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Transmission problem for the Laplace equation and the integral equation method *

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Abstract

We shall study a weak solution in the Sobolev space of the transmission problem for the Laplace equation using the integral equation method. First we use the indirect integral equation method. We look for a solution in the form of the sum of the double layer potential corresponding to the skip of traces on the interface and a single layer potential with an unknown density. We get an integral equation on the boundary. We prove that this equation has a form $(I + M)\varphi = F$ where M is a contractive operator. So, we can obtain a solution of this equation using the successive approximation method. Moreover, we are able to estimate the norm of the operator M and control how quickly this process converges. Then we study the direct integral equation method. We obtain the same integral equation like for the indirect integral equation method. So, we can again calculate a solution using the successive approximation method.

Keywords: single layer potential; double layer potential ; transmission problem; Laplace equation; boundary integral equation; successive approximation

1 Introduction

In this paper we shall study a weak solution of the transmission problem for the Laplace equation

$$\Delta u_+ = 0 \quad \text{in } G_+, \qquad \Delta u_- = 0 \quad \text{in } G_-,$$

$$u_+ - u_- = g, \quad a_+ \frac{\partial u_+}{\partial n} - a_- \frac{\partial u_-}{\partial n} = f \quad \text{on } \partial G_+$$

using the integral equation method. Here $G_+ \subset R^m$, $m > 2$, is a bounded open set with Lipschitz boundary, $G_- = R^m \setminus \overline{G}_+$, a_+ and a_- are given positive constants. (We do not suppose that G_+ or G_- is connected.)

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This problem was studied for $a_+ = a_- = 1$ and G_+, G_- connected in [4]. It was shown that there is unique solution of this problem and $u_+ = \mathcal{D}g + \mathcal{S}f$ in G_+, $u_- = \mathcal{D}g + \mathcal{S}f$ in G_-, where $\mathcal{D}g$ is the double layer potential with density g and $\mathcal{S}f$ is the single layer potential with density f.

For arbitrary a_+, a_- and $f \in L^2(\partial G_+)$, $g = 0$ the transmission problem was studied in [2]. It was shown that a solution of the problem has a form of a single layer potential. [3] studies the transmission problem with arbitrary a_+, a_- and $f \in L^2(\partial G_+)$, $g \in W^{1,2}(\partial G_+)$. A solution is looked for in the form of two single layer potentials: $u_+ = \mathcal{S}\varphi_+$, $u_- = \mathcal{S}\varphi_-$. This attitude is good for the proof of the existence of a solution but it does not tell us how to compute this solution.

We study a weak solution of the transmission problem with a_+, a_- arbitrary and $g \in H^{1/2}(\partial G_+)$, $f \in H^{-1/2}(\partial G_+)$. If we put $u_+ = v_+ + \mathcal{D}g$, $u_- = v_- + \mathcal{D}g$, we get the new problem $\Delta v_+ = 0$, $\Delta v_- = 0$, $v_+ - v_- = 0$ on ∂G_+, $a_+ \partial v_+/\partial n - a_- \partial v_-/\partial n = F$. We look for a solution of this problem in the form of a single layer potential $\mathcal{S}\varphi$ with an unknown density $\varphi \in H^{-1/2}(\partial G_+)$ and reduce the original problem to the integral equation $T\varphi = F$. The same reasoning one can do in more general setting (for G_+, which has not Lipschitz boundary, or in the case of the transmission problem for several media). From these reasons we study the transmission problem for a system of nonoverlapping open sets G_j and positive constants a_j such that the complement of $\cup G_j$ has zero Lebesque measure and $\lambda \equiv \inf a_j > 0$, $\Lambda \equiv \sup a_j < \infty$. It is shown that the problem is uniquely solvable and the solution has a form of a single layer potential $\mathcal{S}\varphi$. Again, we reduce the problem to the equation $T\varphi = F$ on the boundary. This equation is equivalent to the equation $\varphi = [I - 2(\lambda + \Lambda)^{-1}T]\varphi + 2(\lambda + \Lambda)^{-1}F$, where I is the identity operator. Is is shown that $\|I - 2(\lambda + \Lambda)^{-1}T\| \leq (\Lambda - \lambda)/(\Lambda + \lambda) < 1$ and the successive approximation method converges: For a fixed φ_0

$$\varphi_n = [I - 2(\lambda + \Lambda)^{-1}T]\varphi_n + 2(\lambda + \Lambda)^{-1}F, \qquad \varphi_n \to \varphi. \tag{1}$$

Then it is studied the direct integral equation method for the original problem. We have $u_+ = \mathcal{D}g + \mathcal{S}(\partial u_+/\partial n - \partial u_-/\partial n)$ in G_+, $u_- = \mathcal{D}g + \mathcal{S}(\partial u_+/\partial n - \partial u_-/\partial n)$ in G_-. It is shown that $T(\partial u_+/\partial n - \partial u_-/\partial n) = F$, where T and F are the same as in the indirect integral equation method. Therefore we can use the successive approximation method (1).

2 Formulation of the problem

Let $G = G_+ \subset R^m$, $m > 2$, be a bounded open set with Lipschitz boundary. Denote $G_- = R^m \setminus \overline{G}_+$ where $\overline{G}_+$ is the closure of G_+. Denote by n the outward unit normal of G_+. We shall study the transmission problem

$$\Delta u_+ = 0 \quad \text{in } G_+, \qquad \Delta u_- = 0 \quad \text{in } G_-, \tag{2}$$

$$u_+ - u_- = g, \quad a_+ \frac{\partial u_+}{\partial n} - a_- \frac{\partial u_-}{\partial n} = f \quad \text{on } \partial G. \tag{3}$$

Here a_+, a_- are fixed positive constants and $g \in H^{1/2}(\partial G_+)$, $f \in H^{-1/2}(\partial G_+)$.

If $\Omega \subset R^m$ is an open set denote by $W^{1,2}(\Omega)$ the space of all functions $u \in L^2(\Omega)$ such that $\partial_j u \in L^2(\Omega)$ in the sense of distributions for each $j = 1, \ldots, m$ equipped with the norm

$$\|u\|_{W^{1,2}(\Omega)} = \sqrt{\int_\Omega [|u|^2 + |\nabla u|^2] \, d\mathcal{H}_m}.$$

(Here $\mathcal{H}_k$ is the k-dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesgue measure in R^k.)

If Ω is a bounded open set with Lipschitz boundary denote by $H^{1/2}(\partial\Omega)$ the space of traces of $W^{1,2}(\Omega)$ endowed with the norm

$$\|v\|_{H^{1/2}(\partial\Omega)} = \inf\{\|u\|_{W^{1,2}(\Omega)}; u \in W^{1,2}(\Omega), v = u|\partial\Omega\}$$

and by $H^{-1/2}(\partial\Omega)$ the dual space of $H^{1/2}(\partial\Omega)$. If $h \in H^{-1/2}(\Omega)$ then the Neumann problem for the Laplace equation

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial n} = h \quad \text{on } \partial\Omega \tag{4}$$

has a weak formulation: Find $u \in W^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = \langle h, \varphi \rangle \tag{5}$$

for all $\varphi \in W^{1,2}(\Omega)$.

If Ω is an unbounded open set with compact Lipschitz boundary then we can define weak solutions of the Neumann problem for the Laplace equation in the same way. But the condition $u \in W^{1,2}(\Omega)$ is too restrictive for unbounded open sets. There are too many boundary conditions for which the problem is not solvable. So, we use a bit wider space of functions.

If $\Omega \subset R^m$ is a domain denote by $L^{1,2}(\Omega)$ the space of all functions $u \in L^2_{loc}(\Omega)$ such that $\partial_j u \in L^2(\Omega)$ in the sense of distributions for each $j = 1, \ldots, m$. Fix a bounded open set U such that $\overline{U} \subset \Omega$. Then $L^{1,2}(\Omega)$ is a Banach space with the norm

$$\|u\|_{L^{1,2}(\Omega)} = \sqrt{\int_U |u|^2 \, d\mathcal{H}_m + \int_\Omega |\nabla u|^2 \, d\mathcal{H}_m}$$

(see [10], § 1.5.3). Clearly, $W^{1,2}(\Omega) \subset L^{1,2}(\Omega)$. If Ω is a bounded domain with Lipschitz boundary then $W^{1,2}(\Omega) = L^{1,2}(\Omega)$ and both norms are equivalent (see [10], §1.5.2 and [10], §1.5.3). If Ω is an unbounded domain with compact Lipschitz boundary and $u \in L^{1,2}(\Omega)$, then $u \in W^{1,2}(V)$ for each bounded open subset V of Ω. If Ω is an unbounded domain with compact Lipschitz boundary and $h \in H^{-1/2}(\partial\Omega)$, we can define a weak solution of the Neumann problem for the Laplace equation (4) such that we look for $u \in L^{1,2}(\Omega)$ satisfying (5) for all $\varphi \in L^{1,2}(\Omega)$ (compare [11]). Now, we can look for $u_+ \in L^{1,2}(G_+)$, $u_- \in L^{1,2}(G_-)$ solving the problem (2), (3). But the space $L^{1,2}(G_-)$ is too wide. The constant function $u_+ \equiv 1$, $u_- \equiv 1$ is a solution of (2), (3) with the homogeneous boundary conditions $f = g = 0$. To get a uniquely solvable problem we choose another space of functions, which is between $W^{1,2}(\Omega)$ and $L^{1,2}(\Omega)$.

If Ω is an open set denote by $\mathcal{C}_0^\infty(\Omega)$ the space of all infinitely differentiable functions in Ω with compact support. Denote by $\tilde{W}^{1,2}(R^m)$ the closure of $\mathcal{C}_0^\infty(R^m)$ in $L^{1,2}(R^m)$. Then $W^{1,2}(R^m) \subset \tilde{W}^{1,2}(R^m) \subset L^{1,2}(R^m)$ (compare [14], Lemma 6.5). Moreover, the space $L^{1,2}(R^m)$ is the direct sum of $\tilde{W}^{1,2}(R^m)$ and the space of constant functions (see [1], p. 155). If we put

$$\|u\|_{\tilde{W}^{1,2}(R^m)} = \|\nabla u\|_{L^2(R^m)},$$

then this norm is in $\tilde{W}^{1,2}(R^m)$ equivalent with the norm induced from $L^{1,2}(R^m)$ (see [10], §1.5.2 and [10], §1.5.3). According to [6], Lemma 2.2 we have $\tilde{W}^{1,2}(R^m) = \{u \in L^{2m/(m-2)}(R^m); \nabla u \in L^2(R^m; R^m)\}$. For an open set Ω denote by $\tilde{W}^{1,2}(\Omega)$ the space of restrictions of functions from $\tilde{W}^{1,2}(R^m)$ onto Ω. Denote

$$\|u\|_{\tilde{W}^{1,2}(\Omega)} = \inf\{\|v\|_{\tilde{W}^{1,2}(\Omega)}; v = u \text{ on } \Omega\}.$$

Then $\tilde{W}^{1,2}(\Omega)$ is a Banach space. If $u \in \tilde{W}^{1,2}(\Omega)$ then $u \in W^{1,2}(V)$ for every bounded open subset V of Ω. If Ω is a bounded open set with Lipschitz boundary then $\tilde{W}^{1,2}(\Omega) = W^{1,2}(\Omega)$ and both norms are equivalent. If Ω is an unbounded domain with compact Lipschitz boundary then $\|\nabla u\|_{L^2(\Omega)}$ is an equivalent norm in $\tilde{W}^{1,2}(\Omega)$.

We now give a weak formulation of the transmission problem for the Laplace equation (2), (3). We must realize that $-n$ is the unit outward normal of G_-.

We say that $u_+ \in \tilde{W}^{1,2}(G_+)$, $u_- \in \tilde{W}^{1,2}(D_-)$ is a weak solution of the transmission problem for the Laplace equation (2), (3) if $u_+ - u_- = g$ on ∂G_+ in the sense of traces and

$$a_+ \int\limits_{G_+} \nabla u_+ \cdot \nabla v \, \mathrm{d}\mathcal{H}_m + a_- \int\limits_{G_-} \nabla u_- \cdot \nabla v \, \mathrm{d}\mathcal{H}_m = \langle f, v \rangle \quad \forall v \in \tilde{W}^{1,2}(R^m).$$

3 Representation by potentials

Denote by

$$h_m(x) = \frac{|x|^{2-m}}{(m-2)\mathcal{H}_{m-1}(\partial B(0;1))}$$

the fundamental solution of the equation $-\Delta u = 0$ in R^m. If $\Omega \subset R^m$ is a bounded open set with Lipschitz boundary, $f \in H^{-1/2}(\partial\Omega)$, $g \in H^{1/2}(\partial\Omega)$, define

$$\mathcal{S}^\Omega f(x) = \int_{\partial\Omega} h_m(x-y) f(y) \, d\mathcal{H}_m(y)$$

the single layer potential with density f and

$$\mathcal{D}^\Omega g(x) = \int_{\partial G} g(y) \frac{n(y)\cdot(y-x)}{\mathcal{H}_{m-1}(\partial B(0;1))|x-y|^m} \, d\mathcal{H}_m(y)$$

the double layer potential with density g. Then $\mathcal{S}^\Omega f$, $\mathcal{D}^\Omega g$ are harmonic functions in $R^m \setminus \partial\Omega$. Moreover, $\mathcal{S}^\Omega f, \mathcal{D}^\Omega g \in W^{1,2}(\Omega)$, $\mathcal{S}^\Omega f, \mathcal{D}^\Omega g \in W^{1,2}(B(0;r)\setminus\overline{\Omega})$ for each $r > 0$ (see for example [7], Theorem 4.1 and [9], Theorem 2.4). Here $B(x;r)$ denotes the open ball with the center x and the radius r. Since $\mathcal{S}^\Omega f(x) = O(|x|^{2-m})$, $\mathcal{D}^\Omega g(x) = O(|x|^{1-m})$, $|\nabla \mathcal{S}^\Omega f(x)| = O(|x|^{1-m})$, $|\nabla \mathcal{D}^\Omega g(x)| = O(|x|^{-m})$ as $|x| \to \infty$, we infer that $\mathcal{S}^\Omega f, \mathcal{D}^\Omega g \in L^{1,2}(R^m \setminus \overline{\Omega})$. Denote by u_+^Ω the trace of $u \in L^{1,2}(\Omega)$ and by u_-^Ω the trace of $u \in L^{1,2}(R^m \setminus \overline{\Omega})$ on $\partial\Omega$. Then $[\mathcal{S}^\Omega f]_+^\Omega = [\mathcal{S}^\Omega f]_-^\Omega = \mathcal{S}^\Omega f$. Denote

$$K_\Omega g(x) = \lim_{\epsilon \searrow 0} \int_{\partial G \setminus B(x;\epsilon)} g(y) \frac{n(y)\cdot(y-x)}{\mathcal{H}_{m-1}(B(0;1))|x-y|^m} \, d\mathcal{H}_m(y)$$

for $x \in \partial\Omega$. Then $K_\Omega g(x)$ makes sense for almost all $x \in \partial\Omega$ (see [5], Theorem 2.2.13) and K_Ω is a bounded linear operator on $H^{1/2}(\partial\Omega)$ (see [13], Theorem 4.1). Moreover

$$[\mathcal{D}^\Omega g]_+ = \frac{1}{2} g + K_\Omega g, \quad [\mathcal{D}^\Omega g]_- = -\frac{1}{2} g + K_\Omega g \quad \text{on } \partial\Omega. \tag{6}$$

(See for example [5], Theorem 2.2.13.) Denote by K_Ω^* the adjoint operator of K_Ω. Recall that

$$K_\Omega^* g(y) = \lim_{\epsilon \searrow 0} \int_{\partial G \setminus B(y;\epsilon)} g(x) \frac{n(y)\cdot(y-x)}{\mathcal{H}_{m-1}(B(0;1))|x-y|^m} \, d\mathcal{H}_m(x).$$

Then K_Ω^* is a bounded linear operator on $H^{-1/2}(\partial\Omega)$ (see [13], Theorem 4.1). If $\varphi \in \mathcal{C}_0^\infty(R^m)$ then

$$\langle [n^\Omega \cdot \nabla \mathcal{S}^\Omega f]_+, \varphi \rangle = \int_\Omega \nabla \mathcal{S}^\Omega f \cdot \nabla \varphi \, \mathrm{d}\mathcal{H}_m = \langle (1/2) f - K_\Omega^* f, \varphi \rangle, \tag{7}$$

$$\langle [n^{R^m\setminus\overline{\Omega}}\cdot\nabla\mathcal{S}^\Omega f]_-,\varphi\rangle = \int\limits_{R^m\setminus\overline{\Omega}} \nabla\mathcal{S}^\Omega f\cdot\nabla\varphi \,\mathrm{d}\mathcal{H}_m = \langle (1/2)f + K_\Omega^* f,\varphi\rangle \tag{8}$$

(compare [12], [5] or [4]). Since $\mathcal{C}_0^\infty(R^m)$ is a dense subspace of $\tilde{W}^{1,2}(R^m)$, the relations (7), (8) hold for $\varphi\in\tilde{W}^{1,2}(R^m)$.

Let us come back to the transmission problem (2), (3). Put $u_+ = v_+ + \mathcal{D}^G g$, $u_- = v_- + \mathcal{D}^G g$. Then u_+, u_- is a weak solution of the problem (2), (3) if and only if v_+, v_- is a weak solution of the problem

$$\Delta v_+ = 0 \quad \text{in } G^+, \qquad \Delta v_- = 0 \quad \text{in } G_-, \tag{9}$$

$$v_+ - v_- = 0, \quad a_+\frac{\partial v_+}{\partial n} - a_-\frac{\partial v_-}{\partial n} = F \quad \text{on } \partial G_+, \tag{10}$$

where $F = f - a_+[\mathcal{D}^G g/\partial n]_+ + a_-[\mathcal{D}^G g/\partial n]_-$. (Remark that in fact $[\mathcal{D}^G g/\partial n]_+ = [\mathcal{D}^G g/\partial n]_-$.) Since $v_+ = v_-$ on ∂G, the function $v = v_+$ on G_+, $v = v_-$ on G_- must be in $\tilde{W}^{1,2}(R^m)$. We shall look for a solution of the problem (9), (10) in the form of the single layer potential $v = \mathcal{S}^\Omega\varphi$ with an unknown density $\varphi\in H^{-1/2}(\partial\Omega)$. Boundary behavior of a single layer potentials gives that $\mathcal{S}^\Omega\varphi$ is a solution of the problem if

$$a_+\left(\frac{1}{2}\varphi - K_G^*\varphi\right) + a_-\left(\frac{1}{2}\varphi + K_G^*\varphi\right) = F. \tag{11}$$

4 More general problem

The same reasoning we can do in a more general situation - the transmission problem is studied for several domains $G_1,\dots,G_k$ or a set G has not Lipschitz boundary. So, instead of the problem (10) we shall study a more general problem.

Denote

$$\mathcal{E}(R^m) = \{\Delta u; u\in\tilde{W}^{1,2}(R^m)\}.$$

The space $\mathcal{E}(R^m)$ is the so called space of distributions with finite energy. Remark that $\mathcal{E}(R^m) = (\tilde{W}^{1,2}(R^m))'$, the dual space of $\tilde{W}^{1,2}(R^m)$. There is a characterization of the space $\mathcal{E}(R^m)$ using the Fourier transformation: If F is a distribution and $\hat{F}$ its Fourier transformation then $F\in\mathcal{E}(R^m)$ if and only if

$$\sqrt{\int \frac{|\hat{F}(x)|^2}{|x|^2}\,\mathrm{d}\mathcal{H}_m} < \infty. \tag{12}$$

Remark that the expression in (12) gives an equivalent norm on $\mathcal{E}(R^m) = (\tilde{W}^{1,2}(R^m))'$. (See [1] ,[8].) If M is a closed subset of R^m denote by $\mathcal{E}(M)$ the space of all distributions from $\mathcal{E}(R^m)$ supported in M. Then $\mathcal{E}(M)$ is a closed subspace of $\mathcal{E}(R^m)$. If Ω is a bounded open set with Lipschitz boundary then $\mathcal{E}(\overline{\Omega}) = (W^{1,2}(\Omega))'$ (see [12], Remark 7.10) and $\mathcal{E}(\partial\Omega) = H^{-1/2}(\partial\Omega)$ (see [12], Remark 7.11).

We shall study the following generalization of the problem (9), (10):

Let $\Omega \subset R^m$ be an open set with $\mathcal{H}_m(R^m \setminus \Omega) = 0$. Let a function a be constant on each component of Ω and

$$0 < \lambda = \inf_{x\in\Omega} a(x) \le \sup_{x\in\Omega} a(x) = \Lambda < \infty. \tag{13}$$

For a given $F \in \mathcal{E}(\partial\Omega)$ find $v \in \tilde{W}^{1,2}(R^m)$ such that

$$\int\limits_{\Omega} a\nabla v \cdot \nabla w \; \mathrm{d}\mathcal{H}_m = \langle F, w\rangle \qquad \forall w \in \tilde{W}^{1,2}(R^m). \tag{14}$$

Proposition 4.1. *Let $\Omega \subset R^m$ be an open set with $\mathcal{H}_m(\partial\Omega) = 0$, a function a be constant on each component of Ω, a satisfy (13). If $F \in \mathcal{E}(\partial\Omega)$ then there exists unique solution $v \in \tilde{W}^{1,2}(R^m)$ of the generalized transmission problem (14).*

Proof. Put

$$(v, w) = \int\limits_{\Omega} a\nabla v \cdot \nabla w \; \mathrm{d}\mathcal{H}_m.$$

Since

$$\lambda \|v\|^2_{\tilde{W}^{1,2}(R^m)} \le (v, v) \le \Lambda \|v\|^2_{\tilde{W}^{1,2}(R^m)},$$

$(\ ,\ )$ is an inner product which gives a norm equivalent to the norm in $\tilde{W}^{1,2}(R^m)$. Riesz representation theorem gives that there exists unique $u \in \tilde{W}^{1,2}(R^m)$ such that (14) holds for each $w \in \tilde{W}^{1,2}(R^m)$.

5 The indirect integral equation method

For $\varphi \in \mathcal{C}_c^\infty(R^m)$ denote

$$V\varphi(x) = \int\limits_{R^m} h_m(x-y)\varphi(y) \; \mathrm{d}\mathcal{H}_m(y)$$

the Newton potential (or the volume potential) with density φ. The operator $V : \varphi \mapsto V\varphi$ can be extended as a bounded operator from $\mathcal{E}(R^m)$ onto $\tilde{W}^{1,2}(R^m)$. Since $\mathcal{C}_c^\infty(R^m)$ is a dense subset of $\mathcal{E}(R^m)$, this extension is unique. Moreover, $\varphi = V(-\Delta\varphi)$ and V is an isomorphism. (See [8].) If U is a bounded domain with Lipschitz boundary and $\varphi \in \mathcal{E}(\partial U) = H^{-1/2}(\partial U)$ then $V\varphi = \mathcal{S}^U\varphi$, the single layer potential with density φ. If $\varphi \in \mathcal{E}(R^m)$ and U is an open set, then $\Delta V\varphi = 0$ in U if and only if $\varphi \in \mathcal{E}(R^m \setminus U)$, i.e. φ is supported on $R^m \setminus U$.

If $\varphi, \psi \in \mathcal{E}(R^m) = (\tilde{W}^{1,2}(R^m)'$ then $V\psi \in \tilde{W}^{1,2}(R^m)$ and

$$\langle \varphi, V\psi \rangle = (\varphi, \psi)_{\mathcal{E}} = \int_{R^m} \frac{\hat{\varphi}(x)\hat{\psi}(x)}{|x|^2} \, \mathrm{d}\mathcal{H}_m(x) = \int_{R^m} \nabla V\varphi \cdot \nabla V\psi \, \mathrm{d}\mathcal{H}_m. \tag{15}$$

(See [1], [8] and [11], Lemma 5.1.) This gives

$$\|f\|_{\mathcal{E}(R^m)} = \|Vf\|_{\tilde{W}^{1,2}(R^m)} \quad \forall f \in \mathcal{E}(R^m), \tag{16}$$

$$\|w\|_{\tilde{W}^{1,2}(R^m)} = \|\Delta w\|_{\mathcal{E}(R^m)} \quad \forall w \in \tilde{W}^{1,2}(R^m). \tag{17}$$

Let us come back to the generalized transmission problem. Let $\Omega \subset R^m$ be an open set with $\mathcal{H}_m(\partial\Omega) = 0$, a function a be constant on each component of Ω, a satisfy (13). If $F \in \mathcal{E}(\partial\Omega)$ then there exists unique solution $v \in \tilde{W}^{1,2}(R^m)$ of the generalized transmission problem (14). Since $v = V(-\Delta v)$ and $-\Delta v \in \mathcal{E}(R^m)$, we can look for a solution v in the form $V\psi$ with $\psi \in \mathcal{E}(R^m)$. Since $\Delta v = 0$ in Ω, $\psi (= -\Delta v)$ is supported on $R^m \setminus \Omega = \partial\Omega$. Thus $\psi \in \mathcal{E}(\partial\Omega)$, because $\partial\Omega = R^m \setminus \Omega$.

Fix $\psi \in \mathcal{E}(\partial\Omega)$. Define a linear functional $T\psi$ on $\tilde{W}^{1,2}(R^m)$ by

$$\langle T\psi, w \rangle = \int_{R^m} a\nabla w \cdot \nabla V\psi \, \mathrm{d}\mathcal{H}_m, \quad w \in \tilde{W}^{1,2}(R^m). \tag{18}$$

According to (13), (16), (17) and Hölder's inequality

$$|\langle T\psi, w \rangle| \le \Lambda \|\nabla w\|_{L^2(R^m)} \|\nabla V\psi\|_{L^2(R^m)} \le \Lambda \|w\|_{\tilde{W}^{1,2}(R^m)} \|\psi\|_{\mathcal{E}(R^m)}.$$

This gives that $T\psi \in (\tilde{W}^{1,2}(R^m))' = \mathcal{E}(R^m)$ and $\|T\psi\|_{\mathcal{E}(R^m)} \le \Lambda \|\psi\|_{\mathcal{E}(R^m)}$. Fix a component ω of Ω. Then there exists a constant a_ω such that $a = a_\omega$ on ω. Since $\Delta V\psi = 0$ in ω we have for $w \in \mathcal{C}_c^\infty(\omega)$

$$\langle T\psi, w \rangle = \int_{R^m} a\nabla w \cdot \nabla V\psi \, \mathrm{d}\mathcal{H}_m = a_\omega \int_{\omega} \nabla w \cdot \nabla V\psi \, \mathrm{d}\mathcal{H}_m$$

$$= a_\omega \int_{R^m} \nabla w \cdot \nabla V\psi \, \mathrm{d}\mathcal{H}_m = \langle \psi, w \rangle = 0.$$

Hence $T\psi$ is supported in $R^m \setminus \omega$. We infer that $T\psi \in \mathcal{E}(\partial\Omega)$. The operator T is a bounded linear operator on $\mathcal{E}(\partial\Omega)$ with $\|T\| \le \Lambda$.

If $F \in \mathcal{E}(\partial\Omega)$, then $v = V\psi$ is a weak solution of the generalized transmission problem (14) if $T\psi = F$. In particular, in the case of the original problem (9), (10) we have $T\psi = (a_+ + a_-)\psi/2 - (a_+ - a_-)K_G^*\psi$ (see (11)).

We would like to solve the equation $T\psi = F$ using the successive approximation method. To this aim we shall rewrite this equation to the equation

$$\psi = \left(I - \frac{2}{\lambda + \Lambda} T\right)\psi + \frac{2}{\lambda + \Lambda} F,$$

where I is the identity operator.

Theorem 5.1. *Let $\Omega \subset R^m$ be an open set with $\mathcal{H}_m(\partial\Omega) = 0$, a function a be constant on each component of Ω, a satisfy (13). Then*

$$\|I - 2(\lambda+\Lambda)^{-1}T\| \leq \frac{\Lambda - \lambda}{\Lambda + \lambda} < 1. \tag{19}$$

Fix $F, \psi_0 \in \mathcal{E}(\partial\Omega)$. Put

$$\psi_n = \left(I - \frac{2}{\lambda+\Lambda} T\right)\psi_{n-1} + \frac{2}{\lambda+\Lambda} F, \quad n \in N.$$

Then there is $\psi \in \mathcal{E}(\partial\Omega)$ such that $\psi_n \to \psi$ as $n \to \infty$,

$$\|\psi - \psi_n\| \leq \left(\frac{\Lambda - \lambda}{\Lambda + \lambda}\right)^n \frac{\Lambda\|\psi_0\| + \|F\|}{\lambda}. \tag{20}$$

$T\psi = F$ and the single layer potential $V\psi$ is unique solution to the generalized transmission problem (14).

Proof. Since $\lambda \leq a \leq \Lambda$ we have $\lambda - \Lambda \leq \lambda + \Lambda - 2a \leq \Lambda - \lambda$ and thus $|1 - 2a/(\lambda + \Lambda)| \leq (\Lambda - \lambda)/(\Lambda + \lambda)$ in Ω. If $\psi \in \mathcal{E}(\partial\Omega)$ then (15) and Schwarz's inequality give

$$\|[I - 2(\lambda+\Lambda)^{-1}T]\psi\| = \sup_{\|w\|_{\tilde{W}^{1,2}(R^m)} \leq 1} \langle [I - 2(\lambda+\Lambda)^{-1}T]\psi, w\rangle$$

$$= \sup_{\|w\|_{\tilde{W}^{1,2}(R^m)}} \int_{R^m} [\nabla w \cdot \nabla V\psi - 2(\lambda+\Lambda)^{-1} a \nabla w \cdot \nabla V\psi] \, \mathrm{d}\mathcal{H}_m \leq$$

$$\sup_{\|\nabla w\| \leq 1} \int_{R^m} \frac{\Lambda - \lambda}{\Lambda + \lambda} |\nabla w||\nabla V\psi| \, \mathrm{d}\mathcal{H}_m \leq \frac{\Lambda - \lambda}{\Lambda + \lambda} \|\nabla V\psi\|_{L^2(R^m)} = \frac{\Lambda - \lambda}{\Lambda + \lambda} \|\psi\|_{\mathcal{E}(R^m)}.$$

Hence (19) holds.

$$\|\psi_1 - \psi_0\| = \frac{2}{\lambda+\Lambda} \|T\psi_0 + F\| \leq \frac{2}{\lambda+\Lambda} [\Lambda\|\psi_0\| + \|F\|].$$

By the induction

$$\|\psi_{n+1} - \psi_n\| = \|[I - 2(\lambda+\Lambda)^{-1}T](\psi_n - \psi_{n-1})\|$$

$$\leq \frac{\Lambda - \lambda}{\Lambda + \lambda} \|(\psi_n - \psi_{n-1})\| \leq \left(\frac{\Lambda - \lambda}{\Lambda + \lambda}\right)^n \frac{2}{\lambda+\Lambda} [\Lambda\|\psi_0\| + \|F\|].$$

If $k > n$ then

$$\|\psi_k - \psi_n\| \leq \|\psi_k - \psi_{k-1}\| + \ldots + \|\psi_{n+1} - \psi_n\|$$

$$\leq \sum_{j=n}^{\infty} \left(\frac{\Lambda - \lambda}{\Lambda + \lambda}\right)^j \frac{2}{\lambda+\Lambda} [\Lambda\|\psi_0\| + \|F\|] = \left(\frac{\Lambda - \lambda}{\Lambda + \lambda}\right)^n \frac{\Lambda\|\psi_0\| + \|F\|}{\lambda}.$$

Thus ψ_k is a Cauchy sequence, $\psi_k \to \psi$. Letting $k \to \infty$ we get (20) and

$$\psi = \left(I - \frac{2}{\lambda+\Lambda} T\right)\psi + \frac{2}{\lambda+\Lambda} F.$$

6 Direct integral equation method

Let us come back to the original problem (2), (3). If u_+, u_- solve this problem then

$$u_+ = \mathcal{D}^G u_+ + \mathcal{S}^G \frac{\partial u_+}{\partial n} \quad \text{in } G_+, \qquad \mathcal{D}^G u_+ + \mathcal{S}^G \frac{\partial u_+}{\partial n} = 0 \quad \text{in } G_-, \tag{21}$$

$$u_- = -\mathcal{D}^G u_- - \mathcal{S}^G \frac{\partial u_-}{\partial n} \quad \text{in } G_-, \qquad \mathcal{D}^G u_- + \mathcal{S}^G \frac{\partial u_-}{\partial n} = 0 \quad \text{in } G_+. \tag{22}$$

This gives

$$u_+ = \mathcal{D}^G (u_+ - u_-) + \mathcal{S}^G \left(\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} \right) \quad \text{in } G_+,$$

$$u_- = \mathcal{D}^G (u_+ - u_-) + \mathcal{S}^G \left(\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} \right) \quad \text{in } G_-.$$

(Compare [4].) According to (3)

$$u_+ = \mathcal{D}^G g + \mathcal{S}^G \left(\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} \right) \quad \text{in } G_+, \tag{23}$$

$$u_- = \mathcal{D}^G g + \mathcal{S}^G \left(\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} \right) \quad \text{in } G_-. \tag{24}$$

So, it is enough to derive $\partial u_+/\partial n - \partial u_-/\partial n$. If $a_+ = a_- = 1$ we have a solution of this problem (see [4]).

Using boundary properties of potentials (see (7), (8)), we obtain

$$\frac{\partial u_+}{\partial n} = \frac{\partial \mathcal{D}^G g}{\partial n} + \frac{1}{2} \left(\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} \right) - K_\Omega^* \left(\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} \right),$$

$$\frac{\partial u_-}{\partial n} = \frac{\partial \mathcal{D}^G g}{\partial n} - \frac{1}{2} \left(\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} \right) - K_\Omega^* \left(\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} \right).$$

According boundary conditions (3)

$$f = (a_+ - a-) \frac{\partial \mathcal{D}^G g}{\partial n} + \frac{a_+ + a_-}{2} \left(\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} \right) - (a_+ - a_-) K_\Omega^* \left(\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} \right).$$

Putting

$$F = f - (a_+ - a-) \frac{\partial \mathcal{D}^G g}{\partial n}$$

we have

$$T \left(\frac{\partial u_+}{\partial n} - \frac{\partial u_-}{\partial n} \right) = F, \tag{25}$$

where $T\psi = (a_+ + a_-)\psi/2 - (a_+ - a_-) K_G^* \psi$ is the operator, which we studied for the indirect integral equation method. Theorem 5.1 gives that we can use the successive approximation method for solving the equation (25).

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