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ON p DEPENDENENT BOUNDEDNESS OF SINGULAR INTEGRAL OPERATORS

PETR HONZÍK

ABSTRACT. Suppose that Ω is an integrable function on $\mathbf{S}^1$. We study the singular integral operator

$$T_\Omega f = \text{p.v.}\, f * \frac{\Omega(x/|x|)}{|x|^2}.$$

We show that for $\alpha > 0$ the condition

$$\int_I \Omega(\theta)\, d\theta \le C \log^{-1-\alpha} |I| \tag{0.1}$$

for all intervals I in S^1 gives L^p boundedness of T_Ω in the range $|1/2 - 1/p| \le \frac{\alpha}{2(\alpha+1)}$. This condition is weaker than the condition

$$m_\alpha(\Omega) = |\Omega| * \log^{1+\alpha} |\theta| < \infty$$

from [9] and [6].

We also construct an example of an integrable Ω which satisfies $m_\alpha(\Omega) \le \infty$ such that T_Ω is not L^p bounded for $|1/2 - 1/p| \ge \frac{3\alpha+1}{6(\alpha+1)}$. This improves the result of [8] when $\alpha > 1/3$.

1. Introduction

Suppose that Ω is an complex-valued integrable function on the sphere $\mathbf{S}^1$, with mean value zero with respect to the surface measure. We define the Calderón-Zygmund singular integral operator

$$T_\Omega(f)(x) = \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} \frac{\Omega(y/|y|)}{|y|^2} f(x-y)\, dy = \text{p.v.} \int_{\mathbb{R}^2} \frac{\Omega(y/|y|)}{|y|^2} f(x-y)\, dy\,, \tag{1.1}$$

initially for functions f in the Schwartz class $\mathcal{S}(\mathbb{R}^2)$. These operators have been studied in great detail by many authors. It was proved by Calderón and Zygmund in [1] that for Ω odd the operator T_Ω is bounded on L^p for $1 < p < \infty$. For Ω even the same result holds provided $\Omega \in L\log L$, see [2]. This was later extended to $\Omega \in H^1$ in [10] and [5].

If we restrict the range of p, weaker conditions give the L^p boundedness. Specially, for Ω even we can represent the operator as Fourier multiplier

$$m(\Omega)(\xi) := (\text{p.v.}\Omega(x/|x|)|x|^{-2})\hat{}\,(\xi) = \int_{\mathbf{S}^1} \Omega(\theta) \log \frac{1}{|\xi\cdot\theta|}\, d\theta\,, \tag{1.2}$$

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see for example [7]. If $m(\Omega)$ is in L^∞, the operator T_Ω is bounded on L^2.

To extend this result to general p Grafakos and Stefanov in [9] defined

$$m_\alpha(\Omega)(\xi) := \int_{\mathbf{S}^1} |\Omega(\theta)| \log^{1+\alpha} \frac{1}{|\xi \cdot \theta|}\, d\theta\,. \tag{1.3}$$

Using interpolation techniques they proved that if $m_\alpha(\Omega)$ is in L^∞, then T_Ω is bounded on L^p for

$$\left|\frac{1}{2} - \frac{1}{p}\right| < \frac{\alpha}{2(2+\alpha)}.$$

A sharper version of this theorem where $\frac{\alpha}{2(2+\alpha)}$ is replaced by $\frac{\alpha}{2(1+\alpha)}$ was obtained by Fan, Guo, and Pan [6].

The defintion of $m_\alpha(\Omega)$ is not optimal, because it uses absolute value of the function Ω. In a fashion similar to the pass from $L\log L$ to H^1, we are going to extend the result of Fan, Guo, and Pan to functions Ω which oscilate rapidly.

On the negative side, in [8] there was presented an example of a function Ω such that $m_\alpha(\Omega)$ is bounded but the operator T_Ω is unbounded on L^p for p in the range

$$\left|\frac{1}{2} - \frac{1}{p}\right| > \frac{\alpha}{1+\alpha}.$$

Clearly, this only has some meaning for $\alpha < 1$. The example was based on the properties of one dimensional Fourier transform and cannot be further improved. In this work we are going to present a different example, which is based on geometric properties of the plane, and we get a function Ω with m_α bounded such that T_Ω is unbounded on L^p for p in

$$\left|\frac{1}{2} - \frac{1}{p}\right| > \frac{3\alpha+1}{6(1+\alpha)}.$$

This example is better for $\alpha > 1/3$. It shows that for any $\alpha > 0$ there is an operator T_Ω which is unbounded for some p. Moreover, we also construct an example of a function Ω such that T_Ω is L^p bounded for any $p > 1$ but it is not of the weak type $1-1$.

2. Statement of results

Let us denote by $\mathcal{I}_k$ the set of dyadic arcs of length $2\pi 2^{-k}$. We define the conditional expectation

$$\mathbb{E}_k\Omega(x) = \frac{1}{|I_k|}\int_{I_k} \Omega(y)dy$$

where $x \in I_k \in \mathcal{I}_k$ and

$$\mathbb{D}_k\Omega(x) = \mathbb{E}_k\Omega(x) - \mathbb{E}_{k-1}\Omega(x).$$

As $\Omega \in L^1$, we have $\sum \mathbb{D}_k\Omega = \Omega$. Moreover, we define Haar functions

$$H_{k,l} = \frac{2^k}{2\pi}\left(\chi_{(2\pi 2^{-k}l,2\pi 2^{-k-1}(2l+1)]} - \chi_{(2\pi 2^{-k-1}(2l+1),2\pi 2^{-k}(l+1)]}\right)$$

and observe that

$$\mathbb{D}_k\Omega = \sum_{l=0}^{2^k-1} a_l H_{k,l} \tag{2.1}$$

with

$$\|D_k\Omega\|_1 = \sum |a_l|$$

Theorem 2.1. *Let us have Ω even with $\|\Omega\|_1 \le C_1$ and suppose that for some $\alpha > 0$*

$$\int_I \Omega(\theta)\, d\theta \le C_2 \log^{-1-\alpha} |I| \tag{2.2}$$

holds for any dyadic arc I. Then the operator T_Ω is bounded on L^p for any p such that $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{\alpha}{2(1+\alpha)}$.

The condition (2.2) implies $|\mathbb{E}_k\Omega| \lesssim C_2 2^k k^{-1-\alpha}$ and $|\mathbb{D}_k\Omega| \lesssim C_2 2^k k^{-1-\alpha}$, which is all we shall need in the proof.

Quick comparison shows that for a function Ω which does not change sign too much the conditions (1.3) and (2.2) are nearly identical. While (2.2) is somewhat weaker, both conditions lead to the same result in terms of p and α. The situation changes for a function Ω which has a lot of oscilations. This situation is similar to the relationship between the space H^1 and $L\log L$.

We also show some examples of unbounded operators:

Theorem 2.2. *Suppose $\alpha > 0$ then there is an Ω even with $\|\Omega\|_1 \le 1$ and $\|m_\alpha(\Omega)\|_\infty \le 1$ such that the operator T_Ω does not map L^p into $L^{p,\infty}$ for*

$$\frac{1}{p} - \frac{1}{2} > \frac{3\alpha+1}{6(1+\alpha)}$$

and is not bounded on L^p for

$$\left|\frac{1}{p} - \frac{1}{2}\right| > \frac{3\alpha+1}{6(1+\alpha)}$$

.

By the same construction, we also get the following:

Theorem 2.3. *There is an even function Ω with $\|\Omega\|_1 \le 1$ such that the operator T_Ω is not of the weak type $1-1$ for but is bounded on L^p for all $1 < p < \infty$.*

3. Proof of the Theorem 2.1

We are going to write $\sum \mathbb{D}_k\Omega = \Omega$. For each $T_{\mathbb{D}_k\Omega}$ we use interpolation to obtain L^p estimate in terms of k and α. We show that the estimates form convergent series in k.

The easier endpoint is a space near L^1, say $L^{1+\epsilon}$. The H^1 norm of $\mathbb{D}_k\Omega$ is controlled by C_1. This implies that the norm of the operators $T_{\mathbb{D}_k\Omega}$ are uniformly bounded in $L^{1+\epsilon}$ by $C_\epsilon C_1$. Alternatively, by the methods of Stefanov [12] we could even get the weak $1-1$ estimate, which would lead to the same result.

To get the estimate in L^2, we use the formula (1.2). The norm of the multiplier $m(\mathbb{D}_k\Omega)$ is equal to its L^∞ norm. We have both $\|\mathbb{D}_k\Omega\|_1 \leq C_1$ and $|\mathbb{D}_k\Omega| \lesssim C_2 2^k k^{-1-\alpha}$. We see that $\max|a_l| \lesssim C_2 k^{-1-\alpha}$, where a_l are the indices from (2.1). Let us fix a point θ in S^1. We can assume without loss of generality that $\theta \in [0, 2\pi 2^{-k-1}]$. We have that $m(\mathbb{D}_k\Omega)$ is a convolution of $\mathbb{D}_k\Omega$ with the kernel $-\log|\cos\gamma|$. To simplify the notation, we shall replace this kernel by $\log|\sin\gamma|$ and estimate the integral only over the half circle $[0,\pi)$. We will split the integral as

$$\int_0^{2\pi 2^{-k}} \log|\sin(\theta-\gamma)|\mathbb{D}_k\Omega(\gamma)d\gamma + \int_{2\pi 2^{-k}}^{\pi} \log|\sin(\theta-\gamma)|\mathbb{D}_k\Omega(\gamma)d\gamma.$$

A simple calculation shows that the first part is bounded by a constant multiple of a_1, or by constant times $C_2 k^{-1-\alpha}$. To estimate the second part, we write

$$\int_{2\pi 2^{-k}}^{\pi} \log|\sin(\theta-\gamma)|\mathbb{D}_k\Omega(\gamma)d\gamma = \sum_{l=1}^{2^{k-1}} \int a_l H_{k,l}(\gamma)\log|\sin(\theta-\gamma)|d\gamma$$

We use the mean value theorem to estimate the last sum by constant times $\sum_l |a_l|/l$ and observe that this sum only gets bigger if we rearrange the sequence a_l in decreasing order. Thus we may assume that $|a_l| \lesssim C_2 k^{-1-\alpha}$ when $l \lesssim k^{\alpha+1}$ and $|a_l| \leq C_1/l$ otherwise, as $\sum |a_l| \leq C_1$. Thus we get that $m(\mathbb{D}_k\Omega)(\theta)$ is bounded by $Ck^{-\alpha-1}\log k$.

We now interpolate between the two endpoints. The L^p norm of the multiplier $m(\mathbb{D}_k\Omega)$ for $1 < p < 2$ will be bounded by $Ck^{(-1-\alpha)\gamma+\epsilon'}$ where $\gamma + (1-\gamma)/2 = 1/p$. Summing the estimates in k, we see that we get a convergent series if $1/p - 1/2 < \frac{\alpha}{2(1+\alpha)}$.

The singular integral operator T_Ω is initially defined for functions from Schwartz class $\mathcal{S}$. It is clear that for any $S \in \mathcal{S}$ we have $T_\Omega S = \sum T_{D_k\Omega} S$ where the convergence is uniform. (One way to show this is to decompose each kernel into dyadic annuli and estimate each of these using mean value theorem and the fast decay of S.) When $p \leq 2$ this gives the L^p estimate for all functions from $S \in \mathcal{S}$ and it extends to all L^p functions by the definition of T_Ω. For $p > 2$ we get the result by duality.

4. Counterexample

In this section we construct the example from the theorems 2.2 and 2.3. Suppose that α and p are as in theorem 2.2. Fix a large natural number $A >> \alpha$. Take

$$S = \{r/s \;:\; r, s \in \mathbb{N},\; r \leq A/2,\; A/2 < s \leq A,\; s \text{ is a prime}\}.$$

Denote $N = \mathrm{card}(S)$, clearly $A^2 > N \gtrsim A^2/\log A \gtrsim A^{2-\epsilon}$ Denote I_σ an arc of angular measure $2^{-N^{1/(1+\alpha)}}$ centered at slope σ and ω_σ its characteristic function. We define Ω_A as follows:

$$\Omega_A(\xi) = \frac{1}{2\pi} - \sum_{\sigma\in S} \frac{2^{N^{1/(1+\alpha)}}}{N}\omega_\sigma(\xi) - \sum_{\sigma\in S} \frac{2^{N^{1/(1+\alpha)}}}{N}\omega_\sigma(-\xi).$$

It is easy to check that $\|\Omega\|_1$ is bounded by a constant independent of A and α. Moreover, the angular distance of any two slopes from S is bounded from below by a constant multiple of A^2. Than means that $m_\alpha(\Omega_A) \lesssim \log^{1+\alpha} A$.

Let us denote

$$B = [0, \frac{2^{N^{1/(1+\alpha)}}}{100A^3}]^2$$

and put $G = \mathbb{N}^2 \cap B$. Define u to be the characteristic function of a disc of radius $A^{-3}/8$. We denote $g = A^{6/p}u$. We see that $\|g\|_p \approx 1$ and $\|g\|_1 \approx A^{6(1/p-1)}$. Let us put

$$f(x) = \sum_{y\in G} g(x-y).$$

Thus $\|f\|_p \approx |B|^{1/p}$.

Let us define define

$$D = \cup_{s\in S} G + B(0, A^{-3}/4) + \{[r, rs] \,:\, r \in \mathbb{R}\}.$$

We shall observe that

$$|D \cap B| \le 1/2|B|.$$

To see this, consider L to be a vertical line segment of length one. First we see that for fixed $s \in S$

$$\mathrm{card}((G + \{[r, rs] \,:\, r \in \mathbb{R}\}) \cap L) \le A.$$

So, we have

$$\mathrm{card}(\cup_{s\in S}(G + \{[r, rs] \,:\, r \in \mathbb{R}\}) \cap L) \le A^3,$$

and the estimate follows.

We now estimate $T_\Omega f$ for $x \in |D^c \cap B|$. We need to compute

$$T_\Omega f(x) = \int_{R^2} \frac{\Omega(x-y/|x-y|)}{|x-y|^2} f(y)\, dy.$$

The integral is well defined as x is clearly outside of the support of f. Moreover, the choice of x means that the kernel only takes positive values on the support of f. That means we have

$$T_\Omega f(x) = \int_{B(0,\frac{2^{N^{1/(1+\alpha)}}}{100A^3})\setminus B(0,1)} \frac{1}{2\pi|y|^2} f(x-y) dy.$$

So we see that

$$T_\Omega f(x) \gtrsim \|g\|_1 N^{1/(1+\alpha)}/\log N \gtrsim A^\varepsilon,$$

where ε is a positive number dependent on α. As $|D^c \cap B| \ge 1/2|B|$ we see

$$|\{T_\Omega f \ge A^\varepsilon\}| \ge 1/2|B|$$

and so its weak type p-p norm is bigger than constant multiple of A^{ε}.

As $m_\alpha(\Omega_A) \lesssim \log^{1+\alpha} A$ and it is sublinear, we can take $A_k = 2^k$ and put

$$\Omega = \sum \Omega_{A_k}/A_k^{\varepsilon/2} \tag{4.1}$$

this finishes the proof of 2.2 for $p < 2$, for $p > 2$ we get the result from duality.

To prove the Theorem 2.3 we observe that by partially summing and normalizing the series (4.1) we can produce operator T_Ω which is bounded by 1 on L^q for any q such that $|1/q - 1/2| \leq \frac{\alpha}{2(1+\alpha)}$, but has arbitrarily large norm for $|1/q - 1/2| \geq \frac{3\alpha+1}{6(1+\alpha)}$. So, we take $\alpha_k = k$, and we consider a sequence of functions Ω_k such that $\|\Omega_k\|_1 \leq 2^{-k}$, $m_{\alpha_k}\Omega_k \leq 2^{-k}$, there is a $q_k \geq \frac{3\alpha+1}{6(1+\alpha)}$ such the operator T_{Ω_k} has L^{q_k} norm larger than k and the operator is bounded for any $1 < p < \infty$, which clearly holds for partial sum of (4.1). We now put $\Omega = \sum \Omega_k$ and we are done.

5. Notes

While the condition $m_\alpha(\Omega) < \infty$ works also on $\mathbb{R}^d$ for $d \geq 3$, see [6], it is not clear how to extend the theorem 2.1 to higher dimensions.

There also remains a gap between the positive and negative result, for p in the range

$$\frac{\alpha}{2(\alpha+1)} < \left|\frac{1}{2} - \frac{1}{p}\right| < \max\left\{\frac{3\alpha+1}{6(\alpha+1)}, \frac{\alpha}{\alpha+1}\right\}.$$

It was shown in [3], [4] and [11] that the condition $\Omega \in L\log L$ gives weak $1-1$ bound for T_Ω. Our examples show that the condition $m_\alpha(\Omega) < \infty$ is not strong enough to give the weak type estimate. It is a question if there is some similar condition which would be strong enough.

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Petr Honzík, Institute of Mathematics, AS CR, Žitná 25, CZ - 115 67 Praha 1, Czech Republic

E-mail address: honzikp@math.cas.cz, honzik@gmail.com