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Flat sets, ℓ_p-generating and fixing c_0 in nonseparable setting

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Abstract

In terms of a weak* Kadec-Klee asymptotic smoothness, we define p-flat (asymptotically p-flat) sets in Banach spaces and use these concepts in characterizing WCG (Asplund) spaces that are $c_0(\omega_1)$-generated or $\ell_p(\omega_1)$-generated where $p \in (1, +\infty)$. In particular, we obtain that every subspace of $c_0(\omega_1)$ is $c_0(\omega_1)$-generated and every subspace of $\ell_p(\omega_1)$ is $\ell_p(\omega_1)$-generated for every $p \in (1, +\infty)$. As a byproduct of the technology of using PRI, we get an alternative proof of Rosenthal's theorem on fixing $c_0(\omega_1)$.

1 Introduction

In [11] it was proved that *a separable Banach space* $(X, \|\cdot\|)$ *is isomorphic to a subspace of* c_0 *if and only if its norm is* C*-Lipschitz weak***-Kadec-Klee (in short,* C*-LKK***) for some* $C \in (0, 1]$. The norm $\|\cdot\|$ on X is C-LKK* if $\limsup_n \|x^* + x_n^*\| \geq \|x^*\| + C \limsup_n \|x_n^*\|$ whenever $x^* \in X^*$ and (x_n^*) is a weak*-null sequence in X^*. The norm is called LKK* if it is C-LKK* for

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some $C \in (0,1]$. Clearly, the supremum norm on c_0 is 1-LKK*. Recall that a Banach space X has the *Kadec-Klee property* (KK, in short) if, for every $x \in X$ and every weak-null sequence (x_n) in X such that $\|x + x_n\| \to \|x\|$, we have $\|x_n\| \to 0$. X has the *weak*-Kadec-Klee property* (KK*, in short) if the dual norm has the property that, for every $x^* \in X^*$ and every weak*-null sequence (x_n^*) in X^* such that $\|x^* + x_n^*\| \to \|x^*\|$, we have $\|x_n^*\| \to 0$. Clearly, LKK* implies KK*.
In [15], the following moduli of smoothness were introduced. If $(X, \|\cdot\|)$ is a Banach space, $x \in S_X$, Y is a linear subspace of X and $\tau > 0$, put

$$\overline{\rho}(\tau, x, Y) = \sup\{\|x + y\| - 1;\ y \in Y,\ \|y\| \leq \tau\},$$

then

$$\overline{\rho}(\tau, x) = \inf\{\overline{\rho}(\tau, x, Y);\ Y \subset X,\ \dim(X/Y) < \infty\},$$

and finally,

$$\overline{\rho}(\tau) = \sup\{\overline{\rho}(\tau, x);\ x \in S_X\}.$$

It turns out that *the norm $\|\cdot\|$ on X is LKK* if and only if there exists $\tau_0 > 0$ such that $\overline{\rho}(\tau_0) = 0$*, and *it is 1-LKK* if and only if $\overline{\rho}(1) = 0$* (for details and more on the subject see [11], where a non-separable theory is also developed). The geometric description provided by the use of the modulus $\overline{\rho}$ is more clear than the one given by the definition of the C-LKK*-norm above, and can be depicted as B_X being *asymptotically uniformly flat*. Accordingly, *a separable Banach space is LKK*, if and only if B_X is asymptotically uniformly flat, if and only if X isomorphic to a subspace of c_0*; here the latter equivalence is the deep result from [11].
In this paper, we shall use some ideas from [11] to deal with the $c_0(\omega_1)$-generation and the $\ell_p(\omega_1)$-generation of Banach spaces, where $p \in (1, +\infty)$, and to deal with operators from $c_0(\omega_1)$ fixing copies of $c_0(\omega_1)$. We work in the context of nonseparable weakly compactly generated (WCG) Banach spaces. The restriction of the density to the first uncountable cardinal is done for the sake of simplicity. It is plausible that our results hold with milder cardinality restrictions.

In this paper, $(X, \|\cdot\|)$ denotes a Banach space, B_X (resp. S_X) its closed unit ball (resp., its unit sphere). If M is a bounded set in a Banach space X, we denote by $\|\cdot\|_M$ the seminorm in X^* defined by

$$\|x^*\|_M = \sup\{|\langle x, x^* \rangle|;\ x \in M\}, \quad x^* \in X^*. \tag{1}$$

The first infinite ordinal and the first uncountable ordinal are denoted by ω_0 and ω_1, respectively. Sometimes, we identify the interval $[0, \omega_1)$ with ω_1. Throughout the paper, we assume that $\frac{\infty}{\infty} = 1$ and that $\frac{1}{0} = \infty$. Other concepts used in this paper and not defined here can be found, e.g., in [5].

The following concept evolves from the definition of C-LKK* property considered above. It will be used in characterizing WCG Asplund spaces that are generated by $c_0(\omega_1)$ or by $\ell_p(\omega_1)$ for $p \in (1, +\infty)$ (see Theorem 5).

Definition 1 *Let $(X, \|\cdot\|)$ be a Banach space X, let $M \subset X$ be a set, let $p \in (1, +\infty]$, and put $q = \frac{p}{p-1}$. We say that M is* $\|\cdot\|$-asymptotically p-flat *if it is bounded and there exists $C > 0$ such that, for every $f \in X^*$ and every weak*-null sequence (f_n) in X^*, we have*

$$\limsup_{n\to\infty} \|f + f_n\|^q \geq \|f\|^q + C \limsup_{n\to\infty} \|f_n\|_M^q. \tag{2}$$

We say that M is asymptotically p-flat *if there exists an equivalent norm $|||\cdot|||$ on X such that M is $|||\cdot|||$-asymptotically p-flat.*

Remark 2

1. Non-trivial weak*-sull sequences needed in the above definition do exist. Indeed, it is easy to check that 0 belongs to the weak* closure of the dual sphere S_{X^*}. Thus, if X^* is separable, or more generally if (B_{X^*}, w^*) is a Corson (even angelic) compact space, then there exists a `sequence` $(f_n)_{n\in\mathbb{N}}$ in S_{X^*} such that $f_n \to 0$ weak*. For general Banach spaces the existence of such a sequence is guaranteed by a deep Josefson-Niessenzweig theorem [3, Chapter XII]. This remark ensures the right of life for Definition 1.

2. A small effort yields that *a bounded set $M \subset X$ is asymptotically p-flat for some $p \in (1, +\infty]$ if and only if there exists $C > 0$ such that whenewer $\varepsilon \in (0, C^{-q})$, $f \in B_{X^*}$, and (g_n) is a sequence in S_{X^*} such that $g_n \to f$ weak* and $\|f - g_n\|_M \geq \varepsilon$ for all $n \in \mathbb{N}$, then $\|f\|^q \leq 1 - C\varepsilon^q$, where $q = \frac{p}{p-1}$.*

3. Let $(X, \|\cdot\|)$ be a Banach space. Assume that, for some $p \in (1, +\infty]$, B_X is a $\|\cdot\|$-asymptotically p-flat set. Then, $(X, \|\cdot\|)$ has the KK* property.

4. It is easy to check that the unit ball in c_0 is $\|\cdot\|_\infty$-asymptotically ∞-flat, and that the unit ball in ℓ_p is $\|\cdot\|_p$-asymptotically p-flat for all $p \in (1, +\infty)$, with constant $C = 1$.

5. More generally, *if the usual modulus of smoothness of* $(X, \|\cdot\|)$ *is of power type* $p \in (1, 2]$, *then* B_X *is* $\|\cdot\|$*-asymptotically* p*-flat.* To prove this, take such p and put $q = \frac{p}{p-1}$. Then the modulus of rotundity $\delta_{\|\cdot\|}(\varepsilon) := \inf\{1 - \|\frac{x^*+y^*}{2}\|;\ x^*, y^* \in B_{X^*},\ \|x^* - y^*\| \geq \varepsilon\},\ \varepsilon \in (0, 2]$, of the dual norm $\|\cdot\|$ on X^* is of power type q, which means that there exists $K > 0$ such that $\delta_{\|\cdot\|}(\varepsilon) \geq K\varepsilon^q$ for all $\varepsilon \in (0, 2]$. This is a consequence of the basic relationship between both moduli due to Lindenstrauss (see, e.g., [5, Lemma 9.9]). We shall verify the condition from Remark 2.2. So take $\varepsilon,\ f$ and a sequence (g_n) as there. For every $n \in \mathbb{N}$ we have

$$K\|f - g_n\|^q \leq 1 - \|\tfrac{f+g_n}{2}\| \leq 1 - \|\tfrac{f+g_n}{2}\|^q.$$

Thus

$$\|f\|^q \leq \limsup_{n\to\infty} \|\tfrac{f+g_n}{2}\|^q \leq \liminf_{n\to\infty} (1 - K\|f - g_n\|^q) \leq 1 - K\varepsilon^q.$$

Now, Remark 2.2 says that B_X is asymptotically p-flat. ∎

6. If a subset M in a Banach space $(X, \|\cdot\|)$ is $\|\cdot\|$-asymptotically p-flat for some $p \in (1, +\infty]$, then M is also $\|\cdot\|$-asymptotically p'-flat for every $p' \in (1, p)$. This is a straightforward consequence of the fact that $\|\cdot\|_{\ell_q} \geq \|\cdot\|_{\ell_{q'}}$ whenever $1 \leq q < q'$.

7. Let M be a $\|\cdot\|$-compact set in X and (f_n) a weak*-null sequence in X^*. Then, $\lim_n \|f_n\|_M = 0$. Hence, from the w^*-lower semicontinuity of the dual norm, we get that any norm compact set in an arbitrary Banach space is $\|\cdot\|$-asymptotically ∞-flat. The same proof gives that, more generally, any limit set in any Banach space is asymptotically ∞-flat. Recall that a set M in a Banach space X is *limit* if $\lim_n \|f_n\|_M = 0$ whenever (f_n) is a weak*-null sequence in X^*.

8. Lancien [14] proved that *if* K *is a scattered compact of finite height then the unit ball of* $C(K)$ *is an asymptotically* ∞*-flat set in* $C(K)$, by constructing an equivalent norm. Therefore, for instance, the space

JL_0 of Johnson and Lindenstrauss is an example of a space the unit ball of which is asymptotically ∞-flat, though it does not contain any isometric copy of $c_0(\omega_1)$. This space is not weakly Lindelöf determined (see, e.g., [5, Theorem 12.58]).

9. Godefroy, Kalton, and Lancien in [11, Theorem 4.4] proved that *the unit ball of a WCG space X of density character $\leq \omega_1$ is an asymptotically ∞-flat set if and only if X is isomorphic to a subspace of $c_0(\omega_1)$.*

We say that a Banach space X is *generated by a set $M \subset X$* if M is linearly dense in it. X is said to be *generated by a Banach space Y* if there exists a bounded linear operator from Y into X such that $T(Y)$ is dense in X.

In [6] and [7], we studied questions on generating Banach spaces by, typically, Hilbert or superreflexive spaces via the usual moduli of uniform smoothness. Here we continue in this direction by using, in the Asplund setting, weak* uniform Kadec-Klee norms instead. This allows to get a characterization also for $p > 2$, where the former approach cannot work as the usual moduli of smoothness are at most of power type 2.
Below, we strengthen the definition of asymptotically p-flat set to what we call an innerly asymptotically p-flat set. That allows us to go beyond the framework of Asplund spaces required in Theorem 5. We shall see below (Lemma 12) that, under mild assumptions on the space in question, every innerly asymptotically p-flat set is an Asplund set. This fact will then allow us to prove Theorem 7. To be precise, we introduce at this stage the following concept.

Definition 3 *Let $(X, \|\cdot\|)$ be a Banach space X, let $M \subset X$ be a non-empty set, let $p \in (1, +\infty]$, and put $q = \frac{p}{p-1}$. We say that M is* innerly asymptotically p-flat *if it is bounded and there exists $C > 0$ such that,*

$$\limsup_{n\to\infty} \|f + f_n\|_M^q \geq \|f\|_M^q + C \limsup_{n\to\infty} \|f_n\|_M^q. \tag{3}$$

for every $f \in X^*$ and for every sequence (f_n) in X^* such that $\lim_{n\to\infty}\langle x, f_n\rangle = 0$ for every $x \in M$.

Remark 4

1. Notice that, in the above definition, $C \in (0,1]$. Also, being innerly asymptotically p-flat does not depend on a concrete equivalent norm on X. For $M := B_X$, the properties of being innerly asymptotically p-flat and $\|\cdot\|$-asymptotically p-flat coincide.

2. As in Remark 2.6, if a set is innerly asymptotically p-flat for some $p \in (1,+\infty]$, then it is also innerly asymptotically $p\,'$-flat for every $p\,' \in (1,p)$.

3. Again, any norm-compact, more generally, any limit set in X is ∞-flat.

4. The concept of inner asymptotic p-flatness does not inherit to subsets: Fix $p \in (1,+\infty]$ and put $X = c_0$ if $p = 1$ and $X = \ell_p$ otherwise. It is easy to check that B_X is innerly asymptotically p-flat as well as $\|\cdot\|$-asymptotically p-flat. Put $N = \{e_1, e_2, \ldots\}$ where the e_i's are the canonical unit vectors in X; thus $N \subset B_X$. We then have $\|f_1 + f_n\|_N = 1 = \|f_1\|_N = \|f_n\|_N$ for all $n = 2, 3, \ldots$ Thus (3) is violated no matter how small $C > 0$ is. But notice that N is still $\|\cdot\|$-asymptotically p-flat.

5. It is not difficult to check that *the inner asymptotic p-flatness implies the asymptotic p-flatness.* To show this consider a p-flat set $M \subset X$. Put
$$|||f|||^q = \|f\|^q + \|f\|_M^q, \quad f \in X^*. \tag{4}$$
The triangle inequality for the ℓ_q-norm yields that $|||\cdot|||$ is a norm on X^*. Clearly, this norm is equivalent and dual. Take any $f \in X^*$ and any weak*-null sequence (f_n) in X^*. Choose a subsequence (f_{n_i}) of (f_n) such that $\lim_{i\to\infty} \|f_{n_i}\|_M = \limsup_{n\to\infty} \|f_n\|_M$, and that both limits $\lim_{i\to\infty} \|f + f_{n_i}\|$ and $\lim_{i\to\infty} \|f + f_{n_i}\|_M$ exist. Then
$$\begin{aligned}
\limsup_{n\to\infty} |||f + f_n|||^q &\geq \lim_{i\to\infty} |||f + f_{n_i}|||^q \\
&= \lim_{i\to\infty} \|f + f_{n_i}\|^q + \lim_{i\to\infty} \|f + f_{n_i}\|_M^q \\
&\geq \|f\|^q + (\|f\|_M^q + C \lim_{i\to\infty} \|f_{n_i}\|_M^q) = |||f|||^q + C \lim_{i\to\infty} \|f_{n_i}\|_M^q \\
&= |||f|||^q + C \limsup_{n\to\infty} \|f_n\|_M^q.
\end{aligned}$$
Hence M is $|||\cdot|||$-asymptotically p-flat.

As a byproduct of the technology of using PRI, we get an alternative proof of Rosenthal's theorem on fixing $c_0(\omega_1)$.

2 The results

Theorem 5 *Let X be an Asplund space of density ω_1 and let $p \in (1, +\infty)$ be given. Then the following assertions are equivalent.*
(i) *X is WCG and is generated by an asymptotically p-flat subset, resp. by an asymptotically ∞-flat subset.*
(ii) *X is generated by $\ell_p(\omega_1)$, resp. by $c_0(\omega_1)$.*

Corollary 6 *For $p \in (1, +\infty)$, every subspace of $\ell_p(\omega_1)$ is generated by $\ell_p(\omega_1)$.*
Every subspace of $c_0(\omega_1)$ is generated by $c_0(\omega_1)$.

Note that the fact that subspaces of $c_0(\Gamma)$ are WCG goes back to [13].

Theorem 7 *Let X be a general Banach space of density ω_1 and let $p \in (1, +\infty)$ be given. Then the following assertions are equivalent.*
(i) *X is WCG and is generated by an innerly asymptotically p-flat subset, resp. by an innerly asymptotically ∞-flat subset.*
(ii) *X is generated by $\ell_p(\omega_1)$, resp. by $c_0(\omega_1)$.*

Remark 8

1. As a consequence of Theorems 5 and 7, we get that, if a WCG Asplund Banach space X is generated by an asymptotically p-flat set, then it is generated by a (ususally different) innerly asymptotically p-flat set; see also Remark 4.4.

2. Concerning the first statement in Corollary 6, we note that it is not true that "every subspace of an $\ell_p(\omega_1)$-generated space is $\ell_p(\omega_1)$-generated". This is indicated by a Rosenthal's counterexample. He has a non-WCG subspace R of an $L_1(\mu)$ with "big" probability μ. Here $L_1(\mu)$ is $L_2(\mu)$-generated, i.e. $\ell_2(\Gamma)$-generated. Yet R is $\ell_p(\Gamma)$-generated for no $p \in (1, +\infty)$ since it is not WCG.

3. *Given any $p \in (1, +\infty)$, then every subspace of an $\ell_p(\Gamma)$-generated space is a subspace of a Hilbert generated space.* Indeed, this is obvious if $p \leq 2$ since then $\ell_p(\Gamma)$ is $\ell_2(\Gamma)$-generated. If $p > 2$, find a linear bounded operator $T : \ell_p(\Gamma) \to X$, with dense range. Then T^* continuously injects (B_{X^*}, w^*) into a multiple of (the uniform Eberlein compact space) (B_{ℓ_q}, w), and hence (B_{X^*}, w^*) itself is a uniform

Eberlein compact space. Thus $C((B_{X^*}, w^*))$ is Hilbert generated and hence every subspace of X is a subspace of the Hilbert generated space $C((B_{X^*}, w^*))$.

The last result goes back to Rosenthal [16, Remark 1 after Theorem 3.4], [12, Chaper 7].

Theorem 9 *Asume that a Banach space X of density ω_1 admits a linear bounded operator $T : c_0(\Gamma) \to X$ with dense range. Then there exists an uncountable subset $\Gamma_0 \subset \Gamma$ such that T restricted to $c_0(\Gamma_0)$ is an isomorphism.*

Putting together Theorems 7 and 9, we immediately get

Corollary 10 *If a WCG Banach space of density ω_1 is generated by an innerly asymptotically ∞-flat set, then it contains an isomorphic copy of $c_0(\omega_1)$.*

3 Proofs

Proof of Theorem 5

(i)⇒(ii). Let $\|\cdot\|$ be an equivalent norm on X and let $M \subset X$ be a lineraly dense and $\|\cdot\|$-asymptotically p-flat set. Put $q = \frac{p}{p-1}$. A simple gymnastics with M yields a new set — call it again M — which is symmetric, convex, closed, still $\|\cdot\|$-asymptotically p-flat, and such that $M \subset B_{(X,\|\cdot\|)}$. Since X is WCG, putting $M_1 = M,\ M_2 = M_3 = \cdots = B_{(X,\|\cdot\|)}$, and $\varepsilon = \frac{1}{n},\ n \in \mathbb{N}$, in [8, Proposition 15], we get a PRI $(P_\alpha;\ \omega_0 \le \alpha \le \omega_1)$ on $(X, \|\cdot\|)$ such that $(P_\alpha^*;\ \omega_0 \le \alpha \le \omega_1)$ is a PRI on the dual space $(X, \|\cdot\|)^*$, and moreover $P_\alpha(M) \subset M$ for every $\alpha \in (\omega_0, \omega_1)$; recall that $P_{\omega_0} \equiv 0$. Let $C > 0$ witness that M is $\|\cdot\|$-asymptotically p-flat, see Definition 1. Nothing will happen if we take $C \in (0, 1)$.

`Claim 1.` *For every $0 \ne f \in X^*$ and every $\varepsilon > 0$ there is $\gamma_{f,\varepsilon} \in (\omega_0, \omega_1)$ such that*

$$\|f + g\|^q \ge (1 - \varepsilon)\|f\|^q + C\|g\|_M^q \quad \textit{whenever} \quad g \in \operatorname{Ker} P^*_{\gamma_{f,\varepsilon}}.$$

Proof. Fix any $0 \ne f \in X^*$ and any $\varepsilon > 0$. Assume that the claim does not hold. Find then $g_1 \in \operatorname{Ker} P^*_{\omega_0}\ (= X)$ so that $\|f + g_1\|^q < (1-\varepsilon)\|f\|^q + C\|g_1\|_M^q$. Properties of the P_α^*'s guarantee that there is $\alpha_1 \in (\omega_0, \omega_1)$ such that $g_1 \in$

$P^*_{\alpha_1}X^*$. Find then $g_2 \in \operatorname{Ker} P^*_{\alpha_1}$ so that $\|f+g_2\|^q < (1-\varepsilon)\|f\|^q + C\|g_2\|^q_M$. Find $\alpha_2 \in (\alpha_1, \omega_1)$ so that $g_2 \in P^*_{\alpha_2}X^*$. ... Find $g_{n+1} \in \operatorname{Ker} P^*_{\alpha_n}$ so that $\|f+g_{n+1}\|^q < (1-\varepsilon)\|f\|^q + C\|g_{n+1}\|^q_M$. Find then $\alpha_{n+1} \in (\alpha_n, \omega_1)$ so that $g_{n+1} \in P^*_{\alpha_{n+1}}X^*$. ... Thus we get an infinite sequence $g_1, g_2, \ldots \in X^*$ and an increasing sequence $\alpha_1 < \alpha_2 < \cdots < \omega_1$. If $\sup_{n\in\mathbb{N}} \|g_n\| = +\infty$, then

$$1 = \limsup_{n\to\infty} \frac{\|f+g_n\|^q}{\|g_n\|^q} \leq \limsup_{n\to\infty} \frac{(1-\varepsilon)\|f\|^q + C\|g_n\|^q_M}{\|g_n\|^q} \leq C < 1,$$

a contradiction. Therefore, the sequence (g_n) is bounded. It is actually weak*-null. Indeed, put $\lambda = \lim_{n\to\infty} \alpha_n$; we still have $\lambda < \omega_1$. Fix any $x \in X$. Then for every $n \in \mathbb{N}$ we get

$$|\langle x, g_{n+1}\rangle| = |\langle P_\lambda x, g_{n+1}\rangle| = |\langle P_\lambda x - P_{\alpha_n}x, g_{n+1}\rangle| \leq \|P_\lambda x - P_{\alpha_n}x\| \cdot \sup_{n\in\mathbb{N}} \|g_n\|,$$

ane hence $\langle x, g_n\rangle \to 0$ as $n \to \infty$. Therefore, by (2), we have

$$\limsup_{n\to\infty} \|f+g_n\|^q \geq \|f\|^q + C \limsup_{n\to\infty} \|g_n\|^q_M$$

$$\Big(> (1-\varepsilon)\|f\|^q + C \limsup_{n\to\infty} \|g_n\|^q_M \geq \limsup_{n\to\infty} \|f+g_n\|^q \Big),$$

a contradiction.

`Claim 2.` *For every $\alpha \in [\omega_0, \omega_1)$ there exists $\beta_\alpha \in (\alpha, \omega_1)$ such that*

$$\|f+g\|^q \geq \|f\|^q + C\|g\|^q_M \quad \textit{whenever} \quad f \in P^*_\alpha X^* \quad \textit{and} \quad g \in \operatorname{Ker} P^*_{\beta_\alpha}.$$

Proof. Fix any $\alpha \in [\omega_0, \omega_1)$. Let S be a countable dense subset in the (separable) subspace $P^*_\alpha X^*$. Using Claim 1, put then $\beta_\alpha = \sup\{\gamma_{f,1/n};\ f \in S,\ n \in \mathbb{N}\}$. It is easy to check that this ordinal works.

`Claim 3.` *There exists an increasing long sequence $(\delta_\alpha)_{0\leq\alpha\leq\omega_1}$ in $[0, \omega_1]$, with $\delta_0 = \omega_0$ and $\delta_{\omega_1} = \omega_1$, and such that for every $\alpha \in [0, \omega_1)$ we have*

$$\|f+g\|^q \geq \|f\|^q + C\|g\|^q_M \quad \textit{whenever} \quad f \in P^*_{\delta_\alpha} X^* \quad \textit{and} \quad g \in \operatorname{Ker} P^*_{\delta_{\alpha+1}}. \tag{5}$$

Proof. Fix any $\alpha \in (0, \omega_1)$, and assume that we have already constructed oridnals δ_β's for all $\beta \in [0, \alpha)$. If α has a predecessor, say $\alpha - 1$, then, using Claim 2, put $\delta_\alpha = \beta_{\delta_{\alpha-1}}$. If α is a limit ordinal, put simply $\delta_\alpha = \lim_{\beta\uparrow\alpha} \delta_\beta$.

`Claim 4`. *There exists a linear, bounded, injective and weak***-to-weak continuous operator from* X^* *into* $\ell_q(\mathbb{N}\times[0,\omega_1))$.

Proof. For each $\alpha\in[0,\omega_1)$ find a countable dense set $\{v_1^\alpha,v_2^\alpha,\dots\}$ in $\frac{1}{2}(P_{\delta_{\alpha+1}}-P_{\delta_\alpha})(M)$ $(\subset M)$. Define $T:X^*\to\mathbb{R}^{\mathbb{N}\times[0,\omega_1)}$ by

$$Tf(i,\alpha)=2^{-i}f(v_i^\alpha),\quad (i,\alpha)\in\mathbb{N}\times[0,\omega_1),\quad f\in X^*.$$

Clearly, T is linear and weak* to pointwise continuous. T is injective because $(P_{\delta_\alpha};\ \alpha\in[0,\omega_1])$ is clearly a PRI on X. We shall show that the range of T is a subset of the Banach space $\ell_q(\mathbb{N}\times[0,\omega_1))$ and that T is actually a bounded linear operator from X^* to the latter space. Denote by Y the linear span of the set $\bigcup_{0\le\alpha<\omega_1}(P^*_{\delta_{\alpha+1}}-P^*_{\delta_\alpha})X^*$. Take any $f\in Y$. Then we can write f in the form $f=f_1+f_2+\cdots+f_k$ where $f_j\in(P^*_{\delta_{\alpha_j+1}}-P^*_{\delta_{\alpha_j}})X^*,\ j=1,\dots,k$, and $\alpha_1<\alpha_2<\cdots<\alpha_k$. Observing that $\delta_{\alpha_1}<\delta_{\alpha_2}<\cdots<\delta_{\alpha_k}$, we use (5) repeatedly, and thus we get

$$\begin{aligned}\|Tf\|^q_{\ell_q}&=\sum_{i=1}^\infty\sum_{\alpha\in[0,\omega_1)}2^{-iq}|f(v_i^\alpha)|^q=\sum_{i=1}^\infty 2^{-iq}\sum_{j=1}^k|f_j(v_i^{\alpha_j})|^q\\
\le\sum_{i=1}^\infty 2^{-iq}\cdot\sum_{j=1}^k\|f_j\|_M^q&\le\sum_{j=1}^k\|f_j\|^q_M\le\frac{1}{C}\Big(\|f_1\|^q+C\sum_{j=2}^k\|f_j\|^q_M\Big)\\
&\le\frac{1}{C}\Big(\|f_1+f_2\|^q+C\sum_{j=3}^k\|f_j\|^q_M\Big)\le\cdots\\
&\le\frac{1}{C}\|f_1+f_2+\cdots+f_k\|^q=\frac{1}{C}\|f\|^q.\end{aligned}$$

Therefore $T(Y)\subset\ell_q(\mathbb{N}\times[0,\omega_1))$.

Now, it follows easily from the properties of P^*_α's that Y is norm-dense in X^*. Take any $x^*\in X^*$. Find a sequence (f_n) in Y such that $\lim_{n\to\infty}\|f_n-x^*\|=0$. Then $\lim_{n,m\to\infty}\|Tf_n-Tf_m\|_{\ell_q}\le\lim_{n,m\to\infty}\|f_n-f_m\|=0$. Hence the sequence (Tf_n) converges in the ℓ_q-norm to some $h\in\ell_q(\mathbb{N}\times[0,\omega_1))$. Then for every $(i,\alpha)\in\mathbb{N}\times[0,\omega_1)$ we have

$$h(i,\alpha)=\lim_{n\to\infty}Tf_n(i,\alpha)=\lim_{n\to\infty}2^{-i}f_n(v_i^\alpha)=2^{-i}x^*(v_i^\alpha)=Tx^*(i,\alpha).$$

Therefore $Tx^*=h$. We proved that $T(X^*)\subset\ell_q(\mathbb{N}\times[0,\omega_1))$. Further,

$$\|Tx^*\|^q_{\ell_q}=\|h\|^q_{\ell_q}=\lim_{n\to\infty}\|Tf_n\|^q_{\ell_q}\le\lim_{n\to\infty}\tfrac{1}{C}\|f_n\|^q=\tfrac{1}{C}\|x^*\|.$$

Moreover, the mapping T is obviously weak*-to-pointwise continuous. Hence the Banach-Dieudonné theorem guarantees that T is weak*-to-weak continuous. Claim 5 is thus proved.

Finally, from the above, we can conlude that the adjoint operator T^* goes from $\ell_p(\mathbb{N}\times[0,\omega_1))$ into X. And since, T is injective, $T^*(\ell_p(\mathbb{N}\times[0,\omega_1))$ is dense in X and (ii) is proved. ∎

(ii)⇒(i) This will follow immediately from the implication (ii)⇒(i) in Theorem 7 and from Remark 4.5.

Proof of Corollary 6. Let $p\in(1,+\infty]$. Let $(X,\|\cdot\|)$ be a subspace of $\ell_p(\omega_1)$. Let $Q:\ell_q(\omega_1)\to X^*$ be the canonical quotient mapping. The unit ball $B_{\ell_p(\omega_1)}$ is a $\|\cdot\|_p$-asymptotically p-flat set (with constant $C:=1$). We shall prove that B_X is a $\|\cdot\|$-asymptotically p-flat set in X. So take $x^*\in X^*$ and a weak*-null sequence (x_n^*) in X^*. Select first a subsequence $(x_{n_k}^*)$ of (x_n^*) such that $\|x_{n_k}^*\|\to\limsup_{n\to\infty}\|x_n^*\|$ as $k\to\infty$. Let (l_k^*) be a sequence in $\ell_q(\omega_1)$ such that $Ql_k^*=x^*+x_{n_k}^*$ and $\|l_k^*\|=\|x^*+x_{n_k}^*\|$ for all $k\in\mathbb{N}$. The countability of the supports allows us to select a further subsequence $(l_{n_{k_j}}^*)$ of $(l_{n_k}^*)$ that is w^*-convergent to some $l^*\in\ell_q(\omega_1)$. Obviously, $Ql^*=x^*$. Then,

$$\begin{aligned}\limsup_{n\to\infty}\|x^*+x_n^*\|^q &\geq \limsup_{j\to\infty}\|x^*+x_{n_{k_j}}^*\|^q\\ &=\limsup_{j\to\infty}\|l_{k_j}^*\|^q=\limsup_{j\to\infty}\|l^*+(l_{k_j}^*-l^*)\|^q\geq\|l^*\|^q+\limsup_{j\to\infty}\|l_{k_j}^*-l^*\|^q\\ &\geq\|x^*\|^q+\limsup_{j\to\infty}\|x_{n_{k_j}}^*\|^q=\|x^*\|+\limsup_{n\to\infty}\|x_n^*\|^q.\end{aligned}$$

We obtained that B_X is $\|\cdot\|$-asymptotically p-flat. It is enough now to apply Theorem 5. ∎

Remark 11 A compact space K is said to be *angelic* if for every subset $A\subset K$ every point in the closure of A is the limit of a sequence contained in A, see, e.g. [9]. In particular, every Eberlein compact space, or more generally, every Corson compact space is angelic. The proof of Corollary 6 shows that, if $p\in(1,+\infty]$ and X is a subspace of a Banach space Z such that B_Z is asymptotically p-flat and (B_{Z^*},w^*) is angelic, then B_X is also asymptotically p-flat. This time the angelicity of (B_{Z^*},w^*) allows us to select a w^*-convergent subsequence in the former construction. As a byproduct, we get, from Theorem 5, that X is $\ell_p(\omega_1)$-generated.

A subset A of a Banach space X is called *Asplund* if it is bounded and the pseudometric space $(X^*, \|\cdot\|_N)$ is separable for every countable set $N \subset A$ (see, e.g., [4, Definition 1.4.1]).

The following intermediate result will be used in the proof of Theorem 7.

Lemma 12 *Let X be a Banach space such that (B_{X^*}, w^*) is angelic. Then, for all $p \in (1, +\infty]$, every asymptotically p-flat set $M \subset X$ is an Asplund set.*

Proof. Let $N \subset M$ be a countable set. Then, $\mathrm{span}_{\mathbb{Q}}(N)$, the set of all linear rational combinations of elements in N, is also countable. Let $Y := \overline{\mathrm{span}}(N)$. Let $Q : X^* \to Y^*$ be the canonical quotient mapping. Given $y \in \mathrm{span}_{\mathbb{Q}}(N)$, find $\phi(y) := y^* \in S_{Y^*}$ such that $\langle y, y^* \rangle = \|y\|$. The Separation Theorem gives

$$\overline{\Gamma_{\mathbb{Q}}[\phi(\mathrm{span}_{\mathbb{Q}}(N))]}^{w^*} = B_{Y^*},$$

where $\Gamma_{\mathbb{Q}}[\cdot]$ denotes the absolutely rational-convex hull. We shall prove that the (countable) set $\Gamma_{\mathbb{Q}}[\phi(\mathrm{span}_{\mathbb{Q}}(N))]$ is $\|\cdot\|_N$-dense in X^*. This will conclude the proof.

To this end, choose any $x^* \in X^*$. If $x^* \in Y^\perp$, we can find, as $\|\cdot\|$-close (in particular, as $\|\cdot\|_N$-close) to x as we wish, an element which is not in $Y^\perp$. Thus, we may assume, without loss of generality, that $x^* \notin Y^\perp$ and that, for the moment being, $\|Qx^*\| = 1$. Let $y^* := Qx^*$ $(\in S_{Y^*})$. Since Y is separable, we can find a sequence (y_n^*) in $\Gamma_{\mathbb{Q}}[\phi(\mathrm{span}_{\mathbb{Q}}(N))]$ such that $y_n^* \xrightarrow{w^*} y^*$ as $n \to \infty$. For each element $z^* \in \Gamma_{\mathbb{Q}}[\phi(\mathrm{span}_{\mathbb{Q}}(N))]$, choose a single element $\psi(z^*)$ in B_{X^*} such that $Q(\psi(z^*)) = z^*$. Let $x_n^* := \psi(y_n^*)$ for all $n \in \mathbb{N}$. The sequence (x_n^*) has a w^*-cluster point $x_0^* \in B_{X^*}$, hence, by the assumption, there exists a subsequence of (x_n^*) (denoted again by (x_n^*)) such that $x_n^* \xrightarrow{w^*} x_0^*$. Then we have

$$\limsup_{n\to\infty} \|x_n^*\|^q \geq \|x_0^*\|^q + C \limsup_{n\to\infty} \|x_0^* - x_n^*\|_M^q.$$

Obviously, $Qx_n^* = y_n^*$ $(\xrightarrow{w^*} y^*)$. Hence $Qx_0^* = y^*$, and so $\|x_0^*\| = 1$. It follows that $\limsup_n \|x_n^*\|^q = 1$ and we get $\|x_n^* - x_0^*\|_M \to 0$. In particular, $\|\psi(y_n^*) - x_0^*\|_N \to 0$. This proves the assertion for an element $x^* \in X^*$ such that $\|Qx^*\| = 1$, since the sequence (x_n^*) is in the countable set $\psi(\Gamma_{\mathbb{Q}}[\phi(\mathrm{span}_{\mathbb{Q}}(N))])$. A homogeneity argument involving rational multiples of arbitrary elements in X^* concludes the proof. ∎

Proof of Theorem 7

(i)$\Rightarrow$(ii). We shall follow almost word by word the proof of the implication (i)$\Rightarrow$(ii) from Theorem 5, with the following changes. By Lemma 12, M is an Asplund set. Then we can apply [8, Proposition 15] for $M_1 = M_2 = \cdots = M$ and get a PRI $(P_\alpha;\ \alpha \in [\omega_0, \omega_1])$ on $(X, \|\cdot\|)$ such that $\|P_\lambda^* f - P_\alpha^* f\| \to 0$ as $\alpha \uparrow \lambda$ whenever $f \in X^*$ and $\lambda \in (\omega_0, \omega_1]$ is a limit ordinal. In the whole argument, we replace the dual norm $\|\cdot\|$ on X^* by the seminorm $\|\cdot\|_M$. In the proof of Claim 1, the weak* convergence should be replaced by the pointwise convergence on the set M. The proofs of Claims 2 and 3 do not need any change. In the proof of Claim 4, we shall profit from the fact that the properties of P_α^*'s guarantee that Y is dense in X^* in the metric given by $\|\cdot\|_M$. And, as $\|\cdot\|_M \le \|\cdot\|$, we get that the operator T is bounded. The rest of the proof is the same as before.

(ii)$\Rightarrow$(i). Take $p \in (1, +\infty)$. Assume there exists a bounded linear operator $S : \ell_p(\omega_1) \to X$, with dense range. Put $q = \frac{p}{p-1}$ and $M = S(B_{\ell_p(\omega_1)})$. Let $f \in X^*$ and consider a sequence (f_n) in X^* such that $\lim_{n\to\infty}\langle x, f_n\rangle = 0$ for every $x \in M$. Then $S^* f_n \to 0$ weak* and hence

$$\limsup_{n\to\infty} \|f+f_n\|_M^q = \limsup_{n\to\infty} \sup\langle M, f+f_n\rangle^q = \limsup_{n\to\infty} \sup\langle B_{\ell_p(\omega_1)}, S^*f+S^*f_n\rangle^q$$

$$= \limsup_{n\to\infty} \|S^*f+S^*f_n\|_{\ell_q}^q \ge \|S^*f\|_{\ell_q}^q + \limsup_{n\to\infty} \|S^*f_n\|_{\ell_q}^q = \|f\|_M^q + \limsup_{n\to\infty} \|f_n\|_M^q.$$

This shows that the set M is innerly asymptotically p-flat.

The case of inner asymptotical ∞-flatness can be dealt analogously. ∎

Proof of Theorem 9 Let $e_\gamma,\ \gamma \in \Gamma$, denote the canonical unit vectors in $c_0(\Gamma)$. Put $\Gamma_1 = \{\gamma \in \Gamma;\ Te_\gamma \ne 0\}$. Clearly, Γ_1 is uncountable. We observe that the set $\{Te_\gamma;\ \gamma \in \Gamma_1\}$ countably supports all of X^*. Then we can apply, for instance, [8, Proposition 15], with $M_1 = M_2 = \cdots = 0$ and $\varepsilon_n = \frac{1}{n}$, $n \in \mathbb{N}$, and get a PRI $(P_\alpha;\ \alpha \in [\omega_0, \omega_1])$ on $(X, \|\cdot\|)$ such that $P_\alpha(Te_\gamma) \in \{0, Te_\gamma\}$ for every $\alpha \in (\omega_0, \omega_1)$ and every $\gamma \in \Gamma_1$. Put

$$A = \{\alpha \in [\omega_0, \omega_1);\ P_\alpha(Te_\gamma) = 0 \ \text{ and } \ P_{\alpha+1}(Te_\gamma) = Te_\gamma \ \text{ for some } \ \gamma \in \Gamma_1\}.$$

For every $\alpha \in A$ then pick one $\gamma_\alpha \in \Gamma_1$ such that $P_\alpha(Te_{\gamma_\alpha}) = 0$ and $P_{\alpha+1}(Te_{\gamma_\alpha}) = Te_{\gamma_\alpha}$. Let $\Gamma_2 = \{\gamma_\alpha;\ \alpha \in A\}$. This set is uncountable, for otherwise $T(c_0(\Gamma_1))$ would be separable. A simple "countability" argument yields another uncountable set $\Gamma_0 \subset \Gamma_2$ and $\Delta > 0$ such that $\|Te_\gamma\| > \Delta$ for every $\gamma \in \Gamma_0$.

Take any $0 \neq a \in c_0(\Gamma_0)$. Enumerate its support as $\{\delta_1, \delta_2, \ldots\}$ and find $a_1, a_2, \ldots \in \mathbb{R}$ such that $\|\sum_{i=1}^n a_i e_{\delta_i} - a\| \to 0$ as $n \to \infty$. For every $i \in \mathbb{N}$ find $\alpha_i \in [\omega_0, \omega_1)$ such that $P_{\alpha_i+1}(Te_{\delta_i}) = Te_{\gamma_i}$ and $P_{\alpha_i}(Te_{\delta_i}) = 0$. Observe that $\alpha_i \neq \alpha_j$ whenever $i, j \in \mathbb{N}$ are distinct. Then the "orthogonality" of the projections $P_{\alpha_i+1} - P_{\alpha_i},\ i \in \mathbb{N}$, yields that for every fixed $n, j \in \mathbb{N}$, with $n > j$, we have

$$\Big\| \sum_{i=1}^n a_i T e_{\delta_i} \Big\| \geq \frac{1}{2} \Big\| \Big(P_{\alpha_j+1} - P_{\alpha_j} \Big) \Big(\sum_{i=1}^n a_i T e_{\delta_i} \Big) \Big\| = \frac{1}{2} \| a_j T e_{\delta_j} \| \ \Big(\geq \frac{1}{2} |a_j| \Delta \Big).$$

Hence

$$\|Ta\| = \lim_{n\to\infty} \Big\| T \Big(\sum_{i=1}^n a_i e_{\delta_i} \Big) \Big\| = \lim_{n\to\infty} \Big\| \sum_{i=1}^n a_i T e_{\delta_i} \Big\|$$

$$\geq \frac{\Delta}{2} \max\{|a_j|,\ j \in \mathbb{N}\} = \frac{\Delta}{2} \max\{|a_\gamma|;\ \gamma \in \Gamma_0\} = \frac{\Delta}{2} \|a\|.$$

This proves that T is an isomorphism from $c_0(\Gamma_0)$ into X. ∎

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