \boxplus a_{n+1} x_{n+1} \in \mathcal{F}(I\!R^m)$, where $x_0, \dots, x_{n+1}$ lie in an affine hyperplane not containing 0. If $\sum_{k=0}^{n+1} a_k = 0$, then there exists $l \in \mathbb{N} \cup \{0\}, l \leq n$, such that if X, Y are Banach spaces and $f : X \to Y$ is a continuous mapping, then f is compatible with $\mathbf{x}$ if and only if f is a polynomial of degree at most l. If $\sum_{k=0}^{n+1} a_k \neq 0$, then there is no nonzero mapping compatible with $\mathbf{x}$.*

Proof. Since $x_0, \dots, x_{n+1}$ are pairwise linearly independent, Theorem 2.17 applies. Let $A \subset \{0, \dots, n\}$ be a set whose existence is ensured by Theorem 2.17. If $\sum_{k=0}^{n+1} a_k = 0$, then $t \mapsto 1, t \in \mathbb{R}$, is compatible with $\mathbf{x}$ and therefore A is nonempty. Let $l \in A$ be maximal. Since every polynomial from $F_{m,l}$ is compatible with $\mathbf{x}$, by Lemma 2.19 every polynomial from $\Pi_{m,l}$ is also. Hence $A = \{0, \dots, l\}$. This argument also shows that if A is nonempty, then $t \mapsto 1$ is compatible with $\mathbf{x}$, and consequently $\sum_{k=0}^{n+1} a_k = 0$. Hence if $\sum_{k=0}^{n+1} a_k \neq 0$, then there is no nonzero mapping compatible with $\mathbf{x}$. $\square$

Some information on the exact value of l can be derived from the geometrical properties of the set $x_0, \dots, x_{n+1}$. Clearly there is no lower bound on l, since to each $x_0, \dots, x_{n+1}$ we may take $a_0, \dots, a_{n+1}$ such that $\sum_{k=1}^{n+1} a_k \neq 0$, and then there is no nonzero mapping compatible with $\mathbf{x}$. Even if we demand that $\sum_{k=0}^{n+1} a_k = 0$, it is easy to find such $a_0, \dots, a_{n+1}$ so that some $P \in F_{m,1} = (\mathbb{R}^m)^*$ is not compatible with $\mathbf{x}$. Indeed, take $P \in F_{m,1}$ which is not constant on $x_0, \dots, x_{n+1}$ and then find $a_0, \dots, a_{n+1}$ such that $\sum_{k=0}^{n+1} a_k = 0$ and $\sum_{k=0}^{n+1} a_k P(x_k) \neq 0$. However, there is a simple upper bound in terms of the dimension of the affine hull of the points $x_0, \dots, x_{n+1}$. It will be given in Corollary 2.24. In the proof of Lemma 2.22 we will use the following simple fact.

Fact 2.21. *If $M \subset \mathbb{R}^2$ is a union of n distinct lines containing 0, then M is a nullspace of an n-homogeneous polynomial $P : \mathbb{R}^2 \to \mathbb{R}$. Indeed, let $P(x) =$*

$\Pi_{i=1}^n \phi_i(x)$, where $\phi \in (I\!R^2)^*$ are chosen so that their kernels coincide with the given lines.

Lemma 2.22. *Let $x_0, \ldots, x_{n+1} \in \mathbb{R}^m$ and denote $d := \dim(\mathrm{aff}(\{x_0, \ldots, x_{n+1}\}))$. Then there exists $k_0 \in \{0, \ldots, n+1\}$ and a polynomial $P : \mathbb{R}^m \to \mathbb{R}$ of degree at most $n+2-d$ such that $P(x_{k_0}) \neq 0$ and $P(x_k) = 0$ for every $k \in \{0, \ldots, n+1\} \setminus \{k_0\}$.*

Proof. We may WLOG suppose that $m = d$. The case $d = 1$ is trivial. Let $d \geq 2$. We may further suppose WLOG that $x_0, \ldots, x_{d-1}$ are affinely independent, that $M := \mathrm{aff}(\{x_0, \ldots, x_{d-1}\})$ is a hyperplane in $\mathbb{R}^d$ (i.e. it is a subspace), and that $x_{n+1} \notin M$. Using a similar argument as in the proof of Theorem 2.16 we construct a linear mapping $L : \mathbb{R}^d \to \mathbb{R}^2$ such that $L(x_0), \ldots L(x_{d-1})$ lie on a line $p \subset \mathbb{R}^2$, $L(x_{n+1}) \notin p$ and $L(x_{n+1}) \neq L(x_k)$ for all $k \neq n+1$.

Now, there exists $z \in p$ such that the line $q \subset \mathbb{R}^2$ which contains z and $L(x_{n+1})$ does not contain $L(x_k)$ for all $k \neq n+1$. Let $p_1, \ldots, p_r$ be distinct lines which contain z and some $L(x_k), k \neq n+1$. Then $r \leq n+2-d$. By Fact 2.21 (since a translation of a polynomial of degree r is again a polynomial of degree r) there exists a polynomial $Q : \mathbb{R}^2 \to \mathbb{R}$ of degree $r \leq n+2-d$ such that the nullspace of Q is $\bigcup_{i=1}^r p_i$. Then $P := Q \circ L \in \Pi_{m,n+2-d}$ is the desired polynomial. □

Proposition 2.23. *Let $\mathbf{x} = a_0 x_0 \boxplus \cdots \boxplus a_{n+1} x_{n+1} \in \mathcal{F}(I\!R^m)$, and denote $d := \dim(\mathrm{aff}(\{x_0, \ldots, x_{n+1}\}))$. If every $P \in \Pi_{m,k}$ is compatible with $\mathbf{x}$, then $k \leq n+1-d$.*

Proof. By Lemma 2.22 there exists $k_0 \in \{0, \ldots, n+1\}$ and a polynomial $P : \mathbb{R}^m \to \mathbb{R}$ of degree at most $n+2-d$ such that $P(x_{k_0}) \neq 0$ and $P(x_k) = 0$ for every $k \in \{0, \ldots, n+1\} \setminus \{k_0\}$. Then P cannot be compatible with $\mathbf{x}$, since otherwise we would have

$$0 = \langle P, \mathbf{x} \rangle = \sum_{k=0}^{n+1} a_k P(x_k) = a_{k_0} P(x_{k_0}),$$

and therefore $a_{k_0} = 0$, a contradiction. Hence $k \leq n+1-d$. □

Corollary 2.24. *Let $\mathbf{x} = a_0 x_0 \boxplus \cdots \boxplus a_{n+1} x_{n+1} \in \mathcal{F}(I\!R^m)$, where $x_0, \ldots, x_{n+1}$ lie in an affine hyperplane not containing 0 and $\sum_{k=0}^{n+1} a_k = 0$, and let l be as in Theorem 2.20. Denote $d := \dim(\mathrm{aff}(\{x_0, \ldots, x_{n+1}\}))$. Then $l \leq n+1-d$.*

For example, if in Corollary 2.24 the points $x_0, \ldots, x_{n+1}$ are affinely independent, then $d = n+1$ and therefore $l = 0$. Corollary 2.24 also shows that in order to achieve the maximal possible value of l in Theorem 2.20 (i.e. $l = n$), it is necessary that $x_0, \ldots, x_{n+1}$ be collinear; see Theorem 3.4 for more general statement.

3. Generating polynomial identities

In order to generate polynomial identities, we can use the Theorem 2.6 on the generalized Lagrange formula. In fact, the Lagrange formula is an expression of linear dependence of functionals in the dual of $\Pi_{m,d}$. Let $\{x_k\}_{k=1}^N \subset \mathbb{R}^m$ be a basic set of nodes for $\Pi_{m,d}$ and let $\{h_k\}_{k=1}^N \subset \Pi_{m,d}$ be its dual basis. Given $z \in I\!R^m \setminus \{x_k\}_{k=1}^N$ there exist a unique set of coefficients $a_k = a_k(z) \in I\!R$ such that

$$P(z) = \sum_{k=1}^N a_k(z) P(x_k) \text{ for every } P \in \Pi_{m,d}.$$

Then every $P \in \Pi_{m,d}$ is compatible with $a_1(z)x_1 \boxplus \cdots \boxplus a_N(z)x_N \boxplus (-1)z$.

Lemma 3.1. *Let $\{x_k\}_{k=1}^N \subset \mathbb{R}^m$ be a basic set of nodes for $\Pi_{m,d}$, $z \in I\!R^m \setminus \{x_k\}_{k=1}^N$, and let $\mathbf{x} = a_1(z)x_1 \boxplus \cdots \boxplus a_N(z)x_N \boxplus (-1)z$. If every $P \in \Pi_{m,l}$ is compatible with $\mathbf{x}$, then $l \leq d$.*

Proof. Assume WLOG that $a_1(z) \neq 0$. Considering the dual basis of $\{x_k\}_{k=1}^N$ we see that there exists $Q \in \Pi_{m,d}$ such that $Q(x_1) \neq 0$ and $Q(x_k) = 0$ for $k = 2, \dots, N$. Further, it is clear that there exists $R \in \Pi_{m,1}$ (these are the affine functions on $\mathbb{R}^m$) such that $R(x_1) \neq 0$ and $R(z) = 0$. Then clearly $P = QR \in \Pi_{m,d+1}$, $P(x_1) \neq 0$, $P(x_k) = 0$ for $k = 2, \dots, N$ and $P(z) = 0$. But then $\langle P, \mathbf{x} \rangle = a_1 P(x_1) \neq 0$, hence P is not compatible with $\mathbf{x}$, and therefore $l \leq d$. $\square$

The following theorem describes a method of generating polynomial identities which, for prescribed d, characterize polynomials of degree at most d.

Theorem 3.2. *Let $\{x_k\}_{k=1}^N \subset \mathbb{R}^m$ be a basic set of nodes for $\Pi_{m,d}$, $z \in \mathbb{R}^m \setminus \{x_k\}_{k=1}^N$, and let $\mathbf{x} = a_1(z)x_1 \boxplus \cdots \boxplus a_N(z)x_N \boxplus (-1)z$. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be an affine one-to-one mapping such that $0 \notin T(\mathbb{R}^m)$. Then $a_1 = a_1(z), \dots, a_N = a_N(z)$ are the unique coefficients with the following property. Denote $\mathbf{y} = a_1 T(x_1) \boxplus \cdots \boxplus a_N T(x_N) \boxplus (-1)T(z)$. If X, Y are Banach spaces and $f : X \to Y$ is continuous, then f is compatible with $\mathbf{y}$ if and only if f is a polynomial of degree at most d.*

Proof. Since $T(x_1), \dots, T(x_N), T(z)$ lie in an affine hyperplane not containing 0, Theorem 2.20 applies. It follows that it suffices to prove the theorem for $X = \mathbb{R}^n$ and $Y = \mathbb{R}$, and it also follows that the space of those continuous $f : \mathbb{R}^n \to \mathbb{R}$ which are compatible with $\mathbf{y}$ is $\Pi_{n,l}$ for some $l \in \mathbb{N} \cup \{0\}$ or a trivial space. If $P \in \Pi_{n,d}$, then $P \circ T \in \Pi_{m,d}$, so $P \circ T$ is compatible with $\mathbf{x}$, and therefore P is compatible with $\mathbf{y}$. Hence the space of compatible functions is nontrivial and $l \geq d$. Theorem 2.6 also yields the uniqueness part. On the other hand, if $P \in \Pi_{m,l}$, then $P \circ T^{-1} : T(\mathbb{R}^m) \to \mathbb{R}$ can be extended to a member of $\Pi_{n,l}$, which is compatible with $\mathbf{y}$ by the definition of l. It follows from Lemma 2.10 that every polynomial from $\Pi_{m,l}$ is compatible with $\mathbf{x}$. By Lemma 3.1 we conclude that $l \leq d$. $\square$

A special case of Theorem 3.2 in dimension one corresponds to the classical the Lagrange interpolation polynomial.

Theorem 3.3. *(Classical Lagrange interpolation) Let $x_0, \dots, x_{n+1} \in \mathbb{R}$ be distinct. Then there exist a unique set of coefficients $a_0, \dots, a_n \in \mathbb{R} \setminus \{0\}$, such that every $P \in \Pi_{1,n}$ is compatible with $a_0 x_0 \boxplus \cdots \boxplus a_n x_n \boxplus (-1)x_{n+1}$. Moreover*

$$a_k := \prod_{i=0, i\neq k}^{n} \frac{x_{n+1} - x_i}{x_k - x_i}, \; k = 0, \dots, n.$$

The following theorem is a generalization of the equivalence of the conditions (i) and (iv) in Theorem 1.1.

Theorem 3.4. *Let $x_0, \dots, x_{n+1} \in \mathbb{R}^m, m \geq 2, n \in \mathbb{N}$, be distinct points. Then the following statements are equivalent:*

(i) The points $x_0, \dots, x_{n+1}$ lie on a line not containing 0.
(ii) There exist $a_0, \dots a_n \in \mathbb{R} \setminus \{0\}$ such that if X, Y be Banach spaces and $f : X \to Y$ is a continuous mapping, then f is compatible with $\mathbf{x} = a_0 x_0 \boxplus \cdots \boxplus a_n x_n \boxplus (-1)x_{n+1}$ if and only if f is a polynomial of degree at most n.

Moreover the coefficients $a_0, \dots, a_n$ from (ii) are uniquely determined, and if $T : \mathbb{R} \to \mathbb{R}^m$ is an affine one-to-one map and $y_k \in \mathbb{R}, k = 0, \dots, n+1$, are such that $T(y_k) = x_k$, then

$$a_k = \prod_{i=0, i\neq k}^{n} \frac{y_{n+1} - y_i}{y_k - y_i}, \; k = 0, \dots, n.$$

Proof. (i)⇒(ii): Suppose that (i) holds. Since $x_0, \dots, x_{n+1}$ lie on a line not containing 0, there exists an affine one-to-one map $T : \mathbb{R} \to \mathbb{R}^m$ such that there exist $y_k \in \mathbb{R}$ such that $T(y_k) = x_k$ and $0 \notin T(\mathbb{R})$. Combining Theorem 3.2 Theorem 3.3 gives (ii) and also the moreover part.

(ii)⇒(i): Denote $d := \dim(\text{aff}(\{x_0, \dots, x_{n+1}\}))$. It follows from Proposition 2.23 that $n \le n + 1 - d$, and therefore $x_0, \dots, x_{n+1}$ are collinear.

Suppose by contradiction that $x_0, \dots, x_{n+1}$ lie on a line containing 0. It is easy to construct a continuous function $f : \mathbb{R}^m \to \mathbb{R}$ which is not a polynomial but it is linear on every one dimensional subspace of $\mathbb{R}^m$. Let $L \in \mathcal{L}(\mathbb{R}^m)$. As $x_0, \dots, x_{n+1}$ lie in a one dimensional subspace, the same holds for $L(x_0), \dots, L(x_{n+1})$. Hence there exists $P \in \Pi_{m,1}$ such that $P(L(x_k)) = f(L(x_k))$ for all k. Since P is compatible with $\mathbf{x}$, we obtain $0 = \langle P \circ L, \mathbf{x} \rangle = \langle f \circ L, \mathbf{x} \rangle$. Hence f is compatible with $\mathbf{x}$. But this is a contradiction, since f is not a polynomial. □

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