Preprint, Institute of Mathematics, AS CR, Prague. 2010-9-22

On a model in radiation hydrodynamics

Bernard Ducomet Eduard Feireisl* Šárka Nečasová †

CEA/DAM/DIF, F-91297 Arpajon, France

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic

Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, 115 67 Praha 1, Czech Republic

Abstract

We consider a simplified model arising in radiation hydrodynamics based on the Navier-Stokes-Fourier system describing the macroscopic fluid motion, and a transport equation modeling the propagation of radiative intensity. We establish existence of global-in-time existence for the associated initial-boundary value problem in the framework of weak solutions.

Key words: Radiation hydrodynamics, Navier-Stokes-Fourier system, weak solution

1 Introduction

The aim of *radiation hydrodynamics* is to incorporate the effects of radiation in the conventional hydrodynamics framework. There are numerous applications ranging from combustion and high-temperature hydrodynamics to models of gaseous stars in astrophysics. Various degrees of complexity of the mathematical models reflect the effect of coupling between the macroscopic descrition of the fluid and the statistical character of the motion of the massless photons. The reader may consult the monographs by Chandrasekhar [6], Mihalas and Weibel-Mihalas [36], Pomraning [39] for more information on the topic.

*The work of E.F. was supported by Grant 201/09/0917 of GA ČR as a part of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503.

†The work of Š.N. was supported by Grant 201/08/0012 of GA ČR as a part of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503.

Following the recent studies by Buet and Després [5], Golse and Perthame [23], we consider a mathematical model, where the motion of the fluid is governed by the standard field equations of classical continuum fluid mechanics describing the evolution of *the mass density* $\varrho = \varrho(t,x)$, the velocity field $\vec{u} = \vec{u}(t,x)$, and the absolute temperature $\vartheta = \vartheta(t,x)$ as functions of the time t and the Eulerian spatial coordinate $x \in \Omega \subset \mathbb{R}^3$. The effect of radiation, represented by its quanta - massless particles called *photons* traveling at the speed of light c - is incorporated in the *radiative intensity* $I = I(t,x,\vec{\omega},\nu)$, depending on the direction vector $\vec{\omega} \in \mathcal{S}^2$, where $\mathcal{S}^2 \subset \mathbb{R}^3$ denotes the unit sphere, and the frequency $\nu \geq 0$. The collective effect of radiation is then expressed in terms of integral means with respect to the variables $\vec{\omega}$ and ν of quantities depending on I. In particular, the radiation energy E_R is given as

$$E_R(t,x) = \frac{1}{c}\int_{\mathcal{S}^2}\int_0^\infty I(t,x,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu. \tag{1.1}$$

The time evolution of I is described by a transport equation with a source term depending on the absolute temperature, while the effect of radiation on the macroscopic motion of the fluid is represented by extra source terms in the momentum and energy equations evaluated in terms of I.

More specifically, the system of equations to be studied reads as follows:

Equation of continuity:

$$\partial_t \varrho + \mathrm{div}_x(\varrho\vec{u}) = 0 \text{ in } (0,T)\times\Omega; \tag{1.2}$$

Momentum equation:

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x p(\varrho,\vartheta) = \mathrm{div}_x\mathbb{T} - \vec{S}_F \text{ in } (0,T)\times\Omega; \tag{1.3}$$

Energy balance equation:

$$\partial_t\left(\varrho\left(\frac{1}{2}|\vec{u}|^2 + e(\varrho,\vartheta)\right)\right) + \mathrm{div}_x\left(\varrho\left(\frac{1}{2}|\vec{u}|^2 + e(\varrho,\vartheta)\right)\vec{u}\right) + \mathrm{div}_x\Big(p\vec{u} + \vec{q} - \mathbb{T}\vec{u}\Big) = -S_E \text{ in } (0,T)\times\Omega; \tag{1.4}$$

Radiation transport equation:

$$\frac{1}{c}\partial_t I + \vec{\omega}\cdot\nabla_x I = S \text{ in } (0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2. \tag{1.5}$$

The symbol $p = p(\varrho,\vartheta)$ denotes the thermodynamic pressure and $e = e(\varrho,\vartheta)$ is the specific internal energy, interrelated through *Maxwell's equation*

$$\frac{\partial e}{\partial\varrho} = \frac{1}{\varrho^2}\left(p(\varrho,\vartheta) - \vartheta\frac{\partial p}{\partial\vartheta}\right). \tag{1.6}$$

Furthermore, $\mathbb{T}$ is the viscous stress tensor determined by *Newton's rheological law*

$$\mathbb{T} = \mu \left(\nabla_x \vec{u} + \nabla_x^t \vec{u} - \frac{2}{3} \mathrm{div}_x \vec{u} \right) + \eta \ \mathrm{div}_x \vec{u} \ \mathbb{I}, \tag{1.7}$$

where the shear viscosity coefficient $\mu = \mu(\vartheta) > 0$ and the bulk viscosity coefficient $\eta = \eta(\vartheta) \geq 0$ are effective functions of the absolute temperature. Similarly, $\vec{q}$ is the heat flux given by *Fourier's law*

$$\vec{q} = -\kappa \nabla_x \vartheta, \tag{1.8}$$

with the heat conductivity coefficient $\kappa = \kappa(\vartheta) > 0$.

Finally,

$$S = S_{a,e} + S_s, \tag{1.9}$$

where

$$S_{a,e} = \sigma_a \Big(B(\nu, \vartheta) - I \Big), \ S_s = \sigma_s \left(\frac{1}{4\pi} \int_{\mathcal{S}^2} I(\cdot, \vec{\omega}) \ \mathrm{d}\vec{\omega} - I \right), \tag{1.10}$$

and

$$S_E = \int_{\mathcal{S}^2} \int_0^\infty S(\cdot, \nu, \vec{\omega}) \ \mathrm{d}\nu \ \mathrm{d}\vec{\omega}, \ \vec{S}_F = \frac{1}{c} \int_{\mathcal{S}^2} \int_0^\infty \vec{\omega} S(\cdot, \nu, \vec{\omega}) \ \mathrm{d}\nu \ \mathrm{d}\vec{\omega}, \tag{1.11}$$

with the absorption coefficient $\sigma_a = \sigma_s(\nu, \vartheta) \geq 0$, and the scattering coefficient $\sigma_s = \sigma_s(\nu, \vartheta) \geq 0$. More restrictions on the structural properties of constitutive relations will be imposed in Section 2 below.

System (1.2 - 1.5) is supplemented with the boundary conditions:

No-slip, no-flux:

$$\vec{u}|_{\partial\Omega} = 0, \ \vec{q} \cdot \vec{n}|_{\partial\Omega} = 0; \tag{1.12}$$

Transparency:

$$I(t, x, \nu, \vec{\omega}) = 0 \text{ for } x \in \partial\Omega, \ \vec{\omega} \cdot \vec{n} \leq 0, \tag{1.13}$$

where $\vec{n}$ denotes the outer normal vector to $\partial\Omega$.

System (1.2 - 1.13) can be viewed as a toy model in radiation hydrodynamics, the physical foundations of which were described by Pomraning [39] and Mihalas and Weibel-Mihalas [36] in the framework of special relativity, see also [37, 38] for a list of references and a review of related computational works in the relativistic framework. Similar systems have been investigated more recently in astrophysics and laser applications (in the relativistic and inviscid case) by Lowrie, Morel and Hittinger [32], Buet and Després [5], with a special attention to asymptotic regimes, see also Dubroca and Feugeas [10], Lin [30] and Lin, Coulombel and Goudon [31] for related numerical issues.

The *existence* of local-in-time solutions and sufficient conditions for blow up of classical solutions in the non-relativistic inviscid case were obtained by

Zhong and Jiang [42], see also the recent papers by Jiang and Wang [27, 28] for a related one-dimensional "Euler-Boltzmann" type models. Moreover, a simplified version of the system has been investigated by Golse and Perthame [23], where global existence was proved by means of the theory of nonlinear semi-groups. To the best of our knowledge, similar results for *viscous* fluids are restricted to the one-dimensional geometry [1, 13, 14] (however see [10] for a simplified treatment of radiation in the diffusion regime in the physically relevant 3D-case).

Our goal in the present paper is to show that the existence theory for the Navier-Stokes-Fourier system developed in [12], and [17, Chapter 3] can be adapted to problem (1.2 - 1.13). As a complete proof of existence becomes rather involved and nowadays well understood (see [17, Chapter 3]), we focus only on the property of *weak sequential stability* for problem (1.2 - 1.13) in the framework of the weak solutions introduced in [12]. More specifically, we introduce a concept of finite energy weak solution in the spirit of [12] and show that any sequence $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ of solutions to problem (1.2 - 1.13), bounded in the natural energy norm, possesses a subsequence converging to another (weak) solution of the same problem. Such a property highlights the essential ingredients involved in the "genuine" proof of existence that may be carried over by means of the arguments delineated in [17, Chapter 3].

In comparison with the standard Navier-Stokes-Fourier system studied in [17], problem (1.2 - 1.13) features a new principal difficulty due to the apparent discrepancy between the classical (non-relativistic) description of the fluid motion, and the behavior of photons traveling with the speed of light. In particular, in contrast with the Second law of thermodynamics, the associated entropy equation may contain a *negative* production term. This problem, related to the fact that, hypothetically, one might have $|\vec{u}| > c$, has already been observed by Buet and Després [5, Section 2.5]. On the other hand, non-negativity of the entropy production rate plays a crucial role in the approach developed in [12]; whence its adaptation to the present setting requires new ideas. Instead of introducing the radiation entropy, we keep the classical form of the entropy balance equation supplemented with the relevant "radiation" production term proportional to

$$\frac{1}{\vartheta}\Big(\vec{u}\cdot\vec{S}_F - S_E\Big),$$

see Section 2. As pointed out, this term may change sign and, accordingly, we have to establish its "weak continuity" with respect to ϑ, $\vec{u}$, and I contained in $\vec{S}_F$, S_E. Note that this is quite delicate as the velocity field $\vec{u}$ may develop uncontrolled *time oscillations* on the hypothetical vacuum zones where ϱ vanishes. In order to overcome this difficulty, we use higher regularity of the $\omega-$averages of the radiative intensity discovered by Bardos et al. [2] and Golse et al. [24, 25]. For further generalizations and a more complete list of references, see Bournaveas and Perthame [4].

The paper is organized as follows. In Section 2, we list the principal hypotheses imposed on constitutive relations, introduce the concept of weak solution to problem (1.2 - 1.13), and state the main result. Uniform bounds imposed on weak solutions by the data are derived in Section 3. The property of *weak sequential stability* of a bounded sequence of weak solutions is established in Section 4. Finally, we introduce a suitable approximation scheme and discuss the main steps of the proof of existence in Section 5.

2 Hypotheses and main results

The structural hypotheses imposed on constitutive relations are motivated by the general *existence theory* for the Navier-Stokes-Fourier system developed in [17, Chapter 3]. To a certain extent they can be viewed as a suitable compromise between the underlying physical properties of real fluids and the hypotheses required by the mathematical theory.

2.1 Constitutive equations

Motivated by [17], we consider the pressure in the form

$$p(\varrho,\vartheta)=\vartheta^{5/2}P\left(\frac{\varrho}{\vartheta^{3/2}}\right)+\frac{a}{3}\vartheta^4,\ a>0, \tag{2.1}$$

where $P:[0,\infty)\to[0,\infty)$ is a given function with the following properties:

$$P\in C^1[0,\infty),\ P(0)=0,\ P'(Z)>0 \text{ for all } Z\geq 0, \tag{2.2}$$

$$0<\frac{\frac{5}{3}P(Z)-P'(Z)Z}{Z}<c \text{ for all } Z\geq 0, \tag{2.3}$$

$$\lim_{Z\to\infty}\frac{P(Z)}{Z^{5/3}}=p_\infty>0. \tag{2.4}$$

The reader may consult [12], [17] for the physical background of hypotheses (2.1 - 2.4). Note that $\vartheta^{5/2}P(\varrho/\vartheta^{3/2})$ is a general form of the molecular pressure compatible with (1.6), satisfying the universal state equation of a monoatomic gas $p=\frac{2}{3}\varrho e$, see Eliezer et al. [15]. Hypothesis (2.2) reflects positive compressibility of the fluid, while the strangely looking (2.3) is equivalent to positivity and boundedness of the specific heat at constant volume. Hypothesis (2.4) means that the fluid behaves like a Fermi gas in the degenerate area $\varrho >> \vartheta^{3/2}$.

The component $\frac{a}{3}\vartheta^4$ represents the effect of "equilibrium" radiation pressure imposed on the fluid by the collective force of the part of photons that may be considered in thermal equilibrium with the fluid. As a matter of fact, since the radiative transfer equation is linear in I, we tacitly suppose that radiation is a sum of two contributions, where the radiative transfer equation (1.5), together with the sources S_E and $\vec{S}_F$ describes the "out-of-equilibrium" part of the radiation, with a temperature ϑ_r which is a priori distinct from the equilibrium temperature ϑ, while the equilibrium part is described by the ϑ^4 Stefan-Boltzmann correction to the gaseous equation of state. To motivate this kind of splitting, just recall that a difficult problem in high-temperature physics consists in the treatment of interfaces separating two media with different optical properties;

one of them at Local Thermodynamical Equilibrium (LTE) and the other not. This is the case of stellar atmospheres in the astrophysical context, and specially in the studies of complicated radiative phenomena appearing at the surface of the sun, see Mihalas [35]. Another example appears in the ICF (Inertial Confinement Fusion) context, where intense laser beams attack a target producing thermonuclear fusion events through ablation fronts, see [8]. In numerical implementations, one needs to develop effective transmission conditions allowing to compute accurately the flow in both these regimes and it is therefore natural to consider the well-posedness problem for our composite model (cf. [11] for a study of the pure equilibrium system).

In accordance with Maxwell's equation (1.6), the specific internal energy e can be taken in the form

$$e(\varrho,\vartheta) = \frac{3}{2}\vartheta\left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \tag{2.5}$$

whereas the associated specific entropy reads

$$s(\varrho,\vartheta) = M\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{2.6}$$

with

$$M'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0.$$

The transport coefficients μ, η, and κ are continuously differentiable functions of the absolute temperature such that

$$0 < c_1(1+\vartheta) \le \mu(\vartheta),\ \mu'(\vartheta) < c_2,\ 0 \le \eta(\vartheta) \le c(1+\vartheta), \tag{2.7}$$

$$0 < c_1(1+\vartheta^3) \le \kappa(\vartheta) \le c_2(1+\vartheta^3) \tag{2.8}$$

for any $\vartheta \ge 0$.

Finally, we assume that σ_a, σ_s, B are continuous functions of ν, ϑ such that

$$0 \le \sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta) \le c_1,\ 0 \le \sigma_a(\nu,\vartheta)B(\nu,\vartheta) \le c_2, \tag{2.9}$$

$$\sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), \sigma_a(\nu,\vartheta)B(\nu,\vartheta) \le h(\nu),\ h \in L^1(0,\infty), \tag{2.10}$$

and

$$\sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta) \le c\vartheta \tag{2.11}$$

for all $\nu \ge 0$, $\vartheta \ge 0$. Relations (2.9 - 2.11) represent a rather crude "cut-off" hypotheses neglecting the effect of radiation at large frequencies ν and low values of the temperature ϑ. Note, however, that relations similar to (2.11) were derived by Ripoll et al. [40].

2.2 Weak formulation

In the weak formulation of the Navier-Stokes-Fourier system, it is customary to replace the equation of continuity (1.2) by its (weak) *renormalized* version represented by a family of integral identities

$$\int_0^T \int_\Omega \Big(b(\varrho)\partial_t\varphi + b(\varrho)\vec{u}\cdot\nabla_x\varphi + \Big(b(\varrho) - b'(\varrho)\varrho\Big)\mathrm{div}_x\vec{u}\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{2.12}$$

$$= -\int_\Omega b(\varrho_0)\varphi(0,\cdot)\ \mathrm{d}x$$

satisfied for any $\varphi \in C_c^\infty([0,\infty)\times\overline{\Omega})$, and any $b \in C^\infty[0,\infty)$, $b' \in C_c^\infty[0,\infty)$. Note that (2.12) implicitly includes satisfaction of the initial condition

$$\varrho(0,\cdot) = \varrho_0.$$

Similarly, the momentum equation (1.3) is replaced by

$$\int_0^T \int_\Omega (\varrho\vec{u}\cdot\partial_t\varphi + \varrho\vec{u}\otimes\vec{u} : \nabla_x\varphi + p\mathrm{div}_x\varphi)\ \mathrm{d}x\ \mathrm{d}t \tag{2.13}$$

$$= \int_0^T \int_\Omega \mathbb{T} : \nabla_x\varphi + \vec{S}_F\cdot\varphi\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega (\varrho\vec{u})_0\cdot\varphi(0,\cdot)\ \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T)\times\Omega;\mathbb{R}^3)$. As the viscous stress contains first derivatives of the velocity $\vec{u}$, for (2.13) to make sense, the field $\vec{u}$ must belong to a certain Sobolev space with respect to the spatial variable. Here, we require that

$$\vec{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)), \tag{2.14}$$

where (2.14) already includes the no-slip boundary condition (1.12).

As a matter of fact, the total energy balance (1.4) is not suitable for the weak formulation since, at least according to the recent state-of-art, the term $\mathbb{T}\vec{u}$ is not controlled on the (hypothetical) vacuum zones of vanishing density. Following [17], we replace (1.4) by the internal energy equation

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e\vec{u}) + \mathrm{div}_x\vec{q} = \mathbb{T} : \nabla_x\vec{u} - p\mathrm{div}_x\vec{u} + \vec{u}\cdot\vec{S}_F - S_E. \tag{2.15}$$

Furthermore, dividing (2.15) by ϑ and using Maxwell's relation (1.6), we may rewrite (2.16) as the entropy equation

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s\vec{u}) + \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) = \frac{1}{\vartheta}\Big(\mathbb{T} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}\Big) + \frac{1}{\vartheta}\Big(\vec{u}\cdot\vec{S}_F - S_E\Big). \tag{2.16}$$

Finally, similarly to [12], equation (2.16) is replaced in the weak formulation by an *inequality*, specifically,

$$\int_0^T \int_\Omega \left(\varrho s\partial_t\varphi + \varrho s\vec{u}\cdot\nabla_x\varphi + \frac{\vec{q}}{\vartheta}\cdot\nabla_x\varphi\right)\ \mathrm{d}x\ \mathrm{d}t \tag{2.17}$$

$$
\leq - \int_\Omega (\varrho s)_0 \varphi(0,\cdot)\ \mathrm{d}x
$$

$$
- \int_0^T \int_\Omega \frac{1}{\vartheta} \left(\mathbb{T} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} \right) \varphi\ \mathrm{d}x\ \mathrm{d}t - \int_0^T \int_\Omega \frac{1}{\vartheta} \Big(\vec{u} \cdot \vec{S}_F - S_E \Big) \varphi\ \mathrm{d}x\ \mathrm{d}t
$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, $\varphi \geq 0$.

Since replacing *equation* (1.4) by *inequality* (2.17) would certainly result in a formally under-determined problem, system (2.12), (2.13), (2.17) must be supplemented with the total energy balance

$$
\int_\Omega \left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho e(\varrho, \vartheta) + E_R \right) (\tau, \cdot)\ \mathrm{d}x \tag{2.18}
$$

$$
+ \int_0^\tau \int \int_{\partial\Omega \times \mathcal{S}^2,\ \vec{\omega}\cdot\vec{n} \geq 0} \int_0^\infty \vec{\omega} \cdot \vec{n} I(t, x, \vec{\omega}, \nu)\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}\ \mathrm{d}S_x\ \mathrm{d}t
$$

$$
= \int_\Omega \left(\frac{1}{2\varrho_0} |(\varrho \vec{u})_0|^2 + (\varrho e)_0 + E_{R,0} \right)\ \mathrm{d}x,
$$

where E_R is given by (1.1), and

$$
E_{R,0} = \int_{\mathcal{S}^2} \int_0^\infty I_0(\cdot, \vec{\omega}, \nu)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu.
$$

The transport equation (1.5) can be extended to the whole physical space $\mathbb{R}^3$ provided we set

$$
\sigma_a(x, \nu, \vartheta) = 1_\Omega \sigma_a(\nu, \vartheta),\ \ \sigma_s(x, \nu, \vartheta) = 1_\Omega \sigma_s(\nu, \vartheta)
$$

and take the initial distribution $I_0(x, \vec{\omega}, \nu)$ to be zero for $x \in \mathbb{R}^3 \setminus \Omega$. Accordingly, for any fixed $\vec{\omega} \in \mathcal{S}^2$, equation (1.5) can be viewed as a linear transport equation defined in $(0,T) \times \mathbb{R}^3$, with a right-hand side S. With the above mentioned convention, extending $\vec{u}$ to be zero outside Ω, we may therefore assume that both ϱ and I are defined on the whole physical space $\mathbb{R}^3$.

Definition 2.1 *We say that $\varrho, \vec{u}, \vartheta, I$ is a weak solution of problem (1.2 - 1.13) if*

$$
\varrho \geq 0,\ \ \vartheta > 0 \textit{ for a.a. } (t,x) \times \Omega,\ \ I \geq 0 \textit{ a.a. in } (0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty),
$$

$$
\varrho \in L^\infty(0,T; L^{5/3}(\Omega)),\ \ \vartheta \in L^\infty(0,T; L^4(\Omega)),
$$

$$
\vec{u} \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3)),\ \ \vartheta \in L^2(0,T; W^{1,2}(\Omega)),
$$

$$
I \in L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)),\ \ I(t,\cdot) \in L^\infty(0,T; L^1(\Omega \times \mathcal{S}^2 \times (0,\infty)),
$$

and if ϱ, $\vec{u}$, ϑ, I satisfy the integral identities (2.12), (2.13), (2.17), (2.18), together with the transport equation (1.5).

2.3 Main result

The main result of the present paper can be stated as follows.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.6), and that the transport coefficients μ, λ, κ, σ_a, and σ_s comply with (2.7 - 2.11).*

Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to problem (1.2 - 1.13) in the sense of Definition 2.1 such that

$$\varrho_\varepsilon(0,\cdot) \equiv \varrho_{\varepsilon,0} \to \varrho_0 \text{ in } L^{5/3}(\Omega), \tag{2.19}$$

$$\int_\Omega \left(\frac{1}{2}\varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) + E_{R,\varepsilon} \right)(0,\cdot)\ \mathrm{d}x \tag{2.20}$$

$$\equiv \int_\Omega \left(\frac{1}{2\varrho_{0,\varepsilon}} |(\varrho\vec{u})_{0,\varepsilon}|^2 + (\varrho e)_{0,\varepsilon} + E_{R,0,\varepsilon} \right)\ \mathrm{d}x \leq E_0,$$

$$\int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(0,\cdot)\ \mathrm{d}x \equiv \int_\Omega (\varrho s)_{0,\varepsilon}\ \mathrm{d}x \geq S_0,$$

and

$$0 \leq I_\varepsilon(0,\cdot) \equiv I_{0,\varepsilon}(\cdot) \leq I_0,\ |I_{0,\varepsilon}(\cdot,\nu)| \leq h(\nu) \text{ for a certain } h \in L^1(0,\infty).$$

Then

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\mathrm{weak}}([0,T]; L^{5/3}(\Omega)),$$

$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \text{ weakly in } L^2(0,T; W^{1,2}(\Omega)),$$

and

$$I_\varepsilon \to I \text{ weakly-(*) in } L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)),$$

at least for suitable subsequences, where $\{\varrho, \vec{u}, \vartheta, I\}$ is a weak solution of problem (1.2 - 1.13).

A major part of the rest of the paper is devoted to the proof of Theorem 2.1. It consists of three steps. First of all, we establish uniform estimates on the family $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ independent of $\varepsilon \to 0$. Secondly, we observe that the extra forcing terms in (2.13), (2.17) due to the effect of radiation are bounded in suitable Lebesgue norms. In particular, the analysis of the macroscopic variables ϱ_ε, $\vec{u}_\varepsilon$, ϑ_ε is essentially the same as in the case of the Navier-Stokes-Fourier system presented in [17]. Consequently, the proof of Theorem 2.1 reduces to the study of the transport equation (1.5) governing the time evolution of the radiation intensity I_ε. In the last part of the paper, we introduce an approximation scheme similar to that used in [17, Chapter 3] and sketch the main ideas of a complete proof of existence of global-in-time weak solutions to problem (1.2 - 1.13).

3 Uniform bounds

Uniform (*a priori*) bounds form the basis of the existence theory. They are derived from the total energy balance, entropy production equation, and other related physical principles. We follow a line of arguments similar to those of [12].

3.1 Energy estimates

As a direct consequence of the total energy balance (2.18), combined with hypotheses of Theorem 2.1, we obtain

$$\operatorname{ess}\sup_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\vec{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} \leq c, \tag{3.1}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)\|_{L^1(\Omega)} \leq c, \tag{3.2}$$

and

$$\operatorname{ess}\sup_{t\in(0,T)} \|E_{R,\varepsilon}\|_{L^1(\Omega)} \leq c. \tag{3.3}$$

Thus, as the internal energy contains the radiation component proportional to ϑ^4, we deduce from (3.2) that

$$\operatorname{ess}\sup_{t\in(0,T)} \|\vartheta_\varepsilon\|_{L^4(\Omega)} \leq c, \tag{3.4}$$

and, by virtue of hypotheses (2.1 - 2.4),

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon\|_{L^{5/3}(\Omega)} \leq c. \tag{3.5}$$

3.2 Estimates of the radiation intensity

At this stage we focus on the transport equation (1.5). Since the quantity I_ε is non-negative, we have

$$\frac{1}{c}\partial_t I_\varepsilon + \vec{\omega}\cdot\nabla_x I_\varepsilon \leq \sigma_s(\nu,\vartheta_\varepsilon)B(\nu,\vartheta_\varepsilon) + \sigma_a(\nu,\vartheta_\varepsilon)\frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon(\cdot,\vec{\omega})\,\mathrm{d}\vec{\omega} \tag{3.6}$$

as the coefficients σ_s, σ_a are also non-negative. Moreover, making use of the "cut-off" hypothesis (2.9), we deduce a uniform bound

$$0 \leq I_\varepsilon(t,x,\nu,\vec{\omega}) \leq c(T)(1 + \sup_{x\in\Omega,\ \nu\geq 0,\vec{\omega}\in\mathcal{S}^2} I_{0,\varepsilon}) \leq c(T)(1+I_0) \text{ for any } t\in[0,T]. \tag{3.7}$$

Finally, hypothesis (2.10), together with (3.7), yield

$$\|S_{E,\varepsilon}\|_{L^\infty((0,T)\times\Omega)} + \|\vec{S}_{F,\varepsilon}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \leq c, \tag{3.8}$$

which, combined with hypothesis (2.11), implies

$$\left\|\frac{1}{\vartheta_\varepsilon}S_{E,\varepsilon}\right\|_{L^\infty((0,T)\times\Omega)} + \left\|\frac{1}{\vartheta_\varepsilon}\vec{S}_{F,\varepsilon}\right\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \leq c. \tag{3.9}$$

3.3 Dissipative estimates

Since the viscosity coefficients satisfy (2.7), we get

$$\int_0^T \int_\Omega \frac{1}{\vartheta_\varepsilon} \mathbb{T}_\varepsilon : \nabla_x \vec{u}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t \geq c_1 \left\| \nabla_x \vec{u}_\varepsilon + \nabla_x^t \vec{u}_\varepsilon - \frac{2}{3} \mathrm{div}_x \vec{u}_\varepsilon \mathbb{I} \right\|^2_{L^2((0,T)\times\Omega;\mathbb{R}^{3\times 3})}$$

$$\geq c_2 \|\vec{u}_\varepsilon\|^2_{L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))},$$

where we have used the standard Korn's inequality.

On the other hand, in accordance with (3.9)

$$\left| \int_0^T \int_\Omega \frac{1}{\vartheta_\varepsilon} \vec{u}_\varepsilon \cdot \vec{S}_{F,\varepsilon} \ \mathrm{d}x \ \mathrm{d}t \right| \leq c \|\vec{u}_\varepsilon\|_{L^1((0,T)\times\Omega;\mathbb{R}^3)},$$

whence the entropy inequality (2.17) yields the uniform bounds

$$\|\vec{u}_\varepsilon\|_{L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))} \leq c, \tag{3.10}$$

$$\|\nabla_x \vartheta_\varepsilon\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \leq c. \tag{3.11}$$

3.4 Pressure estimates

We start with a simple observation: estimates (3.5), (3.10) imply that the sequences $\{\varrho_\varepsilon \vec{u}_\varepsilon\}_{\varepsilon>0}$, $\{\varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon\}_{\varepsilon>0}$ are bounded in the Lebesgue space $L^p((0,T)\times\Omega)$ for a certain $p>1$. Similarly, combining (3.4), (3.10), (3.11) we get

$$\{\mathbb{T}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^p((0,T)\times\Omega;R^{3\times 3}) \text{ for a certain } p>1.$$

Now, repeating step by step the arguments of [21], we observe that the quantities

$$\varphi(t,x) = \psi(t)\mathcal{B}[\varrho_\varepsilon^\omega], \ \ \psi \in \mathcal{D}(0,T) \text{ for a sufficiently small parameter } \omega > 0,$$

may be used as test functions in the momentum equation (2.13), where $\mathcal{B}[v]$ is a suitable branch of solutions to the boundary value problem

$$\mathrm{div}_x \Big(\mathcal{B}[v] \Big) = v - \frac{1}{|\Omega|} \int_\Omega v \ \mathrm{d}x, \ \mathcal{B}|_{\partial\Omega} = 0. \tag{3.12}$$

Note that the construction of the operator $\mathcal{B}$, described in detail in [22], is based on an integral representation formula due to Bogovskii [3].

The resulting estimate reads

$$\int_0^T \int_\Omega p(\varrho_\varepsilon, \vartheta_\varepsilon)\varrho_\varepsilon^\omega \ \mathrm{d}x \ \mathrm{d}t < c, \text{ with } c \text{ independent of } \varepsilon, \tag{3.13}$$

in particular,

$$\{p(\varrho_\varepsilon, \vartheta_\varepsilon)\}_{\varepsilon>0} \text{ is bounded in } L^p((0,T)\times\Omega) \text{ for a certain } p>1. \tag{3.14}$$

Note that an alternative way to obtain these estimates was proposed in [34].

4 Weak sequential stability

To begin, let us stress that the weak stability property for the studied system is a complex problem that requires a lot of ingredients developed elsewhere. The sequential stability of the densities, for example, is based on the method for solving barotropic Navier-Stokes system proposed by Lions [33] and later developed in [20]. Compactness of the temperature requires tools from the theory of Young measures and depends heavily on the presence of the radiation pressure, see [17, Chapter 3] for details. On the other hand, as these steps are nowadays quite well developed and understood, we restrict ourselves to the part of the proof of weak sequential stability that requires new ideas, in particular, we examine the extra terms in the entropy balance equation (2.17).

4.1 Weak sequential stability of macroscopic thermodynamic quantities

In view of the uniform estimates on the radiation forcing terms S_E, $\vec{S}_F$ established in (3.8), (3.9), strong (pointwise) convergence of the macroscopic thermodynamic quantities $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ can be shown exactly as in [12]. Thus we get

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\rm weak}([0,T];L^{5/3}(\Omega)),\ \ \varrho_\varepsilon \to \varrho \text{ a.a. in } (0,T)\times\Omega, \tag{4.1}$$

$$\vartheta_\varepsilon \to \vartheta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)),\ \ \vartheta_\varepsilon \to \vartheta \text{ a.a. in } (0,T)\times\Omega, \tag{4.2}$$

and

$$\log(\vartheta_\varepsilon) \to \log(\vartheta) \text{ in } L^2((0,T)\times\Omega). \tag{4.3}$$

Moreover,

$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)). \tag{4.4}$$

Relations (4.1 - 4.4), together with the uniform bounds established in Section 3, allow us to pass to the limit in the weak formulation of the Navier-Stokes-Fourier system introduced in Section 2.2, as soon as we show convergence of the sequence $\{I_\varepsilon\}_{\varepsilon>0}$. This will be accomplished in the forthcoming section.

4.2 Convergence of the radiation intensities

Our ultimate goal is to establish convergence of the quantities

$$\frac{1}{\vartheta_\varepsilon}\vec{u}_\varepsilon\cdot\vec{S}_{F,\varepsilon} = \frac{1}{c\vartheta_\varepsilon}\vec{u}_\varepsilon\cdot\int_0^\infty \sigma_a(\nu,\vartheta_\varepsilon)\left(\int_{\mathcal{S}^2}\vec{\omega}\,(B(\nu,\vartheta_\varepsilon)-I_\varepsilon)\ {\rm d}\vec{\omega}\right)\ {\rm d}\nu$$

$$+\frac{1}{c\vartheta_\varepsilon}\vec{u}_\varepsilon\cdot\int_0^\infty \sigma_s(\nu,\vartheta_\varepsilon)\left(\int_{\mathcal{S}^2}\vec{\omega}\left(\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon\ {\rm d}\vec{\omega}\right)-I_\varepsilon\right)\ {\rm d}\vec{\omega}\right)\ {\rm d}\nu$$

and

$$\frac{1}{\vartheta_\varepsilon}S_{E,\varepsilon} = \frac{1}{c\vartheta_\varepsilon}\int_0^\infty \sigma_a(\nu,\vartheta_\varepsilon)\left(\int_{\mathcal{S}^2}(B(\nu,\vartheta_\varepsilon)-I_\varepsilon)\ {\rm d}\vec{\omega}\right)\ {\rm d}\nu.$$

Since $\vartheta_\varepsilon \to \vartheta$ a.a. in $(0,T)\times\Omega$, and

$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$$

the desired result follows from compactness of the velocity averages over the sphere $\mathcal{S}^2$ established by Golse et al. [24, 25], see also Bournaveas and Perthame [4], and hypothesis (2.10). Specifically, we use the following result (see [24]).

Proposition 4.1 *Let $I \in L^q([0,T]\times R^{n+1}\times\mathcal{S}^2)$, $\partial_t I + \omega\cdot\nabla_x I \in L^q([0,T]\times R^{n+1}\times\mathcal{S}^2)$ for a certain $q>1$. In addition, let $I_0 \equiv I(0,\cdot) \in L^\infty(R^{n+1}\times\mathcal{S}^2)$.*
Then

$$\tilde{I} \equiv \int_{\mathcal{S}^2} I(\cdot,\nu)\ \mathrm{d}\vec{\omega}$$

belongs to the space $W^{s,q}([0,T]\times R^{n+1})$ for any s, $0<s<\inf\{1/q, 1-1/q\}$, and

$$\|\tilde{I}\|_{W^{s,q}} \le c(I_0)(\|I\|_{L^q} + \|\partial_t I + \omega\cdot\nabla I\|_{L^q}).$$

A direct application of Proposition 4.1 yields the desired conclusion

$$\int_{\mathcal{S}^2} I_\varepsilon(\cdot,\nu)\ \mathrm{d}\vec{\omega} \to \int_{\mathcal{S}^2} I(\cdot,\nu)\ \mathrm{d}\vec{\omega} \text{ in } L^2((0,T)\times\Omega)$$

and

$$\int_{\mathcal{S}^2} \vec{\omega} I_\varepsilon(\cdot,\nu)\ \mathrm{d}\vec{\omega} \to \int_{\mathcal{S}^2} \vec{\omega} I(\cdot,\nu)\ \mathrm{d}\vec{\omega} \text{ in } L^2((0,T)\times\Omega)$$

for any fixed ν. Note that strong (a.a. pointwise) convergence of the $\omega-$averages is needed as $\vec{u}_\varepsilon$ may fail to converge strongly on hypothetical vacuum zones.

Theorem 2.1 has been proved.

5 Approximations, global-in-time existence

We conclude the paper by proposing an approximation scheme to be used to prove existence of global-in-time weak solutions to problem (1.2 - 1.13). The scheme is essentially the same as in [17, Chapter 3], the extra regularizing terms are put in { }.

- The continuity equation (1.2) is replaced by an "artificial viscosity" approximation
$$\partial_t \varrho + \mathrm{div}_x(\varrho\vec{u}) = \{d\Delta\varrho\},\ d>0, \tag{5.1}$$
to be satisfied on $(0,T)\times\Omega$, and supplemented by the homogeneous Neumann boundary conditions
$$\nabla_x\varrho\cdot\vec{n}|_{\partial\Omega} = 0. \tag{5.2}$$
The initial distribution of the approximate densities is given through
$$\varrho(0,\cdot) = \varrho_{0,\delta}, \tag{5.3}$$

where

$$\varrho_{0,\delta} \in C^{2,\nu}(\overline{\Omega}),\ \nabla_x \varrho_{0,\delta} \cdot \vec{n}|_{\partial\Omega} = 0,\ \inf_{x\in\Omega} \varrho_{0,\delta}(x) > 0, \tag{5.4}$$

with a positive parameter $\delta > 0$.

- The momentum equation is replaced by a Faedo-Galerkin approximation:

$$\int_0^T \int_\Omega \Big((\varrho\vec{u})\partial_t\varphi + (\varrho\vec{u}\otimes\vec{u}) : \nabla_x\varphi + (p + \{\delta(\varrho^\Gamma + \rho^2)\})\ \mathrm{div}\ _x\varphi \Big) =$$

$$\int_0^T \int_\Omega \Big(\{d(\nabla_x\rho\nabla_x\vec{u})\} \cdot \varphi + \mathbb{T}_\delta : \nabla_x\varphi - S_F\varphi \Big) dxdt - \int_\Omega (\rho\vec{u})_0 \cdot \varphi\ dx, \tag{5.5}$$

to be satisfied for any test function $\varphi \in C^1_c([0,T), X_n)$, where

$$X_n \subset C^{2,\nu}(\overline{\Omega}; R^3) \subset L^2(\Omega; R^3) \tag{5.6}$$

is a finite-dimensional space of functions satisfying the no-slip boundary conditions

$$\varphi|_{\partial\Omega} = 0 \tag{5.7}$$

The space X_n is endowed with the Hilbert structure induced by the scalar product of the Lebesque space $L^2(\Omega; R^3)$.

We set

$$\begin{aligned} &\mathbb{T}_\delta = \mathbb{T}_\delta(\vartheta, \nabla_x\vec{u}) = \\ &= (\mu(\vartheta) + \delta\vartheta)\Big(\nabla_x\vec{u} + \nabla_x^T\vec{u} - \tfrac{2}{3}\mathrm{div}\ _x\vec{u}\mathbb{I}\Big) + \lambda(\vartheta)\mathrm{div}\ _x\vec{u}\mathbb{I}. \end{aligned} \tag{5.8}$$

- We replace the entropy equation (1.4) by a modified internal energy balance

$$\partial_t(\varrho e + \{\delta\varrho\vartheta\}) + \mathrm{div}_x\Big((\varrho e + \{\delta\varrho\vartheta\})\vec{u}\Big) - \mathrm{div}_x\nabla_x\mathcal{K}_\delta = \tag{5.9}$$

$$\mathbb{T}_\delta(\vartheta, \nabla_x\vec{u}) : \nabla_x\vec{u} - p\ \mathrm{div}_x\vec{u} + \{d\delta(\Gamma|\varrho|^{\Gamma-2} + 2)|\nabla_x\rho|^2 + \delta\frac{1}{\vartheta^2} - d\vartheta^5\}$$

$$-S_E + \vec{u}S_F$$

to be satisfied in $(0,T)\times\Omega$, together with no-flux boundary conditions

$$\nabla_x\vartheta \cdot \vec{n}|_\partial = 0. \tag{5.10}$$

The initial conditions read

$$\varrho(e + \delta\vartheta)(0,\cdot) = \varrho_{0,\delta}(e(\varrho_{0,\delta}, \vartheta_{0,\delta}) + \delta\vartheta_{0,\delta}), \tag{5.11}$$

where the (approximate) temperature distribution satisfies

$$\vartheta_{0,\delta} \in C^1(\overline{\Omega}),\ \nabla_x\vartheta_{0,\delta} \cdot \vec{n}|_{\partial\Omega} = 0,\ \inf_{x\in\Omega} \vartheta_{0,\delta}(x) > 0. \tag{5.12}$$

By $\mathcal{K}_\delta$ we mean

$$\mathcal{K}_\delta(\vartheta) = \int_1^\vartheta \kappa_\delta dz,\ \kappa_\delta = \kappa + \delta\Big(\vartheta^\Gamma + \frac{1}{\vartheta}\Big). \tag{5.13}$$

- We add the equation for the radiative transfer

$$\frac{1}{c}\partial_t I + \vec{\omega}\cdot\nabla_x I = S \text{ in } (0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2, \tag{5.14}$$

together with the transparency condition (1.13).

Given a family of approximate solutions $\{\varrho_{d,\delta}, \vec{u}_{d,\delta}, \vartheta_{d,\delta}, I_{d,\delta}\}_{d>0,\delta>0}$, we may construct a weak solution of system (1.2 - 1.13) letting successively $d \to 0$, $\delta \to 0$ and using compactness arguments delineated in the previous part of this paper. The reader may consult [17, Chapter 3] for all technical details. The approximate solutions can be constructed by means of a fixed point argument applied to the couple $\vec{u}$, I, similarly to [17, Chapter 3, Section 3.4].

References

[1] A.A. Amosov, Well-posedness in the large of initial and boundary-value problems for the system of dynamical equations of a viscous radiating gas, Sov. Physics Dokl. 30 (1985) 129–131.

[2] C. Bardos, F. Golse, B. Perthame, R. Sentis, The nonaccretive radiative transfer equations: Existence of solutions and Rosseland Approximation, J. of Functional Analysis 77 (1988) 434–460.

[3] M.E. Bogovskii, Solution of some vector analysis problems connected with operators div and grad, Trudy Sem. S.L. Sobolev 80 (1980) 5–40 (in Russian).

[4] N. Bournaveas, B. Perthame, Averages over spheres for kinetic transport equations; hyperbolic Sobolev spaces and Strichartz inequalities, J. Math. Pures Appl. 80 (2001) 517–534.

[5] C. Buet, B. Després, Asymptotic analysis of fluid models for the coupling of radiation and hydrodynamics, J. Quant. Spectroscopy Rad. Transf. 85 (2004) 385–480.

[6] S. Chandrasekhar, Radiative transfer, Dover Publications, New York, 1960.

[7] R. Coifman, Y. Meyer, On commutators of singular integrals and bilinear singular integrals, Trans. Amer. Math. Soc. 212 (1975) 315–331.

[8] R. Dautray, J.P. Watteau, La fusion thermonucléaire inertielle par laser, Eyrolles, Paris, 1993.

[9] R.J. DiPerna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. 98 (1989) 511–547.

[10] B. Dubroca, J.-L. Feugeas, Etude théorique et numérique d'une hiérarchie de modéles aux moments pour le transfert radiatif, C. R. Acad. Sci. Paris, Ser I 329 (1999) 915–920.

[11] B. Ducomet, E. Feireisl, On the dynamics of gaseous stars, Archiv of Rational Mechanics and Analysis 174 (2004) 221–266.

[12] B. Ducomet, E. Feireisl, The equations of magnetohydrodynamics: On the interaction between matter and radiation in the evolution of gaseous stars, Commun. Math. Phys. 266 (2006) 595–629.

[13] B. Ducomet, Š. Nečasová, Global existence of solutions for the one-dimensional motions of a compressible gas with radiation: an infrarelativistic model. Nonlinear Analysis TMA 72 (2010) 3258–3274.

[14] B. Ducomet, Š. Nečasová, Global weak solutions to the 1D compressible Navier-Stokes equations with radiation. Communications in Mathematical Analysis 8 (2010) 23–65.

[15] Eliezer, A. Ghatak, H. Hora, An introduction to equations of states, theory and applications, Cambridge University Press, Cambridge, 1986.

[16] E. Feireisl Dynamics of viscous compressible fluids, Oxford university Press, 2001.

[17] E. Feireisl, A. Novotný, Singular limits in thermodynamics of viscous fluids, Birkhauser, Basel, 2009.

[18] E. Feireisl, On the motion of a viscous, compressible, and heat conducting fluid, Indiana Univ. Math. J. 53 (2004) 1707–1740.

[19] E. Feireisl, On compactness of solutions to the compressible isentropic Navier-Stokes equations when the density is not square integrable, Comment. Math. Univ. Carolinae 42 (2001) 83–98.

[20] E. Feireisl, A. Novotný, H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations of compressible isentropic fluids, J. Math. Fluid Mech. 3 (2001) 358–392.

[21] E. Feireisl, H. Petzeltová, On integrability up to the boundary of the weak solutions of the Navier-Stokes equations of compressible flow. Commun. Partial Differential Equations 25 (2000) 755–767.

[22] G.P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations, I, Springer-Verlag, New York, 1994.

[23] F. Golse, B. Perthame, Generalized solutions of the radiative transfer equations in a singular case, Comm. Math. Phys. 106 (1986) 211–239.

[24] F. Golse, P.L. Lions, B. Perthame, R. Sentis, Regularity of the moments of the solution of a transport equation, J. Funct. Anal. 16 (1988) 110–125.

[25] F. Golse, B. Perthame, R. Sentis, Un résultat de compacité pour les équations de transport et application au calcul de la limite de la valeur propre prinicipale d'un opérateur de transport, C.R. Acad.Sci. Paris, Ser. I. 301 (1985) 341–344.

[26] F. Golse, G. Allaire, Transport et diffusion, Lecture Notes, Ecole polytechnique, 2010.

[27] P. Jiang, D. Wang, Formation of singularities of solutions of the radiative transfer equations in a singular case, Preprint, March 11, 2009.

[28] P. Jiang, D. Wang, Global weak solutions to the Euler-Boltzmann equations in radiation hydrodynamics, Preprint, June 27, 2009.

[29] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Uralceva, Linear and quasilinear equations of parabolic type, AMS, tans. Math. Monograph 23, Providence, 1968.

[30] C. Lin. Mathematical analysis of radiative transfer models, PhD Thesis, 2007.

[31] C. Lin, J.-F. Coulombel, T. Goudon, Shock profiles for non-equilibrium radiative gases, Physica D 218 (2006) 83–94.

[32] R.B. Lowrie, J.E. Morel, J.A. Hittinger, The coupling of radiation and hydrodynamics, The Astrophysical Journal 521 (1999) 432–450.

[33] P.-L. Lions, Mathematical topics in fluid dynamics, Vol.2, Compressible models, Oxford Science Publication, Oxford, 1998.

[34] P.-L. Lions. Bornes sur la densité pour les équations de Navier-Stokes compressibles isentropiques avec conditions aux limites de Dirichlet, C.R. Acad. Sci. Paris, Ser. I 328 (1000) 659–662.

[35] B. Mihalas, Stellar Atmospheres, W.H. Freeman and Cie, 1978.

[36] B. Mihalas, B. Weibel-Mihalas, Foundations of radiation hydrodynamics, Dover Publications, Dover, 1984.

[37] A. Munier, R. Weaver, Radiation transfer in the fluid frame: a covariant formulation Part I: Radiation hydrodynamics, Computer Phys. Rep. 3 (1986) 125–164.

[38] A. Munier, R. Weaver, Radiation transfer in the fluid frame: a covariant formulation Part II: Radiation transfer equation, Computer Phys. Rep. 3 (1986) 165–208.

[39] G.C. Pomraning, Radiation hydrodynamics, Dover Publications, New York, 2005.

[40] J.F. Ripoll, B. Dubroca and G. Duffa, Modelling radiative mean absorption coefficients, Combust. Theory Modelling 5 (2001) 261–274.

[41] L. Tartar, Compensated compactness and applications to partial differential equations, Nonlinear Anal. and Mech., Heriot-Watt Sympos., L.J. Knopp Editor, Research Notes in Math 39, Pitman, Boston (1975) 136–211.

[42] X.Zhong, J. Jiang, Local existence and finite-time blow up in multidimensional radiation hydrodynamics, J. Math.Fluid Mech. 9 (2007) 543–564.