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COMBINATORIAL DIFFERENTIAL GEOMETRY AND IDEAL BIANCHI–RICCI IDENTITIES

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ABSTRACT. We apply the graph complex method of [7] to vector fields depending naturally on a set of vector fields and a linear symmetric connection. We characterize all possible systems of generators for such vector-field valued operators including the classical ones given by normal tensors and covariant derivatives. We also describe the size of the space of such operators and prove the existence of an 'ideal' basis consisting of operators with given leading terms which satisfy the (generalized) Bianchi–Ricci identities without the correction terms.

Plan of the paper. In Sections 1 and 2 we recall classical reduction theorems and the Bianchi-Ricci identities. The main results of this paper, Theorems A–F, are formulated in Section 3. Sections 4, 5 and 6 contain necessary notions and results of the graph complex theory and related homological algebra. Section 7 provides proofs of the statements of Section 3.

1. Classical reduction theorems

In this paper, M will always denote a smooth manifold. The letters X, Y, Z, U, V,..., with or without indices, will denote (smooth) vector fields on M. The local coefficients of a vector field X are smooth functions X^λ in coordinates x^λ, $1 \leq \lambda \leq \dim(M)$, such that $X = X^\lambda \frac{\partial}{\partial x^\lambda}$ where, as usual, the summation over repeated indices is assumed. We also consider a linear connection Γ on M with Christoffel symbols $\Gamma^\lambda_{\mu\nu}$, $1 \leq \lambda, \mu, \nu \leq \dim(M)$, see, for example, [1, Section III.7]. The letter R will denote the curvature $(1,3)$-tensor field of Γ, the symbol ∇ the covariant derivative with respect to Γ, and by $\nabla^{(r)}$ we will denote the sequence of iterated covariant derivatives up to order r, i.e. $\nabla^{(r)} = (\mathrm{id}, \nabla, \dots, \nabla^r)$. In this paper we assume the connection Γ to be *symmetric* (also called *torsion-free*), i.e. $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$. The case of non-symmetric connections will be addressed in a forthcoming paper.

It is well-known that natural (that is invariant with respect to chosen local coordinates, i.e. coordinate-independent) operators of linear symmetric connections on manifolds and of tensor fields which have values in tensor fields can be factorized through the curvature

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tensors, their covariant derivatives, given tensor fields and their covariant derivatives. These results are known as the first (the operators of connections only) and the second reduction theorems.

Let us quote the original Schouten's formulation of the *first reduction theorem.*

1.1**. Theorem.** [12, p. 164] *All differential concomitants of a symmetric connexion are ordinary concomitants of* $R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}$ *and its covariant derivatives.*

Similarly we have the *second reduction theorem.*

1.2**. Theorem.** [12, p. 165] *All differential concomitants of a set of quantities* $\Phi_1, \dots, \Phi_d$ *(indices suppressed) and the symmetric connexion* $\Gamma^\kappa_{\mu\lambda}$ *are ordinary concomitants of* $\Phi_1, \dots, \Phi_d$*,* $R_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}$ *and their covariant derivatives.*

Let us recall that a *differential concomitant* is a polynomial coordinate-independent operator with values in tensor fields depending on certain order derivatives of input fields, while *ordinary concomitants* are zero order operators obtained from input fields by tensorial operations, i.e. by tensor products, permutations of indices and contractions.

Proofs of the above reduction theorems use normal coordinates of Γ centered at a point x_0 of M, see [15]. In such coordinates,

$$\Gamma^\lambda_{\mu\nu}(x^\rho) = x^\rho N^\lambda_{\rho\mu\nu}(x_0) + \frac{1}{2!}\, x^{\rho_1} x^{\rho_2} N^\lambda_{\rho_1\rho_2\mu\nu}(x_0) + \cdots\,, \tag{1.1}$$

where $N_n := (N^\lambda_{\rho_1\cdots\rho_{n-2}\mu\nu})$, $n \geq 3$, are the *normal tensors* satisfying the following identities:

$$N_n(X_{\sigma(1)}, \dots, X_{\sigma(n-2)}, X_{n-1}, X_n) - N_n(X_1, \dots, X_n) = 0 \tag{1.2}$$

for any permutation σ of $(n-2)$ indices,

$$N_n(X_1, \dots, X_{n-2}, X_{n-1}, X_n) - N_n(X_1, \dots, X_{n-2}, X_n, X_{n-1}) = 0 \tag{1.3}$$

and

$$\sum_{\sigma\in\Sigma_n} N_n(X_{\sigma(1)}, \dots, X_{\sigma(n)}) = 0, \tag{1.4}$$

where Σ_n denotes the permutation group of n elements. The independence of a differential concomitant on given local coordinates implies that any differential concomitant of Γ is an ordinary concomitant of the normal tensors N_n, $n \geq 3$. This result is known as the *replacement theorem*, see [14]. The first reduction theorem now follows from the fact that each N_n can be expressed as a linear combination, with real coefficients, of the covariant derivatives of order $(n-3)$ of the curvature tensor R of Γ and a tensor field constructed from covariant derivatives of orders $\leq (n-4)$ of R, [12, p. 162], i.e., if we denote by

$$(\nabla^i R)(X_1, \dots, X_i)(X_{i+1}, X_{i+2})(X_{i+3})$$

the $(1, i+3)$ tensor field $\nabla^i R$ evaluated on vector fields $(X_1, \dots, X_{i+3})$, one can write

$$N_n(X_1, \dots, X_n) = \sum_{\sigma \in \Sigma_n} A_\sigma (\nabla^{n-3} R)(X_{\sigma(1)}, \dots, X_{\sigma(n-3)})(X_{\sigma(n-2)}, X_{\sigma(n-1)})(X_{\sigma(n)}) + l.o.t.\,,$$

with some $A_\sigma \in \mathbb{R}$, where $l.o.t.$ is a $(1,n)$-type ordinary concomitant constructed from $\nabla^{(n-4)} R$.

The second reduction theorem can be proved similarly.

In the language of natural bundles and natural operators, differential concomitants are polynomial natural differential operators, see, for instance, [2, 3, 11, 13]. It is proved in [2, Section 28] that the above reduction theorems are true for all natural differential operators, not only for polynomial ones.

We will study polynomial natural differential operators on vector fields and symmetric linear connections with values in vector fields. By the second reduction theorem such operators of order r with respect to the vector fields are given by their r-th order covariant derivatives. So, the minimal order with respect to the connection is $(r-1)$ but this order can be higher. We have:

1.3. **Corollary.** *All polynomial vector fields depending naturally on vector fields $X_1, \dots, X_d$ (in order r) and a symmetric linear connection Γ (in order $s \geq (r-1)$) are obtained by tensorial constructions from the covariant derivatives (up to the order $(s-1)$) of the curvature tensor of Γ, vector fields $X_1, \dots, X_d$ and their covariant derivatives up to the order r, i.e.*

$$X(\Gamma; X_1, \dots, X_d) = X(\nabla^{(s-1)} R; \nabla^{(r)} X_1, \dots, \nabla^{(r)} X_d)\,.$$

Results of the paper. Classical reduction theorems, as Theorems 1.1 and 1.2 above, describe systems of operators which generate all operators of a given type. One usually also gives a list of relations between these generators. For example, the generating system $\overset{S}{\nabla}{}^{(s-1)} R$ for operators of a symmetric connection given by the symmetrized covariant derivatives of the curvature tensor satisfies the classical (higher order) Bianchi identities (2.11) with non-vanishing right-hand sides.

In this paper we characterize all generating systems of natural operators from a set of vector fields and a connection with values in vector fields, see Theorems A, B and C of Section 3. Theorem D of the same section is a uniqueness result for presentations in a given generating system. Theorem E then states that, for each choice of the leading terms of the generating operators, there exists a streamlined, 'ideal' version of these operators satisfying the (generalized) Bianchi-Ricci identities without the right-hand sides. Finally, in Theorem F, we calculate the size of the spaces of natural operators studied in the paper.

Methods of the paper. Classical proofs of reduction theorems, as those given in [12, Section III.7], as well as proofs that use techniques of natural operators, see [2, Section 28], are based on technically complicated calculations in coordinates. The proofs given in this paper combine the classical methods of normal coordinates with the graph complex method proposed by the second author in [7] which is independent on local calculations and analysis.

While the 'classical' methods are suited for proving that a certain system of operators generate all operators of a given type, the graph-complex method is particularly useful for analyzing the uniqueness of expressing an operator via the generating ones. Therefore, the strength of the graph complex method will be particularly manifest in Theorems B, D and F of Section 3.

Let us close this section by recalling that the graph complex method is a sophisticated version of the 'abstract tensor calculus.' It represents geometric quantities, such as coordinates of a tensor field, via vertices of graphs, with graphs playing the role of contraction schemes for composed geometric objects. The coordinate independence of such expressions is characterized by the vanishing of a graph differential. This brings our method into the realm of homological algebra. See [7, 6] for details.

2. Classical Bianchi and Ricci identities

If Q is a tensor field of the type $(1,k)$, $k\geq 0$, we denote by $(\nabla^r Q)(X_1,\dots,X_r,Z_1,\dots,Z_k)$ the value of the rth covariant derivative of Q evaluated on $(r+k)$ vector fields X_j, Z_i, $j=1,\dots,r$, $i=1,\dots,k$, i.e.

$$(\nabla^r Q)(X_1,\dots,X_r,Z_1,\dots,Z_k) = X^{\nu_1}\cdots X^{\nu_r} Z^{\mu_1}\cdots Z^{\mu_k}\,\nabla_{\nu_1}\cdots\nabla_{\nu_r} Q^{\lambda}_{\mu_1\dots\mu_k}\frac{\partial}{\partial x^\lambda}\,.$$

Let us denote by

$$(\overset{S}{\nabla}{}^r Q)(X_1,\dots,X_r,Z_1,\dots,Z_k) = \frac{1}{r!}\sum_{\sigma\in\Sigma_r}(\nabla^r Q)(X_{\sigma(1)},\dots,X_{\sigma(r)},Z_1,\dots,Z_k)$$

and

$$(\overset{A}{\nabla}{}^r Q)(X_1,\dots,X_r,Z_1,\dots,Z_k) = \frac{1}{r!}\sum_{\sigma\in\Sigma_r}(-1)^{\mathrm{sign}(\sigma)}(\nabla^k Q)(X_{\sigma(1)},\dots,X_{\sigma(r)},Z_1,\dots,Z_k)$$

the symmetrized and the antisymmetrized rth covariant derivatives of Q, respectively.

Then we have the Ricci identity

$$\begin{aligned}(\overset{A}{\nabla}{}^2 Q)(Y,X,Z_1,\dots,Z_k) = -\frac{1}{2}\Big[\, & R(X,Y)(Q(Z_1,\dots,Z_k)) \\ & - Q(R(X,Y)(Z_1),Z_2,\dots,Z_k) - \dots - Q(Z_1,\dots,R(X,Y)(Z_k))\,\Big]\,.\end{aligned}\tag{2.1}$$

From the Ricci identity (2.1) we obtain

$$(\nabla^r Q)(X_1,\dots,X_r,Z_1,\dots,Z_k) = (\overset{S}{\nabla}{}^r Q)(X_1,\dots,X_r,Z_1,\dots,Z_k) + pol(X_1,\dots,X_r,Z_1,\dots,Z_k)\,, \tag{2.2}$$

where pol is a $(1,r+k)$-type ordinary concomitant constructed from $\nabla^{(r-2)}Q$ and $\nabla^{(r-2)}R$.

For vector fields we have

$$(\nabla Y)(X) = \nabla_X Y\,, \qquad (\nabla^2 Z)(Y,X) = \nabla_Y(\nabla_X Z) - \nabla_{\nabla_Y X} Z\,.$$

Identity (2.2) now has the form

$$(\nabla^r Z)(X_1,\dots,X_r) = (\overset{S}{\nabla}{}^r Z)(X_1,\dots,X_r) + pol(X_1,\dots,X_r)\,, \tag{2.3}$$

where pol is a $(1,r)$-type ordinary concomitant constructed from $\nabla^{(r-2)}Z$ and $\nabla^{(r-2)}R$.

For the curvature tensor we have the antisymmetry identity

$$R(X,Y)(Z) = -R(Y,X)(Z)\,, \tag{2.4}$$

the 1st Bianchi identity

$$\underset{X,Y,Z}{\circlearrowright\!\!\sum} R(X,Y)(Z) = 0\,, \tag{2.5}$$

and the 2nd Bianchi identity

$$\underset{U,X,Y}{\circlearrowright\!\!\sum} (\nabla R)(U)(X,Y)(Z) = 0\,, \tag{2.6}$$

where $\circlearrowright\!\!\sum$ denotes the cyclic summation over the indicated vector fields. Identity (2.2) for R has the form

$$(\nabla^r R)(U_1,\dots,U_r)(X,Y)(Z) = (\overset{S}{\nabla}{}^r R)(U_1,\dots,U_r)(X,Y)(Z) + pol(U_1,\dots,U_r,X,Y,Z)\,, \tag{2.7}$$

where pol is a $(1,r+3)$-type ordinary concomitant constructed from $\nabla^{(r-2)}R$.

2.1**. Remark.** The rth order, $r \geq 2$, covariant derivative $\nabla^r R$ satisfies the identities obtained by the covariant derivatives of the 1st and the 2nd Bianchi identities, i.e.

$$\underset{X,Y,Z}{\circlearrowright\!\!\sum} (\nabla^r R)(U_1,\dots,U_r)(X,Y)(Z) = 0\,, \qquad \underset{U_r,X,Y}{\circlearrowright\!\!\sum} (\nabla^r R)(U_1,\dots,U_r)(X,Y)(Z) = 0\,. \tag{2.8}$$

The symmetrized rth order covariant derivative $\overset{S}{\nabla}{}^r R$ however satisfies only the identity obtained by the covariant derivative of the 1st Bianchi identity.

So, for the symmetrized higher order covariant derivatives of the curvature tensor, we have the following (higher order) antisymmetry identity

$$(\overset{S}{\nabla}{}^r R)(U_1,\dots,U_r)(X,Y)(Z) + (\overset{S}{\nabla}{}^r R)(U_1,\dots,U_r)(Y,X)(Z) = 0\,, \tag{2.9}$$

the (higher order) classical 1st Bianchi identity

$$\underset{X,Y,Z}{\overset{S}{\sum\!\!\!\!\!\bigcirc}} (\nabla^r R)(U_1,\dots,U_r)(X,Y)(Z) = 0 \tag{2.10}$$

and the (higher order) classical 2nd Bianchi identity with a non-vanishing right hand side

$$\underset{U_r,X,Y}{\overset{S}{\sum\!\!\!\!\!\bigcirc}} (\nabla^r R)(U_1,\dots,U_r)(X,Y)(Z) = -\underset{U_r,X,Y}{\sum\!\!\!\!\!\bigcirc} pol(U_1,\dots,U_r,X,Y,Z)\,, \tag{2.11}$$

where *pol* is a $(1, r+3)$-type ordinary concomitant from (2.7).

3. Main results

3.1. Operators we consider. Let *Con* be the natural bundle functor of torsion-free linear connections [2, Section 17.7] and T the tangent bundle functor. We will consider natural differential operators $\mathcal{O} : Con \times T^{\otimes d} \to T$ acting on a linear connection and d vector fileds, $d \geq 0$, which are linear in the vector fields variables, and which have values in vector fields. We will denote the space of natural operators of this type by $\mathfrak{Nat}(Con \times T^{\otimes d}, T)$. Some typical operators $Con \times T^{\otimes d} \to T$ are recalled in Example 4.1 on page 16.

Define the *vf-order* (vector-field order) resp. the *c-order* (connection order) of a differential operator $\mathcal{O} : Con \times T^{\otimes d} \to T$ as the order of $\mathcal{O}$ in the vector field variables, resp. the connection variable.

3.2. Traces. Let $\mathcal{O}$ be an operator acting on vector fields $X_1,\dots,X_d$ and a connection Γ, with values in vector fields. Suppose that $\mathcal{O}$ is a linear order 0 differential operator in X_i for some $1 \leq i \leq d$. This means that the local formula $O(\Gamma, X_1,\dots,X_d)$ for $\mathcal{O}$ is a linear function of the coordinates of X_i and does not contain derivatives of the coordinates of X_i. In this situation we define $Tr_i(\mathcal{O}) \in \mathfrak{Nat}(Con \times T^{\otimes(d-1)}, R)$ as the operator with values in the bundle R of smooth functions given by the local formula

$$\begin{aligned} &Tr_i(O)(\Gamma, X_1,\dots,X_{i-1},X_{i+1},\dots,X_d) := \\ &\qquad \mathrm{Trace}(O(\Gamma, X_1,\dots,X_{i-1},-,X_{i+1},\dots,X_d) : \mathbb{R}^n \to \mathbb{R}^n). \end{aligned}$$

It is easy to see that $Tr_i(\mathcal{O})$ is well defined. Whenever we write $Tr_i(\mathcal{O})$ we tacitly assume that the trace makes sense, i.e. that $\mathcal{O}$ is linear order 0 differential operator in X_i.

3.3. Compositions Let $\mathcal{O}' : Con \times T^{\otimes d'} \to T$ and $\mathcal{O}'' : Con \times T^{\otimes d''} \to T$ be operators as in 3.1. Assume that $\mathcal{O}'$ is a linear order 0 differential operator in X_i for some $1 \leq i \leq d'$. In this situation we define the composition $\mathcal{O}' \circ_i \mathcal{O}'' : Con \times T^{\otimes(d'+d''-1)} \to T$ as the operator obtained by substituting the value of the operator $\mathcal{O}''$ for the vector-field variable X_i of $\mathcal{O}'$. As in 3.2, by writing $\mathcal{O}' \circ_i \mathcal{O}''$ we signalize that $\mathcal{O}'$ is of order 0 in X_i.

3.4. Throughout this section, by an *iteration* of differential operators we understand applying a finite number of the following 'elementary' operations:

(i) permuting the vector-fields inputs of a differential operator $\mathcal{O}$,
(ii) taking the pointwise linear combination $k' \cdot \mathcal{O}' + k'' \cdot \mathcal{O}''$, $k', k'' \in \mathbb{R}$,
(iii) performing the substitution $\mathcal{O}' \circ_i \mathcal{O}''$, and
(iv) taking the pointwise product $Tr_i(\mathcal{O}') \cdot \mathcal{O}''$.

We of course assume that the operations in (ii) and (iii) make sense, see 3.2 and 3.3. There are 'obvious' relations between the above operations. The operations $\circ_i$ in (iii) satisfy the 'operadic' associativity and are compatible with permutations in (i), see properties (1.9) and (1.10) in [9, Definition II.1.6]. Other 'obvious' relations are the commutativity of the trace, $Tr_j(\mathcal{O}' \circ_i \mathcal{O}'') = Tr_i(\mathcal{O}'' \circ_j \mathcal{O}')$ and its 'obvious' compatibility with permutations of (i).

The iteration defined above provides a coordinate-independent definition of an ordinary concomitant recalled on page 2, i.e. an operator $\mathcal{O}$ is an iteration of operators $\mathcal{O}_1, \ldots, \mathcal{O}_N$ if and only if it is an ordinary concomitant of $\mathcal{O}_1, \ldots, \mathcal{O}_N$.

3.5. Let us consider, for $n \geq 3$, the induced representation $E^0(n) := \mathrm{Ind}^{\Sigma_n}_{\Sigma_{n-2}\times\Sigma_2}(\mathbf{1}_{n-2} \times \mathbf{1}_2)$, where $\mathbf{1}_{n-2}$ (resp. $\mathbf{1}_2$) is the trivial representation of the symmetric group Σ_{n-2} (resp. Σ_2). Elements of $E^0(n)$ are linear combinations

$$\sum_{\sigma \in \mathrm{Ush}(n-2,2)} \alpha_\sigma \cdot (1_{n-2} \times 1_2)\sigma, \tag{3.1}$$

where $1_{n-2} \times 1_2 \in \mathbf{1}_{n-2} \times \mathbf{1}_2$ is the generator, $\alpha_\sigma \in \mathbb{R}$, and σ runs over all $(n-2,2)$-unshuffles which are, by definition, permutations $\sigma \in \Sigma_n$ such that $\sigma(1) < \cdots < \sigma(n-2)$ and $\sigma(n-1) < \sigma(n)$. Let $E^1(n)$ be the trivial Σ_n-module $\mathbf{1}_n$ and $\vartheta_E : E^0(n) \to E^1(n)$ the equivariant map that sends the generator $1_{n-2} \times 1_2 \in \mathbf{1}_{n-2} \times \mathbf{1}_2$ to $-1_n \in \mathbf{1}_n$. The reason for this notation and sign convention will became clear in Section 6.

Define finally $\mathcal{K}(n) \subset E^0(n)$ to be the kernel of the map ϑ_E. It is clear that $\mathcal{K}(n)$ consists of all expressions (3.1) such that

$$\sum_{\sigma \in \mathrm{Ush}(n-2,2)} \alpha_\sigma = 0. \tag{3.2}$$

Theorem A. *Let $D_n(\Gamma, X_1, \ldots, X_n)$, $n \geq 3$, be differential operators in $\mathfrak{Nat}(Con \times T^{\otimes n}, T)$ whose local expressions are*

$$D^\omega_n\left(\Gamma^\lambda_{\mu\nu}, X^{\delta_1}_1, \ldots, X^{\delta_n}_n\right) = \sum_{\sigma \in \mathrm{Ush}(n-2,2)} \alpha_\sigma \cdot X^{\rho_1}_{\sigma(1)} \cdots X^{\rho_n}_{\sigma(n)} \frac{\partial^{n-2}\Gamma^\omega_{\rho_{n-1}\rho_n}}{\partial x^{\rho_1} \cdots \partial x^{\rho_{n-2}}} + l.o.t. \tag{3.3}$$

where $l.o.t.$ is an expression of differential order $< n-2$, and $\{\alpha_\sigma\}_{\sigma\in\Sigma_n}$ are real constants such that the element (3.1) generates the Σ_n-module $\mathcal{K}(n)$ introduced in 3.5 (which in particular means that (3.2) is satisfied).

Let also $V_n(\Gamma, X_1, \ldots, X_n)$*,* $n \geq 1$*, be differential operators in* $\mathfrak{Nat}(Con \times T^{\otimes n}, T)$ *of the form*

$$V_n^\omega\left(\Gamma_{\mu\nu}^\lambda, X_1^{\delta_1}, \ldots, X_n^{\delta_n}\right) = X_1^{\rho_1} \cdots X_{n-1}^{\rho_{n-1}} \frac{\partial^{n-1} X^{\omega_n}}{\partial x^{\rho_1} \cdots \partial x^{\rho_{n-1}}} + l.o.t.,$$

where l.o.t. is an expression of differential order $< n-1$*.*

Suppose, moreover, that the operator $D_n(\Gamma, X_1, \ldots, X_n)$ *is of vf-order* 0 *and the operator* $V_n(\Gamma, X_1, \ldots, X_n)$ *of order* 0 *in* $X_1, \ldots, X_{n-1}$*. Then each differential operator* $\mathcal{O} : Con \times T^{\otimes d} \to T$ *is an iteration, in the sense of 3.4, of the operators* $\{D_n\}_{n\geq 3}$ *and* $\{V_n\}_{n\geq 1}$*.*

Theorem A, as well as other statements in this Section, are proved in Section 7. Observe that necessarily $V_1(\Gamma, X) = X$, so we may safely discard V_1 from the list of 'generating' operators and consider V_n's only for $n \geq 2$.

3.6**. Remark.** It is a simple exercise on the Littlewood-Richardson rule that the Σ_n-module $\mathcal{K}(n)$ of 3.5 decomposes as $\mathcal{K}(n) \cong \bigoplus_\lambda V_\lambda$, with the summation taken over all two-column Young diagrams $\lambda = (\lambda_1, \lambda_2)$ with $\lambda_1 \geq 2$, $0 < \lambda_2 \leq \lambda_1$, such that $\lambda_1 + \lambda_2 = n$, and where V_λ is the irreducible representation corresponding to λ. Since all irreducible factors of $\mathcal{K}(n)$ have multiplicity one, an element $x \in \mathcal{K}(n)$ is a Σ_n-generator if and only if $\pi_\lambda(x) \neq 0$ for each projection $\pi_\lambda : \mathcal{K}(n) \to V_\lambda$. Therefore the assumption of Theorem A on the coefficients α_σ can in principle be checked.

The operator $D_n(\Gamma, X_1, \ldots, X_n)$ in Theorem A has vf-order 0 so it can be interpreted as a $(1, n)$-tensor field $D_n(\Gamma)$ naturally depending on Γ (with c-order $(n-2)$). Similarly $V_n(\Gamma, X_1, \ldots, X_n)$ can be considered as a $(1, n-1)$-tensor field $V_n(\Gamma, X_n)$ naturally given by Γ and X_n (with order $(n-1)$ with respect to X_n).

Then the set $\{D_n(\Gamma)\}_{n\geq 3}$ and $\{V_n(\Gamma, X_i)\}_{n\geq 1}$, $i = 1, \ldots, d$, is a new system of generating operators for natural vector fields from Corollary 1.3.

There are two 'preferred' choices of the leading terms of the operators D_n in Theorem A, the expression

$$X_1^{\rho_1} \cdots X_n^{\rho_n} \frac{\partial^{n-3}}{\partial x^{\rho_1} \cdots \partial x^{\rho_{n-3}}} \left(\frac{\partial \Gamma^\omega_{\rho_{n-2}\rho_n}}{\partial x^{\rho_{n-1}}} - \frac{\partial \Gamma^\omega_{\rho_{n-1}\rho_n}}{\partial x^{\rho_{n-2}}} \right) \tag{3.4}$$

and the expression

$$\left(\sum_{\sigma \in \mathrm{Ush}(n-2,2)} \frac{n(n-1)}{2} X^{\rho_1}_{\sigma(1)} \cdots X^{\rho_n}_{\sigma(n)} - X_1^{\rho_1} \cdots X_n^{\rho_n} \right) \frac{\partial^{n-2} \Gamma^\omega_{\rho_{n-1}\rho_n}}{\partial x^{\rho_1} \cdots \partial x^{\rho_{n-2}}}. \tag{3.5}$$

The leading term (3.4) is given by the following choice of the coefficients in (3.3):

$$\alpha_\sigma := \begin{cases} -1 & \text{if } \sigma \text{ is the identity,} \\ \ \ 1 & \text{if } \sigma \text{ is the 2-cycle that interchanges } (n-2) \text{ and } (n-1), \text{ and} \\ \ \ 0 & \text{otherwise.} \end{cases}$$

It is fairly easy to prove that the corresponding element in (3.1) generates $\mathcal{K}(n)$. The element (3.1) corresponding to (3.5) can be written as

$$(1_{n-2} \times 1_2) \left(\sum_{\sigma \in \mathrm{Ush}(n-2,2)} \frac{n(n-1)}{2} \sigma - \mathrm{id} \right),$$

so it is the image of the generator $1_{n-2} \times 1_2$ of $E^0(n)$ under the projection $E^0(n) \twoheadrightarrow \mathcal{K}(n)$. This immediately implies that it generates $\mathcal{K}(n)$.

Expression (3.4) is the leading term of the iterated covariant derivative of the curvature resp. of its streamlined version whose existence is proved in Theorem E. The leading term of (3.5) is that of the "normal tensors" of (1.1), see also Example 3.7 below.

3.7. **Example.** Operators having the form required by Theorem A exist. One may, for instance, take

$$K_n(\Gamma, X_1, \ldots, X_n) := (\nabla^{n-3} R)(X_1, \cdots, X_{n-3})(X_{n-2}, X_{n-1})(X_n), \; n \geq 3,$$

for the operators D_n and

$$V_n(\Gamma, X_1, \ldots, X_n) := (\overset{S}{\nabla}{}^{n-1} X_n)(X_1, \cdots, X_{n-1}), \; n \geq 2.$$

It is obvious that the leading term of K_n is expression (3.4). Another realization of the operators D_n is provided by the normal tensors N_n recalled in (1.1) whose leading term is (3.5).

For an operator $\mathcal{O} \in \mathfrak{Nat}(Con \times T^{\otimes n}, T)$ and a permutation $\sigma \in \Sigma_n$ we denote by $\mathcal{O}\sigma$ the operator obtained by permuting the vector-field variables $X_1, \ldots, X_n$ of $\mathcal{O}$ according to Σ_n. This action extends, by linearity, into a right action of the group ring $\mathbb{R}[\Sigma_n]$. We will denote $\mathcal{O}c$ the result of the action of $c \in \mathbb{R}[\Sigma_n]$ on $\mathcal{O}$. The following theorem characterizes all possible systems of generating operators.

Theorem B. *Assume that* $\dim(M) \geq 3$. *Let* $U_n(\Gamma, X_1, \ldots, X_n) \in \mathfrak{Nat}(Con \times T^{\otimes n}, T)$, $n \geq 3$, *be operators of vf-order* 0 *and of c-order* $(n-2)$. *Then the following two conditions are equivalent.*

(i) Each operator $\mathcal{O} \in \mathfrak{Nat}(Con \times T^{\otimes d}, T)$ *of vf-order* 0 *and c-order* $(n-2)$ *is an iteration of the operators* $\{U_u\}_{u \leq n}$.

(ii) For each $n \geq 3$, *there are elements* $c, c_1, \ldots, c_n$ *of the group ring* $\mathbb{R}[\Sigma_n]$ *such that the leading term of the operator*

$$D_n := U_n c + \sum_{1 \leq j \leq n} Tr_j(U_n c_j) X_j \in \mathfrak{Nat}(Con \times T^{\otimes n}, T) \tag{3.6}$$

is of the form required by Theorem A.

3.8**. Example.** Consider the operator

$$U_3(\Gamma, X, Y, Z) := R(X, Y)(Z) + Tr(R(-, Z)(X)Y + Tr(R(-, Z)(Y)X.$$

Then clearly $\frac{1}{2}\{U(\Gamma, X, Y, Z) - U(\Gamma, Y, X, Z)\}$ equals the curvature $R(X, Y)(Z)$, so D_3 defined by (3.6) with $c = \frac{1}{2}\tau_{12}$, $c_1 = c_2 = c_3 = 0$, where τ_{12} is the permutation $(1, 2, 3) \mapsto (2, 1, 3)$, has the leading term required by Theorem A. By Theorem B, U_3 defined above can be a member of a generating series of operators.

3.9**. Example.** Let us illustrate the necessity of the assumption $\dim(M) \geq 3$ in Theorem B. Let $U_3 \in \mathfrak{Nat}(Con \times T^{\otimes 3}, T)$ be the operator defined by

$$U_3(\Gamma, X, Y, Z) := X \cdot Tr(R(Y, -)(Z)) + Y \cdot Tr(R(-, X)(Z)).$$

The leading term of this operator equals

$$X^\omega Y^\mu Z^\nu \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} - Y^\omega X^\mu Z^\nu \frac{\partial \Gamma^\lambda_{\mu\nu}}{\partial x^\lambda} + Y^\omega X^\lambda Z^\nu \frac{\partial \Gamma^\mu_{\mu\nu}}{\partial x^\lambda} - X^\omega Y^\lambda Z^\nu \frac{\partial \Gamma^\mu_{\mu\nu}}{\partial x^\lambda},$$

so it is clearly <u>not</u> of the form required by Theorem A. On the other hand, it can be verified by a straightforward calculation that on a 2-dimensional manifold,

$$U_3(\Gamma, X, Y, Z) = R(X, Y)(Z) + l.o.t.$$

therefore, in dimension 2, the operator U_3 can be a part of a generating series of operators.

Theorem C. *Assume that* $\dim(M) \geq 2d-1$ *and that* $\{D_n\}_{n\geq 3}$*,* $\{V_n\}_{n\geq 1}$ *be as in Theorem A. Let* $\mathcal{O} : Con \times T^{\otimes d} \to T$ *be a differential operator of the vf-order* $a \geq 0$*. Then it has an iterative representation with the following property. Suppose that an additive factor of this iterative representation of* $\mathcal{O}$ *via* $\{D_n\}_{n\geq 3}$ *and* $\{V_n\}_{n\geq 2}$ *contains* $V_{q_1}, \ldots, V_{q_t}$*, for some* $q_1, \ldots, q_t \geq 2$*,* $t \geq 0$*. Then*

$$q_1 + \cdots + q_t \leq a + t.$$

In particular, if $\mathcal{O}$ *is of vf-order* 0*, then there exists an iterative representation that uses only* $\{D_n\}_{n\geq 3}$*.*

Notice that one can prove the particular case of Theorem C for operators of vf-order 0 without the $\dim(M) \geq 2d-1$ assumption by a simple modification of the 'classical' proof of Theorem A given on page 26. We, however, do not know how to use the classical reduction techniques to prove Theorem C in full generality.

3.10**. Example.** It is clear that $[X, Y] = \nabla_X Y - \nabla_Y X$ i.e. if V_2 is as in Example 3.7,

$$[X, Y] = V_2(\Gamma, X, Y) - V_2(\Gamma, Y, X).$$

This shows that the individual summands of an iterative representation of an operator $\mathcal{O}$ may depend on the connection though the operator $\mathcal{O}$ does not. This fact was used in [3], in an other context, as the method of an *auxiliary connection*. Later, it was proved in [7] that operators in $\mathfrak{Nat}(Con \times T^{\otimes d}, T)$ that do not depend on the connection are iterations of the Lie bracket of vector fields.

3.11. **Example.** We show that there is, in general, no relation between the c-order of a differential operator and the c-order of its iterative representation. We have

$$\begin{aligned}[X,[Y,Z]] \;=\; & V_3(\Gamma,X,Y,Z) - V_3(\Gamma,X,Z,Y) + V_2(\Gamma,V_2(\Gamma,X,Y),Z)\\ & - V_2(\Gamma,V_2(\Gamma,X,Z),Y) - V_2(\Gamma,V_2(\Gamma,Y,Z),X)\\ & + V_2(\Gamma,V_2(\Gamma,Z,Y),X) + l.o.t.\end{aligned}$$

While the c-order of $[X,[Y,Z]]$ is 0, the operators V_3 in the right hand side are of the c-order 1.

For $n \geq 3$, $\sigma \in \Sigma_n$ and D_n as in Theorem A, denote by $D_n\sigma$ the operator obtained from D_n by permuting the vector fields variables according to σ. This notation clearly extends to the action of an element $\mathfrak{S}$ of the group ring $\mathbb{R}[\Sigma_n]$.

3.12. **Definition.** We say that $\mathfrak{S} \in \mathbb{R}[\Sigma_n]$ is a *quasi-symmetry* of an operator D_n in (3.3) if

$$\Big(\sum_{\sigma\in\Sigma_n} \alpha_\sigma \sigma\Big)\mathfrak{S} = 0$$

in the group ring $\mathbb{R}[\Sigma_n]$. We say that $\mathfrak{S}$ is a *symmetry* of D_n if $D_n\mathfrak{S} = 0$.

A quasi-symmetry $\mathfrak{S}$ of D_n, by definition, annihilates its leading term, therefore $D_n\mathfrak{S}$ is an operator of c-order $\leq (n-3)$ that does not use the derivatives of the vector field variables. We can express this fact by writing

$$D_n\mathfrak{S}(\Gamma, X_1, \ldots, X_n) = \mathfrak{D}_n^{\mathfrak{S}}(\Gamma, X_1, \ldots, X_n) \tag{3.7}$$

where $\mathfrak{D}_n^{\mathfrak{S}} \in \mathfrak{Nat}(Con \times T^{\otimes n}, T)$ ($\mathfrak{D}$ abbreviating "deviation") is a degree $\leq (n-3)$ operator which is, by Theorem C, an iteration of the operators D_u with $3 \leq u \leq n-1$ (no V_n's). By definition, $\mathfrak{S}$ is a symmetry of D_n if and only if $\mathfrak{D}_n^{\mathfrak{S}} = 0$. We will see, in 3.13 below, that (3.7) offers a conceptual explanation of the Bianchi and Ricci identities.

A similar discussion can be made also for the operators V_n, $n \geq 1$. Since the leading term of V_n is fully symmetric in $X_1, \ldots, X_{n-1}$,

$$V_n(\Gamma, X_{\omega(1)}, \ldots, X_{\omega(n-1)}, X_n) - V_n(\Gamma, X_1, \ldots, X_n) = \mathfrak{D}V_n^{\omega}(\nabla, X_1, \ldots, X_n) \tag{3.8}$$

for any $n \geq 2$ and $\omega \in \Sigma_{n-1}$, where $\mathfrak{D}V^{\omega} \in \mathfrak{Nat}(Con \times T^{\otimes n}, T)$ is an order $\leq (n-2)$ differential operator.

The following theorem states that the iteration of Theorem A is unique up to identities (3.7), (3.8) and the 'obvious' relations.

[September 6, 2008]

Theorem D. *On manifolds of dimension $\geq 2d-1$, the iteration expressing an operator in $\mathfrak{Nat}(Con \times T^{\otimes d}, T)$ via $\{D_n\}_{n\geq 3}$ and $\{V_n\}_{n\geq 2}$ is unique up to relations (3.7) with $\mathfrak{S}$ running over all quasisymmetries of D_n, relation (3.8), and the 'obvious' relations among elementary operations. In particular, (3.7) and (3.8) are the only (quasi)symmetries of the operators $\{D_n\}_{n\geq 3}$ and $\{V_n\}_{n\geq 2}$.*

3.13**. Bianchi and Ricci identities.** The leading term (3.4) enjoys the following symmetries:

(s1) the antisymmetry in X_{n-2} and X_{n-1},
(s2) the cyclic symmetry in X_{n-2}, X_{n-1} and X_n,
(s3) for $n \geq 4$, the cyclic symmetry in X_{n-3}, X_{n-2} and X_{n-1}, and
(s3) for $n \geq 4$, the total symmetry in $X_1, \dots, X_{n-3}$.

We leave as an exercise to express these symmetries via appropriate elements of the group ring $\mathbb{R}[\Sigma_n]$. It is not difficult to prove that (s1)–(s4) generate all symmetries of (3.4).

Let D_n be an operator of the form (3.3) with the leading term (3.4). The (anti)symmetry (s1) leads to the equation

$$D_n\left(\Gamma, X_1, ..., X_{n-2}, X_{n-1}, X_n\right) + D_n\left(\Gamma, X_1, ..., X_{n-1}, X_{n-2}, X_n\right) = \mathcal{D}^{\mathrm{as}}_n(\Gamma, X_1, ..., X_n), \tag{3.9}$$

where the natural differential operator $\mathcal{D}^{\mathrm{as}}_n$ of order $\leq (n-3)$ can be interpreted as the deviation from antisymmetry of D_n. Similarly, (s2) leads to

$$\sum_{\sigma}\!\!\!\!\!\!\circlearrowright\; D_n\left(\Gamma, X_1, \dots, X_{\sigma(n-2)}, X_{\sigma(n-1)}, X_{\sigma(n)}\right) = \mathcal{D}'^{\,\mathrm{cycl}}_n(\Gamma, X_1, \dots, X_n), \tag{3.10}$$

where $\sum\!\!\!\!\!\!\circlearrowright$ is the cyclic summation over the indicated indices and $\mathcal{D}'^{\,\mathrm{cycl}}_n \in \mathfrak{Nat}(Con \times T^{\otimes n}, T)$ is an order $\leq (n-3)$ differential operator. In the same manner, for $n \geq 4$, (s3) gives

$$\sum_{\sigma}\!\!\!\!\!\!\circlearrowright\; D_n\left(\Gamma, X_1, \dots, X_{n-4}, X_{\sigma(n-3)}, X_{\sigma(n-2)}, X_{\sigma(n-1)}, X_n\right) = \mathcal{D}''^{\,\mathrm{cycl}}_n(\Gamma, X_1, \dots, X_n), \tag{3.11}$$

for some order $\leq (n-3)$ operators $\mathcal{D}''^{\,\mathrm{cycl}}_n \in \mathfrak{Nat}(Con \times T^{\otimes n}, T)$. Finally, for $n \geq 5$, the symmetry (s4) implies that

$$\begin{aligned} D_n\left(\Gamma, X_{\omega(1)}, \dots, X_{\omega(n-3)}, X_{n-2}, X_{n-1}, X_n\right) - D_n\left(\Gamma, X_1, \dots, X_n\right) &= \\ &= \mathcal{D}^{\omega}_n(\Gamma, X_1, \dots, X_n) \end{aligned} \tag{3.12}$$

for each $n \geq 4$ and a permutation $\omega \in \Sigma_{n-3}$, with some order $\leq (n-3)$ operators $\mathcal{D}^{\omega}_n \in \mathfrak{Nat}(Con \times T^{\otimes n}, T)$.

Observe that, by Theorem C, the right hand sides of (3.9)–(3.12) are iterations of the operators D_u with $3 \leq u \leq n-1$ (no V_n's). We call (3.8)–(3.12) the *Bianchi-Ricci* identities for the operators $\{D_n\}_{n\geq 3}$ and $\{V_n\}_{n\geq 2}$.

[September 6, 2008]

3.14. **Example.** It is clear that not only the leading term of the operator V_n of Example 3.7, but the operator itself is fully symmetric in $X_1, \ldots, X_{n-1}$, therefore (3.8) for this operator is satisfied with trivial right hand side,

$$V_n(\Gamma, X_{\omega(1)}, \ldots, X_{\omega(n-1)}, X_n) - V_n(\Gamma, X_1, \ldots, X_n) = 0$$

for any $n \geq 2$ and $\omega \in \Sigma_{n-1}$.

Let us inspect symmetries fulfilled by the operators K_n of Example 3.7.

For $n = 3$, (3.9) reduces to the standard antisymmetry of the curvature tensor,

$$R(X,Y)(Z) + R(Y,X)(Z) = 0 \tag{3.13}$$

and, for $n \geq 4$, (3.9) is the iterated covariant derivative of (3.13). Therefore K_n satisfies (3.9) with the trivial right hand side,

$$K_n\left(\Gamma, X_1, \ldots, X_{n-2}, X_{n-1}, X_n\right) + K_n\left(\Gamma, X_1, \ldots, X_{n-1}, X_{n-2}, X_n\right) = 0.$$

Similarly, for $n = 3$, (3.10) means the vanishing of the cyclic sum,

$$R(X,Y)(Z) + R(Y,Z)(X) + R(Z,X)(Y) = 0, \tag{3.14}$$

which is the classical 1st Bianchi identity (2.5) of a torsion-free connection. For $n \geq 4$, (3.10) is the iterated covariant derivative of (3.14), therefore

$$\sum_{\sigma}\!\!\!\!\!\!\circlearrowright K_n\left(\Gamma, X_1, \ldots, X_{n-3}, X_{\sigma(n-2)}, X_{\sigma(n-1)}, X_{\sigma(n)}\right) = 0.$$

For $n = 4$, the left hand side of (3.11) means the cyclic sum,

$$(\nabla R)(U)(X,Y)(Z) + (\nabla R)(X)(Y,U)(Z) + (\nabla R)(Y)(U,X)(Z), \tag{3.15}$$

and, by the classical 2nd Bianchi identity (2.6), it is satisfied with the vanishing right hand side. For $n \geq 5$, (3.11) is the iterated covariant derivative of (3.15), therefore

$$\sum_{\sigma}\!\!\!\!\!\!\circlearrowright K_n\left(\Gamma, X_1, \ldots, X_{\sigma(n-4)}, X_{\sigma(n-3)}, X_{\sigma(n-2)}, X_n\right) = 0.$$

On the other hand, for $n \geq 5$, the left hand side of (3.12) is given by the covariant derivatives of the Ricci identities, and it is <u>nonzero</u>.

3.15. **Symmetries of the normal tensors.** The leading term (3.5) of the normal tensor N_n has the following symmetries:

(s1) the full symmetry in $X_1, \ldots, X_{n-2}$,
(s2) the symmetry in X_{n-1} and X_n, and
(s3) the symmetry described by $\mathfrak{S} := \sum_{\sigma \in \Sigma_n} \sigma$.

One can prove that (s1)–(s3) generate all symmetries of (3.5). Equations (1.2)–(1.4) then say that these symmetries of the leading term in fact extend to symmetries of the operator N_n.

[September 6, 2008]

The following theorem shows that for each choice of the leading terms there exist particularly nice operators $\{D_n\}_{n\geq 3}$ of Theorem A.

Theorem E. *For each choice of the leading terms satisfying (3.2), there exist 'ideal' operators $\{J_n\}_{n\geq 3}$ of the form (3.3), for which all the "generalized" Bianchi-Ricci identities (3.7) are satisfied without the right hand sides. In other words, all quasisymmetries, in the sense of Definition 3.12, are actual symmetries of the operators $\{J_n\}_{n\geq 3}$.*

Observe that, in Theorem E, we do not assume that the element (3.1) related to the leading term generates $\mathcal{K}(n)$, we only assume that it belongs to the kernel of the map ϑ_E.

3.16**. Example.** As we already saw in Example 3.14, the operators V_n introduced in Example 3.7 are 'ideal' in the sense that all their quasi-symmetries are also their symmetries. This is not true for the iterated covariant derivatives K_n of the curvature tensor (see again Example 3.14), neither for their 'naively' symmetrized versions $\widetilde{K}_n := \overset{S}{\nabla}{}^{(n-3)} R$, $n \geq 3$. The ideal versions J_n of these operators (both having the same leading term) which exist by Theorem E, can be constructed by modifying $\widetilde{K}_n$ as

$$J_n = \widetilde{K}_n + P_n,$$

where $P_n(\Gamma, X_1, \dots, X_n)$ is a c-order $\leq (n-3)$ iteration of the operators $\{\widetilde{K}_u\}_{3\leq u\leq n-1}$. While clearly $P_3 = P_4 = 0$, i.e.

$$J_3 = R\,, \quad J_4 = \nabla R,$$

the calculation of the correction term P_n is, for $n \geq 5$, a nontrivial task. To give the reader the taste of the complexity of the calculation, we write an explicit formula for P_5:

$$\begin{aligned}
P(\Gamma, U, V, X, Y, Z) = -\tfrac{1}{2}\Big\{ & 2R(U, R(X,Z)(Y))(V) - 2R(U, R(Y,Z)(X))(V) \\
& + 2R(V, R(X,Z)(Y))(U) - 2R(V, R(Y,Z)(X))(U) + 2R(U, R(X,Y)(V))(Z) \\
& + 2R(V, R(X,Y)(U))(Z) + R(X, R(U,Z)(V))(Y) + R(X, R(V,Z)(U))(Y) \\
& - R(Y, R(U,Z)(V))(X) - R(Y, R(V,Z)(U))(X) + R(U, R(X,Z)(V))(Y) \\
& + R(V, R(X,Z)(U))(Y) - R(U, R(Y,Z)(V))(X) - R(V, R(Y,Z)(U))(X) \\
& + R(Y, R(X,U)(V))(Z) + R(Y, R(X,V)(U))(Z) - R(X, R(Y,U)(V))(Z) \\
& - R(X, R(Y,V)(U))(Z) + R(Y, R(X,U)(Z))(V) + R(Y, R(X,V)(Z))(U) \\
& - R(X, R(Y,U)(Z))(V) - R(X, R(Y,V)(Z))(U) + R(X, R(U,Z)(Y))(V) \\
& + R(X, R(V,Z)(Y))(U) - R(Y, R(U,Z)(X))(V) - R(Y, R(V,Z)(X))(U) \Big\}\,.
\end{aligned}$$

[September 6, 2008]

3.17. **Remark.** Let as remark that the ideal basis $\{J_n\}_{n\geq 3}$ of the type discussed in Example 3.16 can be constructed using the normal tensors $\{N_n\}_{n\geq 3}$ of (1.1) as

$$\begin{aligned} (3.16)\quad & J_n(X_1,\ldots,X_{n-3},X_{n-2},X_{n-1},X_n) := \\ & = N_n(X_1,\ldots,X_{n-3},X_{n-2},X_{n-1},X_n) - N_n(X_1,\ldots,X_{n-3},X_{n-1},X_{n-2},X_n)\,. \end{aligned}$$

It is indeed easy to see that the operators J_n defined in this way have the same leading terms as the operators K_n and that identities (1.2)–(1.4) imply the identities (3.9)–(3.12) with trivial right-hand sides.

On the other hand, the normal tensor N_n is, for each $n \geq 3$, a linear combination

$$N_n(X_1,\ldots,X_n) = \sum_{\sigma\in\Sigma_n} A_\sigma\, J_n(X_{\sigma(1)},\ldots,X_{\sigma(n)}), \tag{3.17}$$

where the real coefficients A_σ are determined by requiring that identities (3.9)–(3.12) with trivial right-hand sides imply identities (1.2)–(1.4).

The size of the space of natural operators $Con \times T^{\otimes d} \to T$ is described in the last theorem of this section:

Theorem F. *On manifolds of dimension $\geq 2d-1$, the vector space $\mathfrak{Nat}(Con \times T^{\otimes d}, T)$ is isomorphic to the graph space $\mathcal{G}\mathrm{r}[\mathcal{K}](d)$ introduced on page 25 of Section 6.*

3.18. **Example.** The calculation of the dimension of $\mathcal{G}\mathrm{r}[\mathcal{K}](d)$ as of a vector space spanned by graphs is a purely combinatorial problem. For instance, for $d = 1$ we get $\dim(\mathcal{G}\mathrm{r}[\mathcal{K}](d)) = 1$, with the corresponding natural operator the identity $X \mapsto X$.

One also easily calculates that, on manifolds of dimension ≥ 3, $\dim(\mathfrak{Nat}(Con \times T^{\otimes 2}, T)) = \dim(\mathcal{G}\mathrm{r}[\mathcal{K}](d)) = 4$. The corresponding generating operators are

$$\nabla_X Y,\ \nabla_Y X,\ X \cdot Tr(\nabla_- Y) \text{ and } Y \cdot Tr(\nabla_- X).$$

Results of this section characterize bases of natural operators in $\mathfrak{Nat}(Con \times T^{\otimes d}, T)$ and state some properties of these bases. Various 'classical' bases are then special cases of these general bases. This is symbolically expressed by Figure 1.

4. Rules of the game

In this section whose bulk is taken from [7] we recall the graph complex describing natural differential operators from $\mathfrak{Nat}(Con \times T^{\otimes d}, T)$. The underlying graded vector space of this complex is spanned by directed, not necessary connected, graphs with three types of vertices plus one special vertex called the anchor, see 4.2. The differential given by vertex replacements is recalled in 4.5. Let us, however, start with an example showing three typical operators from $\mathfrak{Nat}(Con \times T^{\otimes d}, T)$.

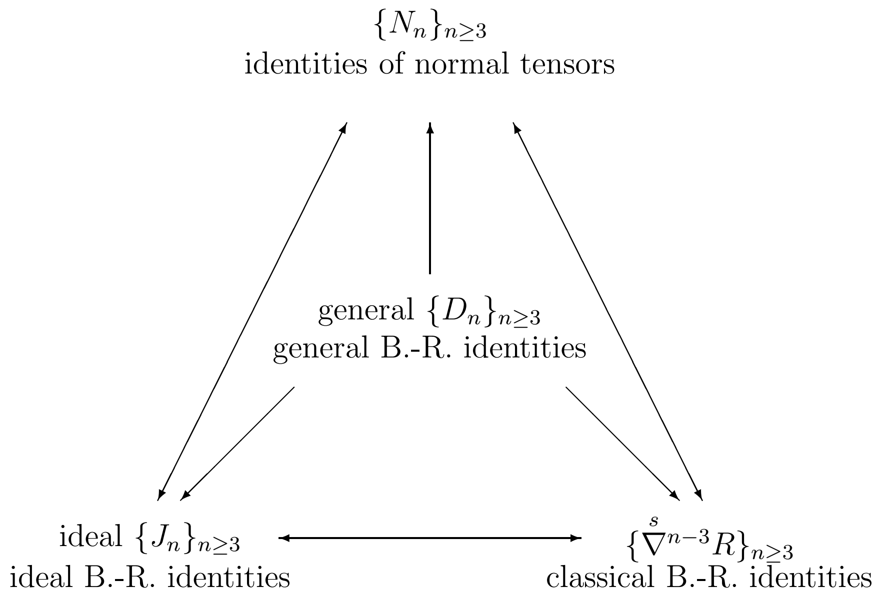


FIGURE 1. Classical bases of operators as specializations of the universal one.

4.1**. Example.** The *Lie bracket* $X, Y \mapsto [X, Y]$ is a natural operator that constructs from two vector fields on M a third one. For the purposes of this paper we consider the Lie bracket as an operator in $\mathfrak{Nat}(Con \times T^{\otimes 2}, T)$. In local coordinates,

$$[X,Y] = \left(X^\mu \frac{\partial Y^\lambda}{\partial x^\mu} - Y^\mu \frac{\partial X^\lambda}{\partial x^\mu} \right) \frac{\partial}{\partial x^\lambda}. \tag{4.1}$$

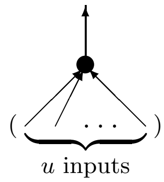
The *covariant derivative* $(\Gamma, X, Y) \mapsto \nabla_X Y$ is another natural differential operator from $\mathfrak{Nat}(Con \times T^{\otimes 2}, T)$. In local coordinates,

$$\nabla_X Y = \left(\Gamma^\lambda_{\mu\nu} X^\mu Y^\nu + X^\mu \frac{\partial Y^\lambda}{\partial x^\mu} \right) \frac{\partial}{\partial x^\lambda}, \tag{4.2}$$

where Γ^i_{jk} are the Christoffel symbols. The *curvature* $R \in \mathfrak{Nat}(Con \times T^{\otimes 3}, T)$ of Γ is a composition of the above operators,

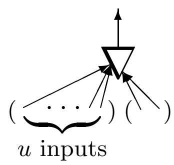
$$R(X,Y)(Z) := \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y] Z.$$

4.2**. The graph complex.** In this paper, by a *graph* we mean a directed (i.e. with oriented edges), not necessary connected, graph whose vertices are of the types described below. Multiple edges, loops and wheels are allowed. Let us recall the construction of the graph complex $\mathfrak{Gr}^*(d)$ describing natural differential operators from d vector fields and a torsion-free linear connection with values in vector fields that are d-multilinear in the vector field variables. Details and motivations can be found in [7] but observe that in that paper we did not assume that the connection is torsion-free. The degree m part $\mathfrak{Gr}^m(d)$ is spanned by graphs with precisely d 'black' vertices

(4.3) $b_u :=$ 

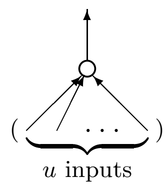
$, u \geq 0,$

labelled by $X, Y, \ldots, X_1, \ldots, X_d$ or $1, \ldots, d$, some number of vertices

(4.4) 

$, u \geq 0.$

precisely m 'white' vertices

(4.5) 

$, u \geq 2,$

and one vertex (the anchor). The braces ( ) in the above pictures mean that the inputs they encompass are fully symmetric, but we will usually omit these braces in the forthcoming text.

In the above graph complex, black vertices (4.3) correspond to derivatives of vector field coordinates,

$$X^{\lambda}_{(\mu_1,\ldots,\mu_u)} := \frac{\partial^u X^{\lambda}}{\partial x^{\mu_1} \cdots \partial x^{\mu_u}},$$

∇-vertices (4.4) to the derivatives of the Christoffel symbols,

$$_{(\omega_1,\ldots,\omega_u)}\Gamma^{\lambda}_{\mu\nu} := \frac{\partial^u \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\omega_1} \cdots \partial x^{\omega_u}},$$

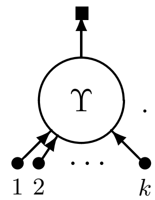
white vertices (4.5) correspond to generators of infinitesimal symmetries and the anchor to the vector-field value of the operator.

4.3**. General connections and vector fields.** A simple modification of material in 4.2 describes operators from general, not necessarily torsion-free, linear connections and vector fields into vector fields. The corresponding graph complex has the same vertices (4.3), (4.5) and as the graph complex $\mathcal{G}r^*(d)$, above but we do not assume that the ∇-vertex (4.4) is symmetric in the two rightmost inputs. The replacement rules are the same as for $\mathcal{G}r^*(d)$ and the obvious analog of Theorem 4.6 holds.

[September 6, 2008]

4.4**. Anchored versus rooted graphs.** Before we proceed to the differential in the graph complex, we need to make a couple of observations on the structure of our graphs. All graphs we have been working with so far had an anchor . For an arbitrary k, $0 \leq k \leq d$, denote by $\mathcal{G}\mathrm{r}^*(d)_k$ the subspace of $\mathcal{G}\mathrm{r}^*(d)$ spanned by graphs Υ with a distinguished subset $\{ {}_1, \ldots, {}_k \}$ of the set of black vertices (4.3) with $u = 0$. Schematically such an Υ looks as

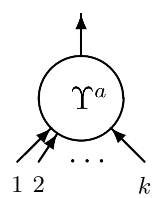
(4.6)



An obvious right Σ_k-action on the space $\mathcal{G}\mathrm{r}^*(d)_k$ permutes the labels of the distinguished vertices. Let $\mathcal{G}\mathrm{r}^*_k := \bigoplus_{d \geq k} \mathcal{G}\mathrm{r}^*(d)_k$. We will call graphs as in (4.6) *anchored k-graphs*.

For each graph $\Upsilon \in \mathcal{G}\mathrm{r}^*_k$, there is a graph Υ^a with k input edges and one output edge, obtained by amputating ■ from the anchor and •'s from the input edges of the distinguished vertices. For instance, the graph Υ in (4.6) gives the amputated graph

(4.7)



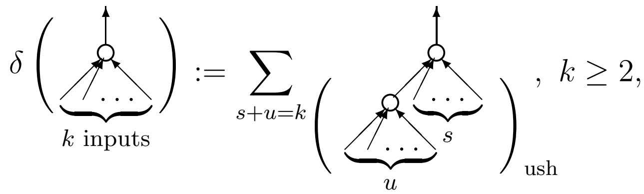
with one output and k numbered input edges. As in operad theory, we will call the output edge of Υ^a the *root* and its inputs the *legs*. Graphs as in (4.7) will be then called *rooted k-graphs*. The operation $\Upsilon \mapsto \Upsilon^a$ is clearly one-to-one and, when convenient, we will make no distinction between anchored graphs and the associated rooted graphs.

Let v be a vertex of a graph $\Lambda \in \mathcal{G}\mathrm{r}^* := \bigoplus_{d \geq 1} \mathcal{G}\mathrm{r}^*(d)$. Suppose that v has k input edges and let $\Upsilon \in \mathcal{G}\mathrm{r}^*_k$ be a rooted k-graph enjoying the same Σ_k-symmetry as the inputs of v. One then may replace the vertex v by Υ, by grafting the root of Υ to the output edge of v and the legs of Υ to the input edges of v. We denote the result by $\Lambda \circ_v \Upsilon \in \mathcal{G}\mathrm{r}^*$ and call this operation the *vertex insertion* or *vertex replacement*. One can clearly extend this by linearity to define $\Lambda \circ_v x$ for $x \in \mathcal{G}\mathrm{r}^*_k$ a linear combination of graphs with appropriate symmetry.

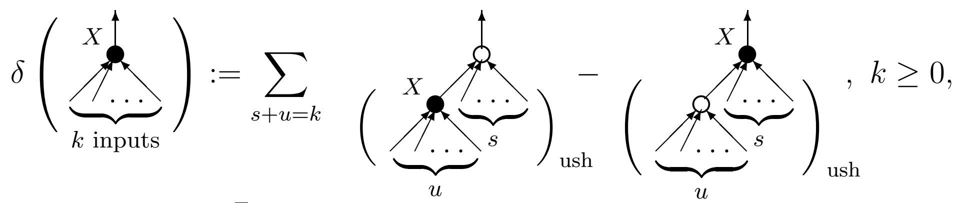
4.5**. The differential.** Let us recall that the graph differential $\delta : \mathcal{G}\mathrm{r}^*(d) \to \mathcal{G}\mathrm{r}^{*+1}(d)$ is the linear map acting on a graph $\Lambda \in \mathcal{G}\mathrm{r}^m(d)$ by the formula

$$\delta(\Lambda) = \sum_{v \in \mathit{Vert}(\Lambda)} \epsilon_v \cdot \Lambda \circ_v \delta(v) \in \mathcal{G}\mathrm{r}^{m+1}(d),\ m \geq 0, \tag{4.8}$$

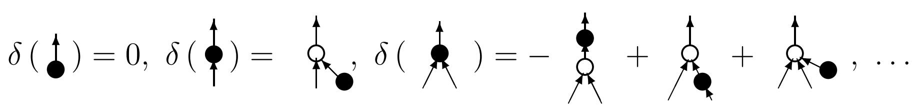
in which $Vert(\Lambda)$ denotes the set of vertices of Λ and $\Lambda \circ_v \delta(v)$ the result of replacing the vertex v by the sum $\delta(v)$ of rooted graphs defined by



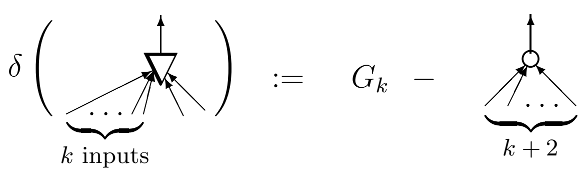
for white vertices,

(4.9) 

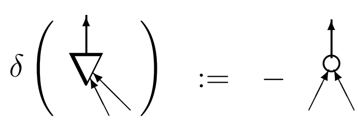
for black vertices and $\delta(\;) = 0$ for the anchor. The braces $(\quad)_{\mathrm{ush}}$ in the right hand sides indicate that the summations over all $(u, s-1)$-unshuffles of the inputs have been performed, see formulas (29) and (30) of [7] for details. For convenience, we write explicitly formula (4.9) for $k = 0, 1, 2$:



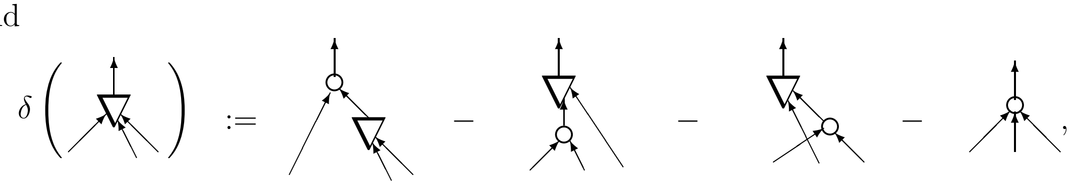
The replacement rule for the ∇-vertices is of the form

(4.10) 

where G_k is a linear combination of 2-vertex trees with one ∇-vertex (4.4), with $u < k$, and one white vertex (4.5) with $u < k + 2$. Explicit formulas for $k = 0, 1$ are



and



which is a graphical form of an equation for the transformation of the Christoffel symbols and their derivative under coordinate changes that can be found in [2, Section 17.7] (but notice a different convention for covariant derivatives used in [2]).

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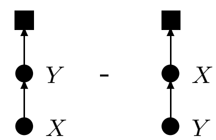
Finally, $\epsilon_v \in \{-1,+1\}$ in (4.8) is a certain sign whose definition can be found in [7, Section 4]. For the purposes of this paper it will be enough to say that, if $\Lambda \in \mathcal{G}\mathrm{r}^0(d)$ (no white vertices), then $\epsilon_v = 1$ for all $v \in \mathit{Vert}(\Lambda)$. The relation between $\mathcal{G}\mathrm{r}^*(d)$ and natural differential operators is described in:

4.6. **Theorem** ([7]). *Each element in* $H^0(\mathcal{G}\mathrm{r}^*(d),\delta) = \mathit{Ker}\left(\delta : \mathcal{G}\mathrm{r}^0(d) \to \mathcal{G}\mathrm{r}^1(d)\right)$ *represents a natural operator* $\mathit{Con} \times T^{\otimes d} \to T$*. On manifolds of dimension* $\geq 2d-1$ *this correspondence is in fact an isomorphism*

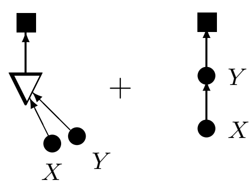
$$H^0(\mathcal{G}\mathrm{r}^*(d),\delta) \cong \mathfrak{Nat}(\mathit{Con} \times T^{\otimes d}, T).$$

4.7. **Example.** In this example taken from [7] we recall graphs representing the Lie bracket, covariant derivative and curvature. The Lie bracket $[X,Y]$ of vector fields X, Y is described by

$[X,Y]$: 

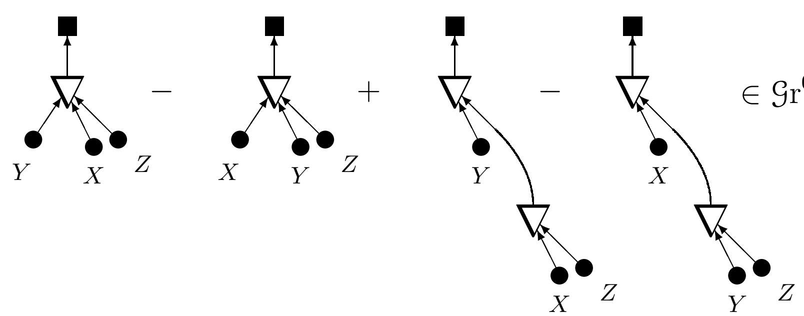
$\in \mathcal{G}\mathrm{r}^0(2)$,

which in the obvious way expresses the local formula (4.1). The covariant derivative is given by the graph

$\nabla_X Y$: 

$\in \mathcal{G}\mathrm{r}^0(2)$,

which is a graphical form of (4.2). Finally, the curvature $R : \mathit{Con} \times T^{\otimes 3} \to T$ is given by the graph

$R(X,Y)(Z)$: 

$\in \mathcal{G}\mathrm{r}^0(3)$.

We recommend as an exercise to verify that all the above graphs belong to the kernel of δ.

5. Auxiliary results

The results of this purely technical section will be used, in Section 6, to construct an explicit basis of the vector space $H^0(\mathcal{G}\mathrm{r}^*(d),\delta)$ recalled in Section 4. Let us consider a bicomplex

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & B^{-3,3} \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & B^{-2,3} \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & B^{-1,3} \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & B^{0,3} \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & 0 \\
 & 0 \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & B^{-2,2} \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & B^{-1,2} \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & B^{0,2} \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & 0 \\
 & & 0 \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & B^{-1,1} \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & B^{0,1} \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & 0 \\
 & & & 0 \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & B^{0,0} \arrow[u, "\delta_{\mathrm{v}}"'] \arrow[r, "\delta_{\mathrm{h}}"] & 0 \\
 & & & & 0 \arrow[u, "\delta_{\mathrm{v}}"'] &
\end{tikzcd}$$

FIGURE 2. The relevant part of a bicomplex $\mathbb{B}$ concentrated in the sector $0 \leq -p \leq q$.

$\mathbb{B} = (B^{*,*}, \delta = \delta_{\mathrm{h}} + \delta_{\mathrm{v}})$, with $B^{*,*} = \bigoplus_{p,q\in\mathbb{Z}} B^{p,q}$ and differentials

$$\delta_{\mathrm{h}} : B^{p,q} \to B^{p+1,q},\ \ \delta_{\mathrm{v}} : B^{p,q} \to B^{p,q+1}.$$

We require, as usual, that

$$\delta_{\mathrm{v}}^2 = 0,\ \ \delta_{\mathrm{h}}^2 = 0 \ \text{ and } \ \delta_{\mathrm{v}}\delta_{\mathrm{h}} + \delta_{\mathrm{h}}\delta_{\mathrm{v}} = 0. \tag{5.1}$$

The associated total complex $\mathrm{Tot}(\mathbb{B}) = (B^*, \delta)$ has $B^* := \bigoplus_{n\in\mathbb{Z}} B^n$ with $B^n := \bigoplus_{p+q=n} B^{p,q}$ and $\delta := \delta_{\mathrm{h}} + \delta_{\mathrm{v}} : B^n \to B^{n+1}$; see [4, §XI.6] for the terminology. Throughout this section we assume that

(i) $\mathbb{B}$ is concentrated in the sector $0 \leq -p \leq q$ (see Figure 2),
(ii) $\mathbb{B}$ is left-bounded in the sense that $B^{p,*} = 0$ for $p \ll 0$, and
(iii) the horizontal cohomology of $\mathbb{B}$ is concentrated on the diagonal $p + q = 0$, that is

$$H^p(B^{*,q}, \delta_{\mathrm{h}}) = 0 \text{ for } p + q \neq 0$$

or, equivalently,

$$H^m(B^*, \delta_{\mathrm{h}}) = 0 \text{ for } m \neq 0.$$

It then follows from a standard spectral sequence argument [4, Theorem XI.6.1] that

$$H^m(\mathrm{Tot}(\mathbb{B})) = 0 \text{ for } m \neq 0$$

while $H^0(\mathrm{Tot}(\mathbb{B}))$ is isomorphic to the direct sum $Z_{\mathrm{h}} := \bigoplus_{r\geq 0} Z^r_{\mathrm{h}}$ (which is finite, by (ii)) of subspaces

$$Z^r_{\mathrm{h}} := Ker(\delta_{\mathrm{h}} : B^{-r,r} \to B^{-r+1,r}). \tag{5.2}$$

Let us indicate how to construct such an isomorphism.

5.1**. Proposition.** *Let $\beta : Z_{\mathrm{h}} = \bigoplus_{r\geq 0} Z^r_{\mathrm{h}} \to \bigoplus_{r\geq 0} B^{-r,r}$ be a linear map such that, for each $r \geq 0$ and $z \in Z^r_{\mathrm{h}}$, $\beta(z)$ is a cocycle in the total complex* $\mathrm{Tot}(\mathbb{B})$ *and has the form*

$$\beta(z) = z + l.o.t. \tag{5.3}$$

with some $l.o.t. \in \bigoplus_{p>r} B^{-p,p}$. Then the map β induces an isomorphism (denoted by the same symbol)

$$\beta : Z_{\mathrm{h}} \xrightarrow{\cong} H^0(\mathrm{Tot}(\mathbb{B})).$$

Proof. Let us interpret $Z_{\mathrm{h}} = \bigoplus_{r\geq 0} Z^r_{\mathrm{h}}$ as a cochain complex concentrated in degree zero, with trivial differential. Define a decreasing filtration

$$Z_{\mathrm{h}} = F'_0 Z_{\mathrm{h}} \supset F'_1 Z_{\mathrm{h}} \supset F'_2 Z_{\mathrm{h}} \supset F'_3 Z_{\mathrm{h}} \supset \cdots$$

of Z_{h} by $F'_s Z_{\mathrm{h}} := \bigoplus_{r\geq s} Z^r_{\mathrm{h}}$. Similarly, define a decreasing filtration

$$B^* = F''_0 B^* \supset F''_1 B^* \supset F''_2 B^* \supset F''_3 B^* \supset \cdots$$

of the total complex $\mathrm{Tot}(\mathbb{B}) = (B^*, d)$ by $F''_s B^* := \bigoplus_{p+q=*} \bigoplus_{q\geq s} B^{p,q}$. With these definitions, β is a map of filtered cochain complexes that induces an isomorphism of the E^1-terms of the associated spectral sequences. The proposition follows from a standard spectral sequence argument [4, Theorem XI.1.1]. □

In Proposition 5.1, l.o.t. is an abbreviation from *lower order terms*. The justification for this terminology will became obvious on page 26 of Section 6. It is not difficult to show that there always exists a map β satisfying the assumptions of the proposition. For further applications we, however, need an explicit construction of this map. It starts by choosing, for each $n \geq 0$, a complementary subspace $D^n \subset B^{-n,n}$ to Z^n_{h} so that

$$B^{-n,n} = Z^n_{\mathrm{h}} \oplus D^n. \tag{5.4}$$

Let $\pi_n : B^{-n,n} \to D^n$ be the projection. For $X^n := Ker(\delta_{\mathrm{v}}\delta_{\mathrm{h}} : B^{-n,n} \to B^{-n+1,n+1})$ we define $U : X^n \to X^{n+1}$ by

$$U(x) := \pi_{n+1}\delta^{-1}_{\mathrm{h}}\delta_{\mathrm{v}}(x),\ x \in X^n. \tag{5.5}$$

We must verify that this definition of U makes sense. Since, for $x \in X^n$, $\delta_{\mathrm{v}}\delta_{\mathrm{h}}x = -\delta_{\mathrm{h}}\delta_{\mathrm{v}}x = 0$ (see the third equation of (5.1)), $\delta_{\mathrm{v}}x$ is a δ_{h}-cocycle, so $\delta_{\mathrm{v}}x \in Im(\delta_{\mathrm{h}} : B^{-n-1,n+1} \to B^{-n,n+1})$ by the acyclicity (iii). The set $\delta_{\mathrm{h}}^{-1}\delta_{\mathrm{v}}(x)$ is therefore non-empty. If $u', u'' \in \delta_{\mathrm{h}}^{-1}\delta_{\mathrm{v}}(x)$, then $\delta_{\mathrm{h}}(u' - u'') = 0$ so $\pi_{n+1}(u') = \pi_{n+1}(u'')$, which means that $U(x) = \pi_{n+1}\delta_{\mathrm{h}}^{-1}\delta_{\mathrm{v}}(x)$ is a one-element set. The condition $\delta_{\mathrm{v}}\delta_{\mathrm{h}}U(x) = 0$ follows from the simple fact that

$$\delta_{\mathrm{h}}U(x) = \delta_{\mathrm{h}}\pi_{n+1}\delta_{\mathrm{h}}^{-1}\delta_{\mathrm{v}}(x) = \delta_{\mathrm{v}}x. \tag{5.6}$$

This shows that indeed $U(x) \in X^{n+1}$. For $z \in Z_{\mathrm{h}}^n \subset X^n$ finally define

$$\beta(z) := z - U(z) + U^2(z) - U^3(z) + \cdots = (id + U)^{-1}(z). \tag{5.7}$$

The above sum is, by assumption (ii), finite.

5.2. **Lemma.** *For each $n \geq 0$ and $z \in Z_{\mathrm{h}}^n$, the element $\beta(z) \in \bigoplus_{p \geq n} B^{-p,p}$ is a degree 0 cocycle of the total complex* $\mathrm{Tot}(\mathbb{B})$*. The map β defined in (5.7) therefore satisfies assumptions of Proposition 5.1.*

Proof. By the definition of the differential in the total complex, one needs to verify that

$$\delta_{\mathrm{h}}z = 0,\ \delta_{\mathrm{h}}U(z) = \delta_{\mathrm{v}}z,\ \delta_{\mathrm{h}}U^2(z) = \delta_{\mathrm{v}}U(z), \ldots$$

The above equations immediately follow from (5.6). □

Let us formulate a simple lemma which will be used in the proof of Corollary 5.4 below.

5.3. **Lemma.** *For $x \in X^n$, the element $U(x)$ defined in (5.5) is characterized by $U(x) \in D^{n+1}$ and $\delta_{\mathrm{h}}U(x) = \delta_{\mathrm{v}}x$.*

Proof. It follows from definition and from (5.6) that $U(x)$ satisfies the conditions of the lemma. On the other hand, suppose we are given an element $a \in D^{n+1}$ such that $\delta_{\mathrm{h}}a = \delta_{\mathrm{v}}x$. Then $a = \pi_{n+1}a = \pi_{n+1}\delta_{\mathrm{h}}^{-1}\delta_{\mathrm{h}}a = \pi_{n+1}\delta_{\mathrm{h}}^{-1}\delta_{\mathrm{v}}x = U(x)$. □

5.4. **Corollary.** *Let G be a group and assume that the bicomplex $\mathbb{B} = (B^{*,*}, \delta = \delta_{\mathrm{h}} + \delta_{\mathrm{v}})$ consists of reductive G-modules. Suppose moreover that the differentials δ_{h} and δ_{v} are G-equivariant. Then there exists a G-equivariant β satisfying assumptions of Proposition 5.1.*

Proof. By the reductivity of the actions, one may obviously assume that the decomposition (5.4) is G-invariant and that the projection π_n is G-equivariant. Moreover, X^n is a G-stable subspace of $B^{-n,n}$ and $\delta_{\mathrm{h}}(U(x)g) = (\delta_{\mathrm{h}}U(x))g = \delta_{\mathrm{v}}(x)g$ for each $x \in X^n$, $g \in G$, because the differentials are assumed to be G-equivariant. Lemma 5.3 then implies that $U(xg) = U(x)g$, that is, U is G-equivariant. Then β defined by (5.7) is G-equivariant, too. □

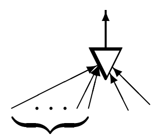
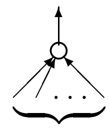
[September 6, 2008]

6. Cohomology of graph complexes

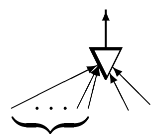
In this part we apply the methods of Section 5 to the graph complex constructed in Section 4. Let us start with $\mathcal{G}\mathrm{r}^*(d)$. The first step is to realize that $(\mathcal{G}\mathrm{r}^*(d), \delta)$ is the total complex of a bicomplex defined as follows. For $p, q \in \mathbb{Z}$, let

$$\mathcal{G}\mathrm{r}^{p,q}(d) := \mathit{Span}\left\{\text{graphs } \Lambda \in \mathcal{G}\mathrm{r}^{p+q}(d);\ \text{the number of } \nabla\text{-vertices } = -p\right\}. \tag{6.1}$$

Define the horizontal differential $\delta_{\mathrm{h}} : \mathcal{G}\mathrm{r}^{p,q}(d) \to \mathcal{G}\mathrm{r}^{p+1,q}(d)$ by

(6.2) $\delta_{\mathrm{h}}\Big($  $\Big) := -$ 

while δ_{h} is trivial on remaining vertices. The vertical differential $\delta_{\mathrm{v}} : \mathcal{G}\mathrm{r}^{p,q}(d) \to \mathcal{G}\mathrm{r}^{p,q+1}(d)$ is defined by requiring that $\delta_{\mathrm{v}} := \delta$ on black vertices (4.3), white vertices (4.5) and the anchor ↥, while

$\delta_{\mathrm{v}}\Big($  $\Big) := G_k,$

where G_k is the same as in (4.10).

6.1**. Lemma.** *The object* $\mathcal{G}\mathrm{r}^{*,*}(d) = (\mathcal{G}\mathrm{r}^{*,*}(d), \delta_{\mathrm{h}} + \delta_{\mathrm{v}})$ *constructed above is a bicomplex whose total complex is the graph complex* $(\mathcal{G}\mathrm{r}^*(d), \delta)$ *recalled in Section 4.*

Proof. The only property which is not obvious are the relations (5.1) which can be verified directly. □

Let us check that the bicomplex $\mathcal{G}\mathrm{r}^{*,*}(d)$ satisfies conditions (i)–(iii) on page 21, Section 5. One immediately sees that (i) is equivalent to the obvious inequality

$$0 \leq \text{number of } \nabla \text{ vertices } \leq \text{number of } \nabla \text{ vertices } + \text{number of white vertices.}$$

Simple graph combinatorics implies that each graph $\Lambda \in \mathcal{G}\mathrm{r}^*(d)$ has at most $d-1$ ∇-vertices, therefore $\mathcal{G}\mathrm{r}^{p,q}(d) = 0$ for $p \leq -d$ so the condition (ii) of Section 5 is also satisfied.

To verify (iii), we need to follow [7] and observe that $(\mathcal{G}\mathrm{r}^*(d), \delta_{\mathrm{v}})$ is a particular case of the following construction. For each collection $(E^*, \vartheta_E) = \{(E^*(s), \vartheta_E)\}_{s\geq 2}$ of right dg-Σ_s-modules $(E^*(s), \vartheta_E)$, one considers the complex $\mathcal{G}\mathrm{r}^*[E^*](d) = (\mathcal{G}\mathrm{r}^*[E^*](d), \vartheta)$ spanned by graphs with d black vertices (4.3), one vertex ↥ and a finite number of vertices decorated by elements of E. The grading of $\mathcal{G}\mathrm{r}^*[E^*](d)$ is induced by the grading of E^* and the differential ϑ replaces E-decorated vertices, one at a time, by their ϑ_E-images, leaving other vertices unchanged. It is a standard fact [10] (see also [5, Theorem 21]) that the assignment $(E^*, \vartheta_E) \mapsto (\mathcal{G}\mathrm{r}^*[E^*](d), \vartheta)$ is a polynomial, hence exact, functor, thus

$$H^*(\mathcal{G}\mathrm{r}^*[E^*](d), \vartheta) \cong \mathcal{G}\mathrm{r}^*[H^*(E, \vartheta_E)](d). \tag{6.3}$$

Let now $(E^*, \vartheta_E) = \{(E^*(s), \vartheta_E)\}_{s\geq 2}$ be such that $E^0(s)$ is spanned by symbols (4.4) with $u + 2 = s$, $E^1(s)$ by symbols (4.5) with $u = s$, and $E^m(s) = 0$ for $m \geq 2$. The differential ϑ_E is defined by replacement rule (6.2). More formally,

$$E^0(s) = \mathrm{Ind}^{\Sigma_s}_{\Sigma_{s-2}\times\Sigma_2}(\mathbf{1}_{s-2} \times \mathbf{1}_2) \text{ and } E^1(s) = \mathbf{1}_s,$$

where $\mathbf{1}_{s-2}$ (resp. $\mathbf{1}_2$, resp. $\mathbf{1}_s$) denotes the trivial representation of the symmetric group Σ_{s-2} (resp. Σ_2, resp. Σ_s). The differential ϑ_E then sends the generator $1 \times 1 \in \mathbf{1}_{s-2} \times \mathbf{1}_2$ into $-1 \in \mathbf{1}_s$. It is clear that, with this particular choice of the collection (E^*, ϑ_E),

$$(\mathcal{G}\mathrm{r}^*(d), \delta_{\mathrm{h}}) \cong (\mathcal{G}\mathrm{r}^*[E^*](d), \vartheta).$$

Since $\vartheta_E : E^0(s) \to E^1(s)$ is onto, the collection $H^*(E, \vartheta_E) = \{H^*(E(s), \vartheta_E)\}_{s\geq 2}$ is concentrated in degree 0, with $H^0(E(s), \vartheta_E)$ the kernel

$$\mathcal{K}(s) := \mathit{Ker}\left(\vartheta_E : E^0(s) \to E^1(s)\right). \tag{6.4}$$

Denoting by $\mathcal{K}$ the collection $\mathcal{K} := \{\mathcal{K}(s)\}_{s\geq 2}$ we conclude that

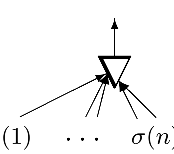
$$H^*(\mathcal{G}\mathrm{r}^*(d), \delta_{\mathrm{h}}) \cong \mathcal{G}\mathrm{r}[\mathcal{K}](d). \tag{6.5}$$

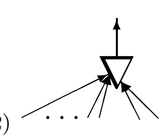
The right hand side is concentrated in degree zero so we omitted the star indicating the grading. In particular, $H^m(\mathcal{G}\mathrm{r}^*(d), \delta_{\mathrm{h}}) = 0$ for $m \neq 0$ which establishes (iii).

The above calculation shows that, for the bicomplex $\mathcal{G}\mathrm{r}^{*,*}(d)$, the cocycle space Z^n_{h} of (5.2) equals

$$Z^n_{\mathrm{h}} = \mathit{Span}\,\{\Lambda \in \mathcal{G}\mathrm{r}[\mathcal{K}](d);\ \Lambda \text{ has precisely } n \text{ vertices decorated by } \mathcal{K}\}. \tag{6.6}$$

Let α_σ, $\sigma \in \mathrm{Ush}(s-2, 2)$, be real coefficients as in Theorem A. If we take the symbol (4.4), with the inputs numbered consecutively from left to right by $\{1, \ldots, s\}$, as the generator of $E^0(s)$, then $\mathcal{K}(s)$ is, as a Σ_s-module, generated by the linear combination

(6.7) $\xi_s := \sum_{\sigma\in\mathrm{Ush}(s-2,2)}$ 

$= \sum_{\sigma\in\mathrm{Ush}(s-2,2)}$ 

$\cdot\, \sigma$.

For each $n \geq 0$ consider the subcomplex $\mathcal{G}\mathrm{r}^*(n+1)_n$ of $\mathcal{G}\mathrm{r}^*(n+1)$ spanned by graphs with a distinguished subset $\{\bullet_1, \ldots, \bullet_n\}$ of the set of black vertices (4.3) with $u = 0$, see 4.4. Suppose we are given, for each $n \geq 0$, cochains $\nu_n \in \mathcal{G}\mathrm{r}^0(n+1)_n$ of the form

$$\nu_n = b_n + l.o.t. \tag{6.8}$$

where b_n denotes the black vertex (4.3) with $u = n$. The abbreviation l.o.t. denotes a linear combination of graphs in $\mathcal{G}\mathrm{r}^0(n+1)_n$ that has at least one ∇-vertex. It is not difficult to verify that each such a graph represents a local expression whose differential order is strictly smaller than the differential order of the local expression represented by b_n (which is $n-1$). This explains why l.o.t. abbreviates "lower order terms."

Similarly, recall from 4.4 that $\mathcal{G}\mathrm{r}^*(n)_n$ denotes the subcomplex of $\mathcal{G}\mathrm{r}^*(n)$ spanned by graphs whose all black vertices belongs to the distinguished subset $\{\overset{\uparrow}{\bullet}_1, \ldots, \overset{\uparrow}{\bullet}_n\}$. Suppose that we are given, for each $n \geq 3$, cochains $\varsigma_n \in \mathcal{G}\mathrm{r}^0(n)$ of the form

$$\varsigma_n = \xi_n + l.o.t. \tag{6.9}$$

where ξ_n is as in (6.7) and l.o.t. a linear combination of graphs with at least two ∇-vertices. It is not difficult to see that ν_n's and ς_n's as above always exist, but we prove a stronger result:

6.2. **Proposition.** *The cocycles $\{\nu_n\}_{n\geq 2}$ and $\{\varsigma_n\}_{n\geq 3}$ can be choosen 'equivariantly,' that is, in such a way that they enjoy the same symmetries as the elements $\{b_n\}_{n\geq 2}$ and $\{\xi_n\}_{n\geq 3}$.*

Proof. Repeating the reasonings in the proof of Lemma 6.1, one easily sees that the obvious modification of the bigrading (6.1) turns the graph complex $\mathcal{G}\mathrm{r}^*(n+1)_n$ into a bicomplex satisfying conditions (i)–(iii) on page 21. The symmetric group Σ_n permutes the distinguished vertices of graphs spanning $\mathcal{G}\mathrm{r}^*(n+1)_n$. This action satisfies the requirements of Corollary 5.4 which therefore gives a Σ_n-equivariant β satisfying assumptions of Proposition 5.1. The element $\nu_n := \beta(b_n)$ is then an 'equivariant' cocycle in that it is, as b_n, Σ_n-stable. An 'equivariant' ς_n can be constructed in the same fashion, considering $\mathcal{G}\mathrm{r}^*(n)_n$ instead of $\mathcal{G}\mathrm{r}^*(n+1)_n$. □

7. Proofs of Theorems A–F

Proof of Theorem A. We will in fact give two proofs, one using the classical reduction theorem, and one based on the graph complex method. Let us start with the 'classical' proof.

By Example 3.7, expression (3.1) corresponding to the leading term of the iterated covariant derivative $\nabla^{n-2}R$ of the curvature tensor belongs to the kernel $\mathcal{K}(n)$, for each $n \geq 3$. Since, by assumption, expressions (3.1) corresponding to the leading terms of the operators D_n generate $\mathcal{K}(n)$, one clearly has, for each $n \geq 3$,

$$\nabla^{n-2}R(X_1, \ldots, X_n) = \sum_{\sigma\in\Sigma_n} A_\sigma D_n(X_{\sigma(1)}, \ldots, X_{\sigma(n)}) + l.o.t.$$

with some real coefficients A_σ. Similarly, the leading term of any operator V_n as in Theorem A equals, by Example 3.7 and (2.3), the leading term of the iterated covariant derivative $(\nabla^{n-1} X_n)(X_1, \ldots, X_{n-1})$, therefore

$$(\nabla^{n-1} X_n)(X_1, \ldots, X_{n-1}) = V(\Gamma, X_1, \ldots, X_n) + l.o.t.$$

Theorem A now follows from Theorem 1.2 and obvious induction on the differential degree.

Let us give another proof of Theorem A based on the method of graph complexes. The drawback of this proof is that it requires the 'stability' $\dim(M) \geq 2d-1$. On the other hand, it is completely independent on local calculations. Moreover, we will need to set up the stage for graph-ical proofs of other statements.

Each iteration in the sense of 3.4 is clearly a linear combination of terms given by contracting 'free' indices of the local coordinate expressions of the operators $\{D_n\}_{n\geq 3}$ and $\{V_n\}_{n\geq 2}$. Each such a contraction is determined by a 'contraction scheme,' which is a graph as in 4.2 with vertices of the following two types:

– vertices d_n, $n \geq 3$, with n linearly ordered input edges and one output, and
– vertices v_n, $n \geq 0$, labeled $1, \ldots, d$, with n linearly ordered edges and one output.

Observe that we allowed vertices v_n also for $n = 0$ as places where order zero vector field variables are to be inserted. Denote by $\mathfrak{Cont}(d)$ the space spanned by the above contraction schemes. Consider the diagram

$$\mathfrak{Gr}[\mathcal{K}](d) \overset{\pi}{\twoheadleftarrow} \mathfrak{Cont}(d) \xrightarrow{\Psi} \mathfrak{Gr}^0(d) \tag{7.1}$$

in which the maps π and Ψ are defined as follows.

The map π replaces each vertex d_n of a contraction scheme $K \in \mathfrak{Cont}(d)$ by ξ_n and each vertex v_n by b_n – recall (6.7) resp. (4.3) for the definition of ξ_n resp. b_n. The map Ψ replaces each vertex d_n by the cocycle $\varsigma_n \in \mathfrak{Gr}^0(n)$ representing, in the correspondence of Theorem 4.6, the operator D_n, and each vertex v_n by the cocycle $\nu_n \in \mathfrak{Gr}^0(n+1)$ representing the operator V_{n+1}. Therefore $\Psi(K)$ is the cocycle representing the iteration determined by K.

The map π is an epimorphism. One may establish this fact by constructing a section (= right inverse) $s : \mathfrak{Gr}[\mathcal{K}](d) \to \mathfrak{Cont}(d)$ of π as follows. Recall [8] that a graph with vertices decorated by a collection $F = \{F(s)\}_{s\geq 0}$ of right Σ_s-modules $F(s)$ (F-graph for short) is an equivalence class of graphs whose vertices have linearly ordered inputs and are decorated by elements of F. The equivalence identifies graphs that differ only by the orders of the inputs and actions of the corresponding permutations at the decorations. The space $\mathfrak{Gr}[\mathcal{K}](d)$ is then spanned by F-graphs with $F := \mathcal{K} \oplus 1$, where $\mathcal{K}$ defined in (6.4) is generated by the sequence $\{\xi_n\}_{n\geq 3}$ (we of course put $\mathcal{K}(s) = 0$ for $s = 0, 1, 2$) and 1 is the collection of trivial representations generated by the elements $\{b_n\}_{n\geq 0}$.

Assume that the graphs $\Lambda_1,\ldots,\Lambda_b$ form a basis of $\mathcal{G}\mathrm{r}[\mathcal{K}](d)$. Choose a representative $\widetilde{\Lambda}_i$ of each Λ_i, $1 \leq i \leq b$, in the equivalence relation described in the previous paragraph. Define $s(\Lambda_i)$ as the contraction scheme obtained from $\widetilde{\Lambda}_i$ by replacing each vertex ξ_n by d_n and each vertex b_n by v_n, preserving the linear orders of the inputs. The identity $\Psi \circ s = id$ for the map s defined in this way is obvious.

The composition $\beta := \Psi \circ s : \mathcal{G}\mathrm{r}[\mathcal{K}](d) \to \mathcal{G}\mathrm{r}^0(d)$ is easy to describe; $\beta(\Lambda_i)$ is the graph obtained from $\widetilde{\Lambda}_i$ by replacing each vertex ξ_n by the graph ς_n representing the operator D_n and each vertex b_n of Λ_i by the graph ν_n representing the operator V_{n+1}, $1 \leq i \leq b$. One easily sees that β satisfies assumptions of Propositions 5.1, with $B^{*,*}$ the graph bicomplex $\mathcal{G}\mathrm{r}^{*,*}(d)$ defined by (6.1) and Z_{h} equalling, by (6.6), the space $\mathcal{G}\mathrm{r}[\mathcal{K}](d)$. Therefore $\beta = \Psi \circ s$ induces an isomorphism $\mathcal{G}\mathrm{r}[\mathcal{K}](d) \cong H^0(\mathcal{G}\mathrm{r}^*(d),\delta)$. In particular, the map Ψ is an epimorphism onto $Ker(\delta : \mathcal{G}\mathrm{r}^0(d) \to \mathcal{G}\mathrm{r}^1(d)) = H^0(\mathcal{G}\mathrm{r}^*(d),\delta)$. This, along with Theorem 4.6, proves Theorem A.

Proof of Theorem B. Operators $\{D_n\}_{n\geq 3}$ defined by (3.6) are clearly iterations, in the sense of 3.4, of the operators $\{U_n\}_{n\geq 3}$. If the leading terms of the operators $\{D_n\}_{n\geq 3}$ are as in Theorem A, then each operator $\mathcal{O}$ is an iteration of the operators $\{D_n\}_{n\geq 3}$ and hence also of the operators $\{U_n\}_{n\geq 3}$. This proves (ii) $\Longrightarrow$ (i). Let us prove the oposite implication.

Fix $n \geq 3$ and write $U_n = L_n + l.o.t.$ (i.e., L_n is the leading term of U_n). Let $E_n = Q_n + l.o.t.$ be an arbitrary operator whose leading term satisfies the assumptions of Theorem A. If (ii) is fulfilled, then, in particular, the operator E_n is an iteration of the operators $\{U_u\}_{u\leq n}$. A simple reasoning based on the c-order implies that the leading term Q_n of E_n is obtained from L_n by linear combinations of successive applications of the operations $\mathcal{O} \mapsto \mathcal{O}\sigma$, $\sigma \in \Sigma_n$, and $\mathcal{O} \mapsto Tr_j(\mathcal{O})X_j$, with some $1 \leq j \leq n$, i.e. by using elementary iterations (i), (ii) and (iv) of 3.4 only, with $\mathcal{O}' := \mathcal{O}$ and $\mathcal{O}''(\Gamma, X_1,\ldots,X_n) := X_j$ in (iv).

Traces commute with the symmetric group action in the sense that, for each $\sigma \in \Sigma_n$ and $1 \leq j \leq n$, there exists some $\tilde{\sigma}_j \in \Sigma_n$ such that

$$(Tr_j(\mathcal{O})X_j)\sigma = Tr_{\sigma^{-1}(j)}(\mathcal{O}\tilde{\sigma}_j)X_{\sigma^{-1}(j)}. \tag{7.2}$$

The explicit description of the permutation $\tilde{\sigma}_j$ is not important for this proof and we leave it as an exercise for the reader. The 'commutativity' (7.2) implies that one may always move the symmetric group action inside the trace and write

$$Q_n = L_n c + \sum_{1\leq j\leq n} Tr_j(L_n c_j)X_j \tag{7.3}$$

for some $c, c_1,\ldots,c_n \in \mathbb{R}[\Sigma_n]$.

While the leading terms of natural differential operators need not be invariant under general coordinate changes, they are still invariant under the action of the general linear group GL_n. This means (see [6]) that the expressions in (7.3) are represented by linear combination of graphs. These graphs have the anchor $\blacksquare$, vertices $\{\bullet_1, \dots, \bullet_n\}$, and one vertex (4.4) with $u := n-2$. Each such a graph has $n+1$ edges, but, since the ∇-vertex is fully symmetric in the first $(n-2)$ and the last two inputs, the stability dimension (= the minimal dimension of the underlying space for which a GL_n-invariant operator uniquely determines a linear combination of graphs) is 3, see [6, Proposition 4.9].

So both sides of (7.3) are represented by the same linear combinations of graphs, i.e. they are given by the same contraction schemes of indices. We conclude that, if $\dim(M) \geq 3$, the leading term of the operator D_n defined by (3.6), with $c, c_1, \dots, c_n$ as in (7.3), is of the form required by Theorem A.

Proof of Theorem C. One may assign to each graph $\Lambda \in \mathcal{G}\mathrm{r}[\mathcal{K}](d)$ the (formal) vf-order (where, as on page 6, vf abbreviates 'vector field') defined by the summation

$$ord_{\mathrm{vf}}(\Lambda) := \sum_{v \in Vert(\Lambda)} ord_{\mathrm{vf}}(v), \tag{7.4}$$

where

$$ord_{\mathrm{vf}}(v) := \begin{cases} 0, & \text{if } v \text{ is } \xi_n,\ n \geq 3, \text{ and} \\ n, & \text{if } v \text{ is } b_n,\ n \geq 0. \end{cases}$$

The vf-order of a contraction scheme $G \in \mathcal{C}\mathrm{ont}(d)$ can be defined similarly, with the role of vertices ξ_n played by d_n, and the role of vertices b_n by v_n. Therefore, if a contraction scheme has vertices $v_{p_1}, \dots, v_{p_t}$ for some $p_1, \dots, p_t \geq 0$ (plus possibly some other vertices of either types), then

$$p_1 + \cdots + p_t \leq ord_{\mathrm{vf}}(G). \tag{7.5}$$

Finally, the vf-order of a graph Λ in $\mathcal{G}\mathrm{r}^0(d)$ is given by formula (7.4) in which we define now

$$ord_{\mathrm{vf}}(v) := \begin{cases} 0, & \text{if } v \text{ is a } \nabla\text{-vertex, and} \\ n, & \text{if } v \text{ is } b_n,\ n \geq 0. \end{cases}$$

The vf-order of an element of $\mathcal{G}\mathrm{r}[\mathcal{K}](d)$ (resp. $\mathcal{C}\mathrm{ont}(d)$, resp. $\mathcal{G}\mathrm{r}^0(d)$) is then the maximum of vf-orders of its linear constituents. It is clear that the (formal) vf-order of a cocycle in $\mathcal{G}\mathrm{r}^0(d)$ equals the vf-order of the operator it represents.

We are going to show that the isomorphism $\beta = \Phi \circ s$ constructed in the proof of Theorem A preserves the vf-order. As before, let $\{\Lambda_i\}_{1 \leq i \leq b}$ be a basis of $\mathcal{G}\mathrm{r}[\mathcal{K}](d)$. Recall from page 28 that β acts by replacing ξ_n-vertices of Λ_i by ς_n and b_n-vertices of Λ_i by ν_n. Another observation we need is that

$$ord_{\mathrm{vf}}(\varsigma_n) = 0,\ n \geq 3, \tag{7.6}$$

while $\nu_n = b_n + \eta_n$, where η_n is a graph such that

$$ord_{\mathrm{vf}}(\eta_n) < ord_{\mathrm{vf}}(b_n),\ n \geq 0. \tag{7.7}$$

Equation (7.6) expresses that ς_n represents the operator D_n which is, by assumption, of order 0 in the vector field variables. Inequality (7.7) is a consequence of the fact that η_n is a linear combination of graphs having at least one ∇-vertex and that, by simple graph combinatorics, the vf-order of each such a graph is strictly less than the vf-order of b_n, compare the remark following (6.8).

The above implies that $\beta(\Lambda_i) = \Lambda_i + \epsilon_i$, where $ord_{\rm vf}(\epsilon_i) < ord_{\rm vf}(\Lambda_i)$ for each $1 \leq i \leq b$, thus also, for an arbitrary linear combination $y \in \mathcal{G}{\rm r}[\mathcal{K}](d)$ of the basis elements, $\beta(y) = y + \varepsilon$, where $ord_{\rm vf}(\varepsilon) < ord_{\rm vf}(y)$. We conclude that then indeed $ord_{\rm vf}(y) = ord_{\rm vf}(\beta(y))$. The fact that the section s constructed in the proof of Theorem A also preserves the vf-order, is obvious.

Let $\mathcal{O} \in \mathfrak{Nat}(Con \times T^{\otimes d}, T)$ be a differential operator represented by a cocycle $c \in \mathcal{G}{\rm r}^0(d)$, $y := \beta^{-1}(c)$ and $C := s(y)$. Acording to our constructions, $C \in \mathcal{C}{\rm ont}(d)$ describes an iteration of $\{D_n\}_{n\geq 3}$ and $\{V_n\}_{n\geq 1}$ representing $\mathcal{O}$. Since both β and s preserve the vf-order, one has $ord_{\rm vf}(C) = ord_{\rm vf}(\mathcal{O})$. Theorem C now immediately follows from formula (7.5).

Remaining proofs. A moment's reflection convinces us that the kernel of the map Ψ in diagram (7.1) is generated by contraction schemes expressing relations (3.7) and (3.8). This is precisely the content of Theorem D. The 'ideal' tensors in Theorem E are natural operators corresponding to the cocycles $\{\varsigma_n\}_{n\geq 3}$ and $\{\nu_n\}_{n\geq 2}$ constructed in Proposition 6.2. Theorem F is a combination of Theorem 4.6 and isomorphism (6.5).

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